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# Coincidence point theorems for cyclic multi-valued and hybrid contractive mappings

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ABSTRACT. In this paper, we consider the existence theorem of coincidence point for a pair of single-valued and multi-valued mapping that are concerned with the concepts of cyclic contraction type mapping. Some illustrative examples and remarks are also discussed.

## 1. INTRODUCTION

It is well known that, in the case of single-valued mappings, the Banach contraction principle (see [4]) is one of the most powerful tools in nonlinear analysis. It has been extended and generalized in many directions. One of the most significant extensions is due to Kannan [18], who considered a contraction condition that does not force the mapping to be continuous, as in the case of Banach contraction principle. Another important generalization has been established by Kirk et al. [25] who introduced the notion of cyclic operators, which is a natural generalization of the Banach contraction principle. They proved the following fixed point result.

**Theorem 1.1.** *Let $A$ and $B$ be two nonempty closed subsets of a complete metric space. Suppose $T : A \cup B \to A \cup B$ satisfies the following conditions:*

*i)* $T(A) \subseteq B$ *and* $T(B) \subseteq A$,

*ii) there is $r \in (0, 1)$ such that*

$$d(Tx, Ty) \leq rd(x, y), \quad \text{for all} \quad x \in A, y \in B.$$

*Then $T$ has a unique fixed point in $A \cap B$.*

This theorem represents one of the important acquirements of fixed point theory for cyclic mappings. In the same paper [25], Theorem 1.1 has been extended to the case of arbitrary finite non-empty subsets of the metric space. For the other generalizations of Banach contraction principle towards the cyclic type direction, one may see [10, 20, 21, 22, 31, 32, 35, 34, 38, 40, 43].

By considering the Pompeiu-Hausdorff metric $H(\cdot, \cdot)$ on the class of closed bounded subsets $\mathcal{CB}(X)$ of a complete metric space $(X, d)$, Nadler [30] obtained the following fixed point theorem for multi-valued contractive type mappings.

**Theorem 1.2.** [30] *Let $(X, d)$ be a complete metric space and $T : X \to \mathcal{CB}(X)$. Assume that there exists $r \in [0, 1)$ such that*

$$H(Tx, Ty) \leq rd(x, y) \quad \text{for all} \quad x, y \in X. \tag{1.1}$$

*Then there exists $z \in X$ such that $z \in Tz$.*

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Following Nadler's theorem, the fixed point theory for multi-valued mapping has developed in many directions, see, for instance [2, 7, 13, 14, 15, 16, 17, 19, 26, 27, 29, 36, 42] and the papers cited there, and has important applications in many branches in nonlinear analysis, for instance, control theory, differential equations, economics etc., see [23, 28, 39, 41].

In 2008, Kikkawa and Suzuki [26] provided the significant improvement of Nadler's result by considering the following condition, now the so-called Suzuki type contractive condition.

**Theorem 1.3.** [26] *Let $(X,d)$ be a complete metric space and $T: X \to \mathcal{CB}(X)$. Define a strictly decreasing function $\eta : [0,1) \to (\frac{1}{2}, 1]$ by*

$$\eta(r) = \frac{1}{1+r},$$

*and assume that there exists $r \in [0,1)$ such that*

$$\eta(r)D(x,Tx) \leq d(x,y) \ \textit{implies} \quad H(Tx,Ty) \leq rd(x,y) \tag{1.2}$$

*for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.*

On the other hand, Damjanović and Dorić [14] obtained the fixed point theorem for multi-valued generalization of the well-known Kannan's fixed point theorem from the case of single-valued mappings.

**Theorem 1.4.** [14] *Let $(X,d)$ be a complete metric space and $T : X \to \mathcal{CB}(X)$. Define a non-increasing function $\phi : [0,1) \to (0,1]$ by*

$$\phi(r) := \begin{cases} 1, & \textit{if } 0 \leq r < \frac{\sqrt{5}-1}{2}; \\ 1-r, & \textit{if } \frac{\sqrt{5}-1}{2} \leq r < 1. \end{cases}$$

*Assume that*

$$\phi(r)D(x,Tx) \leq d(x,y) \ \textit{implies} \ H(Tx,Ty) \leq r\max\{D(x,Tx), D(y,Ty)\}, \tag{1.3}$$

*for all $x, y \in X$. Then, there exists $z \in X$ such that $z \in Tz$.*

Later, in 2011, by focusing on the contractive condition part of above theorems, Dorić and Lazović [16] presented another generalization of both Nadler's result and Kannan's result by considering a Ćirić type strong quasi-contractive condition [12], see also [5].

**Theorem 1.5.** [16] *Let $(X,d)$ be a complete metric space and $T : X \to \mathcal{CB}(X)$. Assume that there exists $r \in [0,1)$ such that the function $\varphi : [0,1) \to (0,1]$ which is defined by*

$$\varphi(r) := \begin{cases} 1, & \textit{if } 0 \leq r < \frac{1}{2}; \\ 1-r, & \textit{if } \frac{1}{2} \leq r < 1, \end{cases}$$

*satisfies the following condition: if $\varphi(r)d(x,Tx) \leq d(x,y)$ then*

$$H(Tx,Ty) \leq r\max\left\{d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty)+D(y,Tx)}{2}\right\},$$

*for all $x, y \in X$. Then, there exists $z \in X$ such that $z \in Tz$.*

In this work, motivated by Dorić and Lazović [16] results, we introduce a new class of hybrid pair of single-valued and multi-valued mappings and establish some hybrid coincidence and common fixed point theorems in complete metric spaces. Moreover, some illustrative examples and remarks are also discussed.

## 2. Main results

Let $(X,d)$ be a metric space. We denote by $\mathcal{CB}(X)$ for the family of nonempty closed bounded subsets of $X$. The Pompeiu-Hausdorff metric induced by the metric $d$, $H(\cdot,\cdot)$, is defined by

$$H(A,B) = \max\left\{\sup_{a\in A} D(a,B), \sup_{b\in B} D(A,b)\right\}, \quad \text{for } A,B \in \mathcal{CB}(X),$$

where $D(a,B) = \inf_{b\in B} d(a,b)$ is the distance from a point $a$ to a set $B \in \mathcal{CB}(X)$. It is well known that $(\mathcal{CB}(X),H)$ is a metric space. Moreover, $(\mathcal{CB}(X),H)$ is a complete metric space if it is induced by a complete metric space $(X,d)$.

The following lemmas will be needed to prove our main results.

**Lemma 2.1.** [30] *For $A,B \in \mathcal{CB}(X)$ and for $a \in A$ and $q > 1$, there exists an element $b \in B$ such that $d(a,b) \le qH(A,B)$.*

**Lemma 2.2.** [1] *Let $A$ be a nonempty subset of metric space $(X,d)$. Then $D(x,A) \le d(x,y) + D(y,A)$ for any $x,y \in X$.*

We now recall the concepts of fixed point, coincidence point and common fixed point. Let $(X,d)$ be a metric space, $f : X \to X$ be a single-valued mapping and $T : X \to \mathcal{CB}(X)$ be a multi-valued mapping. An element $x \in X$ is called

i) a *fixed point* of $T$ if $x \in Tx$.
ii) a *coincidence point* of $f$ and $T$ if $fx \in Tx$.
iii) a *common fixed point* of $f$ and $T$ if $x = fx \in Tx$.

For the mappings $f : X \to X$ and $T : X \to \mathcal{CB}(X)$, we will denote by $F(T)$, $C(f,T)$ and $F(f,T)$ the set of all fixed points, coincidence points and commom fixed points, respectively. Let $T : X \to \mathcal{CB}(X)$ be a multi-valued mapping. For each $A \subseteq X$, we put

$$T(A) = \bigcup_{a\in A} Ta. \tag{2.4}$$

Now, we present a coincidence point theorem by considering the following non-increasing function $\varphi : [0,1) \to (0,1]$ defined by Dorić and Lazović [16], that is,

$$\varphi(r) := \begin{cases} 1, & \text{if } 0 \le r < \frac{1}{2}; \\ 1-r, & \text{if } \frac{1}{2} \le r < 1. \end{cases} \tag{2.5}$$

**Theorem 2.6.** *Let $(X,d)$ be a complete metric space, $f : X \to X$ be a single-valued mapping and $T : X \to \mathcal{CB}(X)$ be a multi-valued mapping. Let $A_1, A_2, \ldots, A_m$ be nonempty subsets of $X$ such that $T(A_i) \subseteq f(A_{i+1})$, for each $i = 1,\ldots,m-1$ and $T(A_m) \subseteq f(A_1)$. Assume that the following conditions are satisfied:*

*i) There is $\bar{j} \in \{1,\ldots,m\}$ such that $f(A_{\bar{j}})$ is a closed set.*
*ii) There exists $r \in [0,1)$ such that $\varphi(r)D(fx,Tx) \le d(fx,fy)$ implies*

$$H(Tx,Ty) \le r\max\left\{d(fx,fy), D(fx,Tx), D(fy,Ty), \frac{D(fx,Ty)+D(fy,Tx)}{2}\right\}, \tag{2.6}$$

*for $x \in A_i, y \in A_{i+1}$, where $i \in \{1,\ldots,m\}$ and $A_{m+1} = A_1$.*

*Then there is $z \in A_{\bar{j}}$ such that $z \in C(f,T)$. In addition, if $ffz = fz$ and either $f(A_{\bar{j}+1})$ or $f(A_{\bar{j}-1})$ is a closed set then $fz \in F(f,T)$.*

*Proof.* Let $r_1$ be a real number with $0 \leq r < r_1 < 1$. Consider $x_1 \in A_1$. By assumption (i), we have $T(x_1) \subseteq f(A_2)$. So there exists a point $x_2$ in $A_2$ such that $fx_2 \in Tx_1 \subseteq f(A_2)$. Since $\varphi(r) < 1$, we have

$$\varphi(r)D(fx_1, Tx_1) \leq D(fx_1, Tx_1) \leq d(fx_1, fx_2).$$

By using the contractive condition (2.6), we obtain that

$$\begin{aligned} H(Tx_1, Tx_2) &\leq r \max\left\{d(fx_1, fx_2), D(fx_1, Tx_1), D(fx_2, Tx_2), \frac{D(fx_1, Tx_2) + D(fx_2, Tx_1)}{2}\right\} \\ &= r \max\left\{d(fx_1, fx_2), D(fx_2, Tx_2), \frac{D(fx_1, Tx_2)}{2}\right\} \\ &\leq r \max\left\{d(fx_1, fx_2), D(fx_2, Tx_2), \frac{d(fx_1, fx_2) + D(fx_2, Tx_2)}{2}\right\} \\ &= r \max\big\{d(fx_1, fx_2), D(fx_2, Tx_2)\big\}. \end{aligned}$$

Using this one, together with the fact that $D(fx_2, Tx_2) \leq H(Tx_1, Tx_2)$, we have

$$D(fx_2, Tx_2) \leq r \max\big\{d(fx_1, fx_2), D(fx_2, Tx_2)\big\}.$$

If $\max\big\{d(fx_1, fx_2), D(fx_2, Tx_2)\big\} = D(fx_2, Tx_2)$, then we have

$$D(fx_2, Tx_2) \leq rD(fx_2, Tx_2) < D(fx_2, Tx_2),$$

which is a contradiction. Thus, $\max\big\{d(fx_1, fx_2), D(fx_2, Tx_2)\big\} = d(fx_1, fx_2)$ and it follows that

$$D(fx_2, Tx_2) \leq H(Tx_1, Tx_2) \leq rd(fx_1, fx_2). \tag{2.7}$$

Since $fx_2 \in Tx_1$ and $\frac{r_1}{r} > 1$, by using Lemma2.1 together with (2.7), there exists $x_3 \in A_3$ with $fx_3 \in Tx_2$ such that

$$d(fx_2, fx_3) \leq r_1 d(fx_1, fx_2).$$

By continuing this process, we construct a sequence $\{fx_n\}$ in $X$ such that

$$fx_{n+1} \in Tx_n \quad \text{and} \quad d(fx_{n+1}, fx_{n+2}) \leq r_1 d(fx_n, fx_{n+1}), \tag{2.8}$$

where $(x_n, x_{n+1}) \in (A_n, A_{n+1})$. Next, from (2.8), we have

$$\sum_{n=1}^{\infty} d(fx_n, fx_{n+1}) \leq \sum_{n=1}^{\infty} r_1^{n-1} d(fx_1, fx_2) < \infty,$$

since $r_1 \in (0, 1)$, this implies that $\{fx_n\}$ is a Cauchy sequence in the complete metric space $(X, d)$. Subsequently, let $u \in X$ such that $\lim_{n\to\infty} fx_n = u$. Moreover, by constructing method of $\{fx_n\}$, we have $u \in \overline{f(A_i)}$ for all $i \in \{1, \dots, m\}$. Thus, since $f(A_{\bar{j}})$ is a closed set and $\cap_{i=1}^m \overline{f(A_i)} \subseteq \overline{f(A_{\bar{j}})} = f(A_{\bar{j}})$, there exists $z \in A_{\bar{j}}$ such that $fz = u$. Moreover, we can find a subsequence $\{fx_{n(k)}\}$ of $\{fx_n\}$ such that $\{fx_{n(k)}\} \subseteq f(A_{\bar{j}})$ and $\lim_{k\to\infty} fx_{n(k)} = fz$. By considering, such an index $\bar{j}$, we now show that

$$D(fz, Tx) \leq r \max\big\{d(fz, fx), D(fx, Tx)\big\}, \tag{2.9}$$

for all $x \in A_{\bar{j}-1} \cup A_{\bar{j}+1}$ such that $fx \in X \setminus \{fz\}$. Assume that $x \in A_{\bar{j}-1}$ and $fx \in X \setminus \{fz\}$. Note that, there is a natural number $n_1 \in \mathbb{N}$ such that $d(fz, fx_{n(k)}) \leq \frac{1}{3} d(fz, fx)$ for all $k \geq n_1$. Now, for each $k \geq n_1$ we consider

$$\begin{aligned} \varphi(r)D(fx_{n(k)}, Tx_{n(k)}) &\leq D(fx_{n(k)}, Tx_{n(k)}) \leq d(fx_{n(k)}, fx_{n(k)+1}) \\ \leq d(fx_{n(k)}, fz) + d(fz, fx_{n(k)+1}) &\leq d(fx, fz) - d(fx_{n(k)}, fz) \leq d(fx_{n(k)}, fx). \end{aligned}$$

Thus, we have $\varphi(r)D(fx_{n(k)}, Tx_{n(k)}) \leq d(fx_{n(k)}, fx)$ for all $k \geq n_1$. Then, in view of (2.6), we get

$$H(Tx_{n(k)}, Tx) \leq r \max\Big\{ d(fx_{n(k)}, fx), D(fx_{n(k)}, Tx_{n(k)}), D(fx, Tx), \frac{D(fx_{n(k)}, Tx) + D(fx, Tx_{n(k)})}{2} \Big\}. \tag{2.10}$$

Since $fx_{n(k)+1} \in Tx_{n(k)}$, we have $D(fx_{n(k)+1}, Tx) \leq H(Tx_{n(k)}, Tx)$ and $D(fx_{n(k)}, Tx_{n(k)}) \leq d(fx_{n(k)}, fx_{n(k)+1})$. Using this fact, from (2.10), we obtain

$$D(fx_{n(k)+1}, Tx) \leq r \max\Big\{ d(fx_{n(k)}, fx), d(fx_{n(k)}, fx_{n(k)+1}), D(fx, Tx), \frac{D(fx_{n(k)}, Tx) + d(fx, fx_{n(k)+1})}{2} \Big\},$$

for all $k \geq n_1$. Now, letting $k \to \infty$, we have

$$D(fz, Tx) \leq r \max\Big\{ d(fz, fx), D(fx, Tx), \frac{D(fz, Tx) + d(fx, fz)}{2} \Big\},$$

for all $x \in A_{\bar{j}-1}$, such that $fx \in X \setminus \{fz\}$, which is equivalent to

$$D(fz, Tx) \leq r \max\big\{ d(fz, fx), D(fx, Tx) \big\}.$$

Similarly, for the case $x \in A_{\bar{j}+1}$, we can show that (2.9) holds. This proves the claim.

Finally, we will show that $z \in C(f, T)$. We divide the proof into the following two cases.

<u>Case I.</u> $0 \leq r < \frac{1}{2}$.

Suppose on the contrary, that $fz \notin Tz$. Let $a \in A_{\bar{j}-1}$ such that $fa \in Tz$ be such that $2rd(fz, fa) < D(fz, Tz)$. Note that we also have $fa \neq fz$ Now, we consider in the case that $fa \neq fz$. Thus, in view of (2.9), we have

$$D(fz, Ta) \leq r \max\big\{ d(fz, fa), D(fa, Ta) \big\}. \tag{2.11}$$

On the other hand, since $\varphi(r)D(fz, Tz) \leq D(fz, Tz) \leq d(fz, fa)$, we have

$$\begin{aligned} H(Tz, Ta) &\leq r \max\Big\{ d(fz, fa), D(fz, Tz), D(fa, Ta), \frac{D(fz, Ta) + D(fa, Tz)}{2} \Big\} \\ &\leq r \max\big\{ d(fz, fa), D(fa, Ta) \big\}. \end{aligned} \tag{2.12}$$

So,

$$D(fa, Ta) \leq H(Tz, Ta) \leq r \max\big\{ d(fz, fa), D(fa, Ta) \big\}.$$

Observe that, if $\max\big\{ d(fz, fa), D(fa, Ta) \big\} = D(fa, Ta)$, we would have

$$D(fa, Ta) \leq rD(fa, Ta) < D(fa, Ta)$$

which is a contradiction. Thus, by this observation and by (2.11) and (2.12), we have

$$D(fz, Ta) \leq rd(fz, fa) \text{ and } H(Tz, Ta) \leq rd(fz, fa).$$

This implies

$$D(fz, Tz) \leq D(fz, Ta) + H(Ta, Tz) \leq rd(fz, fa) + rd(fz, fa) = 2rd(fz, fa) < D(fz, Tz),$$

which is a contradiction. Thus we must have $fz \in Tz$, as required.

<u>Case II.</u> $\frac{1}{2} \leq r < 1$.

First, we will show that for all $x \in A_{\bar{j}-1}$, such that $fx \in X \setminus \{fz\}$, one has

$$\varphi(r)D(fx, Tx) \leq d(fx, fz). \tag{2.13}$$

Let $x \in A_{\bar{j}-1}$, such that $fx \in X/\{fz\}$ be given. Observe that,

$$D(fx, Tx) \le d(fx, fz) + D(fz, Tx).$$

Thus, by (2.9), it follows that

$$D(fx, Tx) \le d(fx, fz) + r \max\big\{d(fz, fx), D(fx, Tx)\big\}. \tag{2.14}$$

If $\max\big\{d(fz, fx), D(fx, Tx)\big\} = d(fz, fx)$, from (2.14) we get

$$D(fx, Tx) \le d(fx, fz) + rd(fx, fz) = (1 + r)d(fx, fz).$$

So,

$$\varphi(r)D(fx, Tx) = (1 - r)D(fx, Tx) \le \frac{1}{1 + r}D(fx, Tx) \le d(fx, fz),$$

and we conclude that (2.13) holds. Now, if

$$\max\big\{d(fz, fx), D(fx, Tx)\big\} = D(fx, Tx),$$

from (2.14) and we get

$$D(fx, Tx) \le d(fx, fz) + rD(fx, Tx),$$

which implies that

$$\varphi(r)D(fx, Tx) = (1 - r)D(fx, Tx) \le d(fx, fz),$$

we have that (2.13) holds, too. By using the assumption (2.6), we obtain that

$$H(Tx, Tz) \le r \max\left\{d(fx, fz), D(fx, Tx), D(fz, Tz), \frac{D(fx, Tz) + D(fz, Tx)}{2}\right\}, \tag{2.15}$$

for all $x \in A_{\bar{j}-1}$, such that $fx \in X \setminus \{fz\}$.

If $\{fx_{n(k)}\} \subseteq f(A_{\bar{j}-1})$ satisfies $\lim_{n\to\infty} fx_{n(k)} = fz$, then by (2.15) we obtain

$$\begin{aligned} D(fz, Tz) = \lim_{k\to\infty} D(fx_{n(k)+1}, Tz) \le \lim_{k\to\infty} H(Tx_{n(k)}, Tz) \le \lim_{k\to\infty} r \max\Big\{&d(fx_{n(k)}, fz), \\ D(fx_{n(k)}, Tx_{n(k)}), D(fz, Tz), \frac{D(fx_{n(k)}, Tz) + D(fz, Tx_{n(k)})}{2}&\Big\} \\ \le \lim_{k\to\infty} r \max\Big\{d(fx_{n(k)}, fz), D(fx_{n(k)}, fx_{n(k)+1}), D(fz, Tz),& \\ \frac{D(fx_{n(k)}, Tz) + D(fz, fx_{n(k)+1})}{2}\Big\} = rD(fz, Tz).& \end{aligned}$$

Since $r < 1$, we can conclude that $D(fz, Tz) = 0$. This means that $z \in C(f, T)$.

Now, assume that $ffz = fz$ and either $f(A_{\bar{j}+1})$ or $f(A_{\bar{j}-1})$ is a closed set. We will show that $fz \in F(f, T)$.

Note that, since $z \in C(f, T)$ and $ffz = fz$, we have $ffz = fz \in Tz$. Let us consider for the case $fz \in f(A_{\bar{j}+1})$. We see that

$$\varphi(r)D(fz, Tz) \le D(fz, Tz) \le d(fz, ffz).$$

Thus, we using (2.6), we obtain

$$\begin{aligned} H(Tz, Tfz) &\le r \max\left\{d(fz, ffz), D(fz, Tz), D(ffz, Tz), \frac{D(fz, Tfz) + D(ffz, Tz)}{2}\right\} \\ &= rD(ffz, Tfz). \end{aligned}$$

By using the above result together with the fact that $ffz \in Tz$, we get $D(ffz, Tfz) \le rD(ffz, Tfz)$. Since $r \in [0, 1)$, this implies $D(ffz, Tfz) = 0$. Thus, by the hypothesis $fz = ffz$, we conclude that $fz \in Tfz$. This means $fz \in F(f, T)$, as required.

The proof for the case of $fz \in f(A_{\bar{j}-1})$ is similar to the above case, so it is omitted. □

In the following, we give an example of a pair of a single-valued and a multi-valued mapping that satisfies all the hypotheses of Theorem 2.6.

**Example 2.1.** Consider the set of real numbers, $\mathbb{R}$, with the usual metric and let $T:\mathbb{R}\to\mathcal{CB}(\mathbb{R})$ and $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$Tx=\begin{cases}\left[\frac{19}{10},\frac{-2x+25}{10}\right], & \text{if } x\le\frac{7}{2},\\ \left[\frac{-2x+25}{10},\frac{19}{10}\right], & \text{if } x>\frac{7}{2},\end{cases}$$

and $fx=\dfrac{x+2}{2}$, for all $x\in X$, respectively. Let us choose $A_1=\left[0,\frac{5}{2}\right]$, $A_2=\left(1,\frac{7}{2}\right)$ and $A_3=\left[\frac{3}{2},4\right]$. We have

$$T(A_1)=\left[\frac{19}{10},\frac{5}{2}\right],T(A_2)=\left(\frac{9}{5},\frac{23}{10}\right),T(A_3)=\left[\frac{7}{4},\frac{11}{5}\right],$$
$$f(A_1)=\left[1,\frac{9}{4}\right],f(A_2)=\left(\frac{3}{2},\frac{11}{4}\right),f(A_3)=\left[\frac{7}{4},3\right].$$

Therefore, $T(A_1)\subseteq f(A_2), T(A_2)\subseteq f(A_3)$ and $T(A_3)\subseteq f(A_1)$ and it is clear that $f(A_1)$ and $f(A_3)$ are closed set.

We will show that the mappings $f$ and $T$ satisfy condition ii) of Theorem 2.6 with $r=\frac{1}{2}$. For each $x\in A_1$ and $y\in A_2$, such that $\varphi(r)D(fx,Tx)\le d(fx,fy)$, we have

$$\begin{aligned}H(Tx,Ty)&=H\left(\left[\frac{19}{10},\frac{-2x+25}{10}\right],\left[\frac{19}{10},\frac{-2y+25}{10}\right]\right)\le\left|\frac{-2x+25}{10}-\frac{-2y+25}{10}\right|\\ &=\frac{1}{5}|x-y|\le\frac{1}{4}|x-y|=\frac{1}{2}\left|\frac{x+2}{2}-\frac{y+2}{2}\right|=\frac{1}{2}d(fx,fy)\\ &\le\frac{1}{2}\max\left\{d(fx,fy),D(fx,Tx),D(fy,Ty),\frac{D(fx,Ty)+D(fy,Tx)}{2}\right\}.\end{aligned}$$

Similarly, one proves that the contraction condition holds if $x\in A_2$ and $y\in A_3$.

Hence, all requirements of the Theorem 2.6 are satisfied. In fact, we have $C(f,T)=\left[\frac{9}{5},\frac{15}{7}\right]$. Moreover, there is $z=2\in\left[\frac{9}{5},\frac{15}{7}\right]$ such that $ffz=fz$ and thus $fz=2\in F(f,T)$.

The next example shows that, under the hypotheses of Theorem 2.6, the assumption $ffz=fz$ is essential for guaranteeing the existence of a common fixed point of $f$ and $T$.

**Example 2.2.** Consider the set of real numbers, $\mathbb{R}$, with the usual metric and let $T:\mathbb{R}\to\mathcal{CB}(\mathbb{R})$ and $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$Tx=\begin{cases}\left[0,\frac{-5x+1}{10}\right], & \text{if } x\le\frac{2}{5},\\ \left[\frac{-5x+1}{10},0\right], & \text{if } x>\frac{2}{5},\end{cases}$$

and

$$fx=\frac{6x-1}{10},\quad \text{for all}\quad x\in X.$$

One can check that $F(f,T)=\emptyset$. Let us consider $A_1=\left(-2,\frac{2}{5}\right)$, $A_2=[0,2]$. So, $f(A_2)$ is a closed set and

$$T(A_1)=\cup_{a\in A}Ta=\left(-\frac{1}{10},\frac{11}{10}\right)\subseteq\left[-\frac{1}{10},\frac{11}{10}\right]=f(A_2),$$

and

$$T(A_2)=\cup_{b\in B}Tb=\left[-\frac{9}{10},\frac{1}{10}\right]\subseteq\left(-\frac{13}{10},\frac{7}{50}\right)=f(A_1).$$

Next, we show that the mappings $T$ and $f$ satisfy condition ii) of Theorem 2.6 with $r = \frac{5}{6}$. Indeed, for each $x \in A_1$ and $y \in A_2$, such that $\varphi(r)D(fx, Tx) \leq d(fx, fy)$, we have

$$H(Tx, Ty) = H\left(\left[0, \frac{-5x+1}{10}\right], \left[\frac{-5y+1}{10}, 0\right]\right) \leq \left|\frac{-5x+1}{10} - \frac{-5y+1}{10}\right| = \frac{1}{2}|x-y|$$

$$= \frac{5}{6}\left|\frac{6x-1}{10} - \frac{6y-1}{10}\right| = \frac{5}{6}d(fx, fy)$$

$$\leq \frac{5}{6}\max\left\{d(fx, fy), D(fx, Tx), D(fy, Ty), \frac{D(fx, Ty) + D(fy, Tx)}{2}\right\}.$$

This proves the claim. Thus, all requirements of Theorem 2.6 are satisfied and we have $C(f, T) = \left[\frac{1}{6}, \frac{2}{11}\right]$ but $ffz \neq fz$, for all $z \in \left[\frac{1}{6}, \frac{2}{11}\right]$.

The following special case of our main result given by Theorem 2.6 is important by itself.

**Theorem 2.7.** *Let $(X, d)$ be a complete metric space and $T : X \to \mathcal{CB}(X)$ be a multi-valued mapping. Let $A_1, A_2, \ldots, A_m$ be nonempty subsets of $X$ such that $T(A_i) \subseteq A_{i+1}$, for each $i = 1, \ldots, m-1$ and $T(A_m) \subseteq A_1$. If the following conditions are satisfied :*

*i) There is $\bar{j} \in \{1, \ldots, m\}$ such that $A_{\bar{j}}$ is a closed set.*
*ii) There exists $r \in [0, 1)$ such that $\varphi(r)D(x, Tx) \leq d(x, y)$ implies*

$$H(Tx, Ty) \leq r\max\left\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2}\right\}, \tag{2.16}$$

*when $x \in A_i, y \in A_{i+1}$, where $i \in \{1, \ldots, m\}$, $A_{m+1} = A_1$ and the function $\varphi$ is defined as in (2.5).*

*Then, $F(T) \neq \emptyset$.*

**Remark 2.1.** Taking $A_1 = A_2 = \ldots = A_m = X$ in Theorem 2.7, we get Theorem 1.5. However, if each $A_i$ is a proper subset of $X$, it is worth to point out that Theorem 2.7 is a genuine generalization of Theorem 1.5. Moreover, Theorem 2.7 also improves the important results with were presented by Ciric in [13].

The following example shows the generality of Theorem 2.7, by comparing with Theorem 1.3.

**Example 2.3.** Consider $X = (-\infty, 4]$ equipped with the absolute valued metric distance. Define $T : X \to \mathcal{CB}(X)$ by

$$Tx = \begin{cases} \left[\frac{6-x}{2}, \frac{x+9}{6}\right], & \text{if } x < 4, \\ \left[\frac{5}{2}, \frac{13}{5}\right], & \text{if } x = 4. \end{cases}$$

For $A_1 = \left[0, \frac{13}{5}\right]$ and $A_2 = [1, 4]$, we observe that

$$T(A_1) = \left[\frac{3}{2}, 3\right] \subseteq [1, 4] = A_2$$

and

$$T(A_2) = \left(1, \frac{5}{2}\right] \cup \left[\frac{5}{2}, \frac{13}{5}\right] = \left(1, \frac{13}{5}\right] \subseteq \left[0, \frac{13}{5}\right] = A_1.$$

Moveover, we can show that the condition (2.16) is satasified with $r = \frac{3}{5}$ . Therefore, all assumptions of Theorem 2.7 are satisfied and $F(T) = \left[\frac{9}{5}, 2\right]$.

However, $T$ does not satisfy Kikkawa and Suzuki's condition (Theorem 1.3). Indeed, for $x = 3$ and $y = 4$, we have

$$\frac{1}{1+r}D(3, T3) = \frac{1}{1+r}D(3, \left[\frac{3}{2}, 2\right]) = \frac{1}{1+r} \cdot 1 \leq 1 = d(3, 4),$$

but $H(T3, T4) = H\left(\left[\frac{3}{2}, 2\right], \left[\frac{5}{2}, \frac{13}{5}\right]\right) = 1 > r(1) = rd(3, 4)$.

## 3. Conclusions

In this work, which basically rely on the results of Dorić and Lazović [16], we introduce and study a new class of multi-valued mappings induced by the cyclic concept. Some coincidence and fixed point theorems, examples and remarks are discussed.

It is important to point out that Theorem 2.6 is obtained by requiring only the assumption that $f(A_{\bar{j}})$ is a closed set, which is a weaker condition than those appearing in the existing literature.

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