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A new hybrid algorithm for global minimization of best proximity points in Hilbert spaces

RAWEEROTE SUPARATULATORN[1] and SUTHEP SUANTAI[2]

ABSTRACT. The purpose of this paper is to introduce a new hybrid algorithm for finding a global minimization of best proximity points for a new class of mappings, called best proximally nonexpansive (BPNE), which is weaker than nonself nonexpansive mappings and then prove strong convergence of the proposed method under some suitable conditions in real Hilbert spaces. Finally, some numerical experiment is also given for demonstrating our main result.

1. INTRODUCTION

Various problems arising in different areas of science, applied science, economics, physics and engineering, can be modeled as fixed point equations of the form $x = Tx$, where $T : X \to X$ is a nonlinear operator. So fixed point theory plays very important role in solving existence and uniqueness of solutions of those problems. In the case that T is nonself mapping, the above fixed point equation may have no solution. For more precisely, suppose $T : A \to B$ with $A \cap B = \emptyset$, where A and B are two subsets of a metric space (X, d), in this case, $d(A, B) \le d(x, Tx)$ for all $x \in A$, where $d(A, B)$ is the gap distance between A and B, i.e., $d(A, B) := \inf\{\|a - b\| : a \in A \text{ and } b \in B\}$. It is natural to ask how can we find a point $x^* \in A$ such that the function $f(x^*) = d(x^*, Tx^*)$ attains its minimum value $d(A, B)$. Such point x^* is a global minimization of above function f and it is called a best proximity point of T in A.

It is well-known that the existence of best proximity points for some nonlinear mappings can be applied to solve equilibrium, for example, see [15, 20]. It was shown by Pirbavafa and Vaezpour [20] that existence of equilibrium pair in free abstract of economies can be guaranteed by best proximity point theory. Since then, the concept of best proximity point attracted the attention of many mathematicians, see for instance [2, 24, 3, 4, 5].

From the past 15 years, many mathematicians paid attention on proving existence of best proximity point of various kind of nonlinear mappings satisfying some contractive conditions, see [9, 25, 21, 22, 10], and both existence and approximation of best proximity point for nonexpansive mappings were investigated extensively by many authors, see [16, 11, 13].

For approximating fixed point of nonexpansive mappings, many iterative methods were introduced and studies. Many of them provided only weak convergence. However, by using the hybrid method in mathematical programming, Nakajo and Takahashi [19] introduced the hybrid projection method and obtained its strong convergence to a fixed point of such nonexpansive mappings in a real Hilbert space. After that Martinez-Yanes and Xu [17] employed the idea of Nakajo and Takahashi to obtain strong convergence theorems for nonexpansive mappings in a real Hilbert space. By modification the hybrid

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Corresponding author: Suthep Suantai; suthep.s@cmu.ac.th

method introduced in [19], Takahashi et al. [28] proposed a shrinking projection method for fixed point of nonexpansive mappings in a real Hilbert space. They obtained the following theorem:

Theorem 1.1. [28] *Let H be a Hilbert space and let C be a nonempty closed and convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $u \in H$. For $C_1 = C$ and $x_1 = P_{C_1}u$, define a sequence $\{x_n\}$ in C as follows:*

$$\begin{cases} y_n &= \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} &= P_{C_{n+1}}u, \end{cases}$$

where $0 \leq \alpha_n \leq \alpha < 1$ for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $z = P_{F(T)}u$.

Recently, Jacob et al. [14] introduced hybrid methods for best proximity point of nonself nonexpansive mappings in a real Hilbert space and proved strong convergence theorems of best proximity points. It is observed that, in practice, it is quite hard to compute the set C_n and Q_n and $x_{n+1} = P_{C_n \cap Q_n}x_0$ in the algorithm introduced by Jacob et al. [14], so it is natural to ask how can we modify this method to the easier one with convenience for implementation in the practice.

Motivated by above research works, we introduce a new hybrid algorithm for finding best proximity points of best proximally nonexpansive mappings in a real Hilbert space and prove some strong convergence theorems under some control conditions.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\|\cdot\|$. Let A and B be two nonempty closed and convex subsets of H. A mapping $T : A \to B$ is said to be nonself nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in A$. The fixed point set of T is denoted by $F(T)$, that is, $F(T) = \{x \in A : x = Tx\}$. Denote by P_A the metric projection from X onto A and the best proximity point set of T in A is denoted by $Best_A(T)$, that is,

$$Best_A(T) = \{x \in A : \|x - Tx\| = d(A, B)\}.$$

Through out this paper, we denote A_0 and B_0 the following sets.

$$A_0 := \{x \in A : \|x - y\| = d(A, B), \text{for some } y \in B\},$$

$$B_0 := \{y \in B : \|x - y\| = d(A, B), \text{for some } x \in A\},$$

and we use $\rightharpoonup$ for weak convergence and $\to$ for strong convergence. For a sequence $\{x_n\}$ in H, the weak ω-limit set of $\{x_n\}$ is denoted by $\omega_w(x_n) := \{x : \exists \text{ a subsequence } \{x_{n_j}\} \text{ of } \{x_n\} \text{ such that } x_{n_j} \rightharpoonup x\}$. In [16], the authors discussed sufficient conditions which guarantee the nonemptiness of A_0 and B_0. Further, in [2], it has been ascertained that in the framework of a normed linear space E, we have

$$A_0 \subseteq \partial A \text{ and } B_0 \subseteq \partial B,$$

where ∂C denotes the boundary of C for any $C \subseteq E$, when the distance between A and B is nonzero. It is easy to see that A_0 and B_0 are closed and convex subsets of A and B, respectively, if A and B are closed and convex.

We recall the following notion of best proximity point:

Definition 2.1. An element $x \in A$ is said to be a best proximity point of the nonself mapping $T : A \to B$ if it satisfies the condition that

$$\|x - Tx\| = d(A, B).$$

If the underlying mapping is self-mapping, a best proximity point becomes a fixed point.

Definition 2.2. [21, 22, 30] Let A and B be two nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property if

$$\left.\begin{aligned} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{aligned}\right\} \Longrightarrow d(x_1, x_2) = d(y_1, y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$. The pair (A, B) is said to have the weak P-property if $x_1, x_2 \in A_0$, $y_1, y_2 \in B_0$,

$$\left.\begin{aligned} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{aligned}\right\} \Longrightarrow d(x_1, x_2) \leq d(y_1, y_2).$$

Example 2.1. [21, 22] Any pair (A, B) of nonempty closed and convex subsets of a real Hilbert space H has the P-property.

Definition 2.3. [25] Let A and B be two nonempty subsets of a metric space (X, d). Then (A, B) is said to satisfy the property UC if

$$\left.\begin{aligned} d(x_n, y_n) \to d(A, B) \\ d(x'_n, y_n) \to d(A, B) \end{aligned}\right\} \Longrightarrow d(x_n, x'_n) \to 0$$

for all sequences $\{x_n\}$ and $\{x'_n\}$ in A and for every sequence $\{y_n\}$ in B.

Example 2.2. [9] Let A and B be nonempty subsets of a uniformly convex Banach space. Assume that A is convex. Then (A, B) has the property UC.

Definition 2.4. [1] Let A be a nonempty subset of a Banach space X and let $T : A \to X$ be a mapping. Then, T is said to be demiclosed at $y \in X$ if, for any sequence $\{x_n\}$ in A such that $x_n \rightharpoonup x \in A$ and $Tx_n \to y$ imply $Tx = y$.

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.1. [29] *There holds the identity in a real Hilbert space H, for $u, v \in H$:*

$$\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u - v, v\rangle.$$

Lemma 2.2. [29] *Let A be a closed and convex subsets of a real Hilbert space H. Given $x \in H$ and $z \in A$. Then $z = P_A x$ if and only if there holds the relation:*

$$\langle x - z, y - z\rangle \leq 0,$$

for all $y \in A$.

Lemma 2.3. [17] *Let A be a closed and convex subsets of a real Hilbert space H and given points $x, y, z \in H$. Given also a real number $a \in \mathbb{R}$. The set*

$$D := \{v \in A : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$$

is closed and convex.

Lemma 2.4. [12] *Let A be a closed and convex subsets of a real Hilbert space H and let $T : A \to A$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$. If a sequence $\{x_n\}$ in A is such that $x_n \rightharpoonup z$ and $\|x_n - Tx_n\| \to 0$, then $z = Tz$.*

Lemma 2.5. [17] *Let A be a closed and convex subsets of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $z = P_A u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset A$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - z\|,$$

for all $n \in \mathbb{N}$. Then $x_n \to z$.

3. Main results

We first introduce the concept of C-nonexpansive mappings which is more general than those of nonexpansive and quasi-nonexpansive mappings.

Definition 3.5. Let A and B be two nonempty subsets of a Banach space X and C a subset of A. A mapping $T : A \to B$ is said to be C-nonexpansive if

$$\|Tx - Tz\| \leq \|x - z\|,$$

for all $x \in A$ and $z \in C$. If $C = Best_A(T) \neq \emptyset$, we say that T is best proximally nonexpansive mapping.

Remark 3.1. It is note that if T is nonself nonexpansive, then it is C-nonexpansive for every subset C of A, and if $C = F(T) \neq \emptyset$, then every C-nonexpansive is quasi-nonexpansive.

Example 3.3. Consider l_2 with the usual norm. Let $A = \{x \in l_2 : x = \alpha e_1 + (1-\alpha)e_2, \alpha \in [0,1]\}$ and $B = \{x \in l_2 : x = \beta e_3 + (1-\beta)e_4, \beta \in [0,1]\}$, where e_i is the sequences whose the i^{th} term is 1 and the other terms are zero, for $i = 1,2,3,4$. Define $T : A \to B$ by

$$Tx = \begin{cases} \alpha e_3 + (1-\alpha)e_4, & \text{if} \quad \alpha \in [0, \frac{1}{2}] \\ \frac{1}{2}(e_3 + e_4), & \text{if} \quad \alpha \in (\frac{1}{2}, \frac{3}{4}) \\ \frac{3}{4}e_3 + \frac{1}{4}e_4, & \text{if} \quad \alpha \in [\frac{3}{4}, 1], \end{cases}$$

for all $x = \alpha e_1 + (1-\alpha)e_2 \in A$. It is not hard to see that $Best_A(T) = \{\frac{1}{2}(e_1 + e_2)\}$ and T is a best proximally nonexpansive mapping but it is not nonself nonexpansive because $\|Tu - Tv\| = \frac{\sqrt{2}}{4} > \frac{3\sqrt{2}}{20} = \|u - v\|$ when $u = \frac{3}{4}e_1 + \frac{1}{4}e_2$ and $v = \frac{3}{5}e_1 + \frac{2}{5}e_2$.

In [26], Suzuki showed in a metric space that if (A, B) has the weak P-property, then there exists a nonexpansive mapping Q from B_0 to A_0 such that $d(u, Qu) = d(A, B)$ for every $u \in B_0$. For the case of normed space, we obtain the following results which are crucial for proving our main results.

Lemma 3.6. *Let A, B be two nonempty closed and convex subsets of a normed space X. Then*
(i) $\|x - P_B x\| = d(A, B)$ for all $x \in A_0$, and
(ii) $\|y - P_A y\| = d(A, B)$ for all $y \in B_0$.

Proof. (i) Let $x \in A_0$. Then there exists $z \in B$ such that $\|x - z\| = d(A, B)$. We see that

$$\begin{aligned} \|x - P_B x\| &= \inf_{u \in B} \|x - u\| \\ &\leq \|x - z\| \\ &= d(A, B). \end{aligned}$$

Therefore, $\|x - P_B x\| = d(A, B)$.
(ii) By the same manner, we have $\|P_A y - y\| = d(A, B)$. □

Lemma 3.7. *Let A, B be two nonempty subsets of a uniformly convex Banach space X such that A is closed and convex. Suppose that $T : A \to B$ is a mapping such that $T(A_0) \subseteq B_0$. Then $F(P_A \circ T\,|_{A_0}) = F(P_A \circ T) \cap A_0 = Best_A(T)$.*

Proof. Let x be an element in $F(P_A \circ T) \cap A_0$. By Lemma 3.6 (ii), we have

$$\begin{aligned} \|x - Tx\| &\leq \|x - (P_A \circ T)x\| + \|(P_A \circ T)x - Tx\| \\ &= 0 + d(A, B) \\ &= d(A, B). \end{aligned}$$

Therefore, $\|x - Tx\| = d(A, B)$ and hence $x \in Best_A(T)$.

Conversely, let x be an element in $Best_A(T)$, so $x \in A_0$. Then $\|x - Tx\| = d(A, B)$ By Lemma 3.6 (ii), we get $\|(P_A \circ T)x - Tx\| = d(A, B)$. By property UC of (A, B), we get $\|x - (P_A \circ T)x\| = 0$, that is, $x \in F(P_A \circ T)$, and hence $x \in F(P_A \circ T) \cap A_0$. □

Lemma 3.8. *Let A, B be two nonempty subsets of a Hilbert space H such that A is closed and convex. Suppose that $T : A \to B$ is a best proximally nonexpansive mapping such that $T(A_0) \subseteq B_0$. Then $P_A \circ T \mid_{A_0}$ is quasi-nonexpansive mapping.*

Proof. Let $x \in A_0$ and $p \in F(P_A \circ T \mid_{A_0})$. By Lemma 3.7, we know $F(P_A \circ T \mid_{A_0}) = Best_A(T)$. Then

$$\begin{aligned} \|(P_A \circ T \mid_{A_0})x - (P_A \circ T \mid_{A_0})p\| &\leq \|T \mid_{A_0} x - T \mid_{A_0} p\| \\ &\leq \|x - p\|. \end{aligned}$$

Therefore, $P_A \circ T \mid_{A_0}$ is quasi-nonexpansive mapping. □

We now modify the shrinking projection method which was introduced by Takahashi et al. [28] for finding a best proximity point of best proximally nonexpansive mappings by introducing the following algorithm:

Algorithm 1:

Initialization Step. Choose $u \in H$ arbitrarily and set $x_1 = P_{C_1}u$, and $n = 1$.

Iterative Step 1. For x_n, compute y_n and C_{n+1} by using

$$\begin{cases} y_n &= \alpha_n P_B x_n + (1 - \alpha_n)Tx_n, \\ C_{n+1} &= \{z \in C_n : \|P_A y_n - z\| \leq \|x_n - z\|\}, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1)$.

Iterative Step 2. Compute x_{n+1} by using

$$x_{n+1} = P_{C_{n+1}}u.$$

The value of n is then set to $n + 1$, and then go to Iterative Step 1.

Theorem 3.2. *Let A, B be two nonempty closed and convex subsets of a real Hilbert space H. Suppose that $T : A \to B$ is best proximally nonexpansive mapping such that $T(A_0) \subseteq B_0$ and $Best_A(T) \neq \emptyset$. For $C_1 = A_0$ and let $\{\alpha_n\}$ be a sequence in $[0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Assume that $I - (P_A \circ T \mid_{A_0})$ is demiclosed at zero. If $u \in H$, then the sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 1 converge strongly to z and Tz, respectively, where z is a best proximity point of T in A and $z = P_{Best_A(T)}u$.*

Proof. Let u be an element in H. Firstly, we show by induction that $Best_A(T) \subset C_n$ for all $n \in \mathbb{N}$. It is obvious that $Best_A(T) \subset A_0 = C_1$. Assume that $Best_A(T) \subset C_k$ for some $k \in \mathbb{N}$. For $v \in Best_A(T) \subset C_k$ and by Lemma 3.7, we have

$$\begin{aligned} \|P_A y_k - v\| &= \|P_A y_k - (P_A \circ T)v\| \\ &\leq \|y_k - Tv\| \\ &\leq \alpha_k \|P_B x_k - Tv\| + (1 - \alpha_k)\|Tx_k - Tv\| \\ &\leq \alpha_k \|P_B x_k - Tv\| + (1 - \alpha_k)\|x_k - v\|. \end{aligned}$$

By Lemma 3.6 (i), $\|P_B x_k - x_k\| = d(A, B)$. Since (A, B) has the P-property, we get $\|P_B x_k - Tv\| = \|x_k - v\|$ which implies by above inequality that

$$\|P_A y_k - v\| \leq \|x_k - v\|.$$

Hence, $v \in C_{k+1}$. By induction, we can conclude that $Best_A(T) \subset C_n$ for all $n \in \mathbb{N}$. By Lemma 2.3, we obtain C_n is closed and convex for all $n \in \mathbb{N}$. This follows that $\{x_n\}$ is well-defined. By Lemma 3.7, we know $F(P_A \circ T \mid_{A_0}) = Best_A(T)$. We know from

Lemma 3.8 that $P_A \circ T\mid_{A_0}$ is quasi-nonexpansive, so from Lemma 2.7 in [27], we obtain that $F(P_A \circ T\mid_{A_0})$ is closed and convex, which implies by Lemma 3.7 that $Best_A(T)$ is closed and convex. From $x_{n+1} = P_{C_{n+1}}u$ and $Best_A(T) \subset C_n$ for all $n \in \mathbb{N}$, we have

$$\|x_{n+1} - u\| \leq \|z - u\|, \tag{3.1}$$

where $z = P_{Best_A(T)}u$. Thus, $\{\|x_n - u\|\}$ is bounded. From $x_n = P_{C_n}u$, we obtain

$$\langle u - x_n, x_n - y\rangle \geq 0,$$

for all $y \in C_n$. For $n \in \mathbb{N}$, we have

$$\begin{aligned}
0 &\leq \langle u - x_n, x_n - x_{n+1}\rangle \\
&= \langle u - x_n, x_n - u + u - x_{n+1}\rangle \\
&= -\|u - x_n\|^2 + \langle u - x_n, u - x_{n+1}\rangle \\
&\leq -\|u - x_n\|^2 + \|u - x_n\|\|u - x_{n+1}\|
\end{aligned}$$

and hence,

$$\|x_n - u\| \leq \|x_{n+1} - u\|.$$

Since $\{\|x_n - u\|\}$ is bounded, $\lim_{n\to\infty} \|x_n - u\|$ exists. By Lemma 2.1, we have

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - u) - (x_n - u)\|^2 \\
&= \|x_{n+1} - u\|^2 - \|x_n - u\|^2 - 2\langle x_{n+1} - x_n, x_n - u\rangle \\
&\leq \|x_{n+1} - u\|^2 - \|x_n - u\|^2
\end{aligned}$$

and so $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. On the other hand, $x_{n+1} \in C_{n+1}$, which implies

$$\begin{aligned}
\|P_A y_n - x_n\| &\leq \|P_A y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\
&\leq 2\|x_n - x_{n+1}\| \\
&\to 0
\end{aligned}$$

as $n \to \infty$. Further, we have

$$\begin{aligned}
\|Tx_n - x_n\| &\leq \|Tx_n - (P_A \circ T)x_n\| + \|(P_A \circ T)x_n - P_A y_n\| + \|P_A y_n - x_n\| \\
&\leq d(A, B) + \|Tx_n - y_n\| + \|P_A y_n - x_n\| \\
&= d(A, B) + \alpha_n\|P_B x_n - Tx_n\| + \|P_A y_n - x_n\| \\
&\to d(A, B)
\end{aligned}$$

as $n \to \infty$. Since (A, B) has the property UC, P-property and $\|P_B x_n - x_n\| = d(A, B) = \|(P_A \circ T)x_n - Tx_n\|$, it follows that $\|(P_A \circ T)x_n - x_n\| = \|P_B x_n - Tx_n\| \to 0$ as $n \to \infty$. Since $I - (P_A \circ T\mid_{A_0})$ is demiclosed at zero and Lemma 3.7, $\omega_w(x_n) \subset F(P_A \circ T\mid_{A_0}) = Best_A(T)$. This together with (3.1) and Lemma 2.5, we obtain that $\{x_n\}$ converges strongly to z which is a best proximity point of T in A. This completes the proof. $\square$

The following result is directly obtained by Lemma 2.4, Remark 3.1 and Theorem 3.2.

Corollary 3.1. *Let A, B be two nonempty closed and convex subsets of a real Hilbert space H. Suppose that $T : A \to B$ is nonself nonexpansive mapping such that $T(A_0) \subseteq B_0$ and $Best_A(T) \neq \emptyset$. For $C_1 = A_0$ and let $\{\alpha_n\}$ be a sequence in $[0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. If $u \in H$, then the sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 1 converge strongly to z and Tz, respectively, where z is a best proximity point of T in A and $z = P_{Best_A(T)}u$.*

4. Numerical result

In this section, we demonstrate numerical result of Theorem 3.2 as in the following example.

Example 4.4. Consider $\mathbb{R}^2$ with the usual norm. Let $A = \{(x, y) \in \mathbb{R}^2 : x \leq -1 \text{ and } 0 \leq y \leq 2\}$ and $B = \{(x, y) \in \mathbb{R}^2 : x \geq 1 \text{ and } y \leq 1\}$. Let $T : A \to B$ be defined as

$$T(x, y) = \begin{cases} (-x, \frac{3}{5}), & \text{if} \quad y = 0 \\ (-x, \frac{5}{4}y^2), & \text{if} \quad 0 < y < \frac{\sqrt{2}}{2} \\ (-x, \sqrt{1 - y^2}), & \text{if} \quad \frac{\sqrt{2}}{2} \leq y \leq 1 \\ (-x, \sqrt{y - 1}), & \text{if} \quad 1 < y \leq 2, \end{cases}$$

for all $(x, y) \in A$. Then $A_0 = \{(-1, y) \in A : 0 \leq y \leq 1\}$, $B_0 = \{(1, y) \in B : 0 \leq y \leq 1\}$ and $d(A, B) = 2$. It is easy to show that T is best proximally nonexpansive mappings such that $T(A_0) \subseteq B_0$ but it is not nonself nonexpansive because it is discontinuous. Suppose the sequences $\{x_n\}$ generated by Algorithm 1 by choosing $\alpha_n = \frac{1}{n+1}$, for all $n \in \mathbb{N}$. For the initial point $u = (3, 2)$ and set $C_1 = A_0$. We obtain the following numerical experiments for best proximity point of T in A.

n	x_n	y_n	C_{n+1}
1	(-1.0000, 1.0000)	(1.0000, 0.5000)	{-1.0000}×[0.0000, 0.7500]
2	(-1.0000, 0.7500)	(1.0000, 0.6910)	{-1.0000}×[0.0000, 0.7205]
3	(-1.0000, 0.7205)	(1.0000, 0.7002)	{-1.0000}×[0.0000, 0.7104]
4	(-1.0000, 0.7104)	(1.0000, 0.7051)	{-1.0000}×[0.0000, 0.7078]
5	(-1.0000, 0.7078)	(1.0000, 0.7067)	{-1.0000}×[0.0000, 0.7072]
6	(-1.0000, 0.7072)	(1.0000, 0.7070)	{-1.0000}×[0.0000, 0.7071]
7	(-1.0000, 0.7071)	(1.0000, 0.7071)	{-1.0000}×[0.0000, 0.7071]

TABLE 1. Numerical experiments in Example 4.4.

From our main theorem, Theorem 3.2, we can conclude that the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to a best proximity point z of T in A, where $z = (-1, \frac{\sqrt{2}}{2})$ and we observe from above table that the iterate sequence $x_7 = (-1.0000, 0.7071)$ is an approximation of the best proxity point of T in A with absolute error ≈ 0.000002.

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[1]DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
PH. D. PROGRAM IN MATHEMATICS
CHIANG MAI UNIVERSITY, CHIANG MAI 50200, THAILAND
Email address: raweerote.s@gmail.com

[2]DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
CENTER OF EXCELLENCE IN MATHEMATICS AND APPLIED MATHEMATICS
CHIANG MAI UNIVERSITY, CHIANG MAI 50200, THAILAND
Email address: suthep.s@cmu.ac.th