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Dedicated to Prof. Juan Nieto on the occasion of his 60^{th} anniversary

Weighted G-Drazin inverse for operators on Banach spaces

DIJANA MOSIĆ

ABSTRACT. We define an extension of weighted G-Drazin inverses of rectangular matrices to operators between two Banach spaces. Some properties of weighted G-Drazin inverses are generalized and some new ones are proved. Using weighted G-Drazin inverses, we introduce and characterize a new weighted pre-order on the set of all bounded linear operators between two Banach spaces. As an application, we present and study the G-Drazin inverse and the G-Drazin partial order for operators on Banach space.

1. INTRODUCTION

Let X and Y be arbitrary Banach spaces. We use $\mathbf{B}(X,Y)$ to denote the set of all bounded linear operators from X to Y. Set $\mathbf{B}(X) = \mathbf{B}(X,X)$. For $A \in \mathbf{B}(X,Y)$, the notations $N(A)$ and $R(A)$ stand for the null space and the range of A, respectively.

An operator $A \in \mathbf{B}(X,Y)$ is relatively regular if there exists some $B \in \mathbf{B}(Y,X)$ such that $ABA = A$. The operator B is called an inner inverse of A and it is not unique. By $A\{1\}$ we denote the set of all inner inverses of A. Recall that $A \in \mathbf{B}(X,Y)$ is relatively regular if and only if $N(A)$ and $R(A)$ are closed and complemented subspaces of X and Y, respectively. In the case that X and Y are Hilbert spaces, A is relatively regular if and only if $R(A)$ is closed.

Let $W \in \mathbf{B}(Y,X)$ be a fixed nonzero operator. An operator $A \in \mathbf{B}(X,Y)$ is Wg–Drazin invertible if there exists a unique $B \in \mathbf{B}(X,Y)$ such that

$$AWB = BWA, \quad BWAWB = B \quad \text{and} \quad A - AWBWA \text{ is quasinilpotent}.$$

The Wg–Drazin inverse B of A will be denoted by $A^{d,W}$ [5]. In the case that $A-AWBWA$ is nilpotent in the above definition, $A^{d,W} = A^{D,W}$ is the W-weighted Drazin inverse of A [3, 16]. When $X = Y$ and $W = I$, then $A^d = A^{d,W}$ is the generalized Drazin inverse (or the Koliha-Drazin inverse) of A [8] and $A^D = A^{D,W}$ is the Drazin inverse of A. The symbol $\mathbf{B}(X)^d$ denotes the set of all generalized Drazin invertible operators of $\mathbf{B}(X)$. The group inverse is a particular case of Drazin inverse for which the condition $A - ABA$ is nilpotent is replaced with $A = ABA$. By $A^{\#}$ will be denoted the group inverse of A.

For $A \in \mathbf{B}(X,Y)$ and $W \in \mathbf{B}(Y,X)$, the following conditions are equivalent [5]:

(1) A is Wg-Drazin invertible and $A^{d,W} = B \in \mathbf{B}(X,Y)$,
(2) $AW \in \mathbf{B}(Y)^d$ with $(AW)^d = BW$,
(3) $WA \in \mathbf{B}(X)^d$ with $(WA)^d = WB$.

Then, the Wg-Drazin inverse $A^{d,W}$ of A satisfies

$$A^{d,W} = ((AW)^d)^2 A = A((WA)^d)^2.$$

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Lemma 1.1. [5] *Let $A \in \mathbf{B}(X,Y)$ and $W \in \mathbf{B}(Y,X)\backslash\{0\}$. Then A is Wg-Drazin invertible if and only if there exist topological direct sums $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$ such that*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}, \tag{1.1}$$

where $A_i \in \mathbf{B}(X_i, Y_i)$, $W_i \in \mathbf{B}(Y_i, X_i)$, for $i = 1, 2$, with A_1, W_1 invertible, and W_2A_2 and A_2W_2 quasinilpotent in $\mathbf{B}(X_2)$ and $\mathbf{B}(Y_2)$, respectively. The Wg-Drazin inverse of A is given by

$$A^{d,W} = \begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \tag{1.2}$$

with $(W_1A_1W_1)^{-1} \in \mathbf{B}(X_1, Y_1)$ and the (2,2) *matrix block satisfies that $0 \in \mathbf{B}(X_2, Y_2)$.*

For recent results related to the (generalized) Drazin and (generalized) weighted Drazin inverse see [11, 12, 14, 17, 18, 20, 21].

Various kinds of pre-orders (i.e. reflexive and transitive binary relations) and partial orders were defined using various generalized inverses [1, 9, 10].

Let $A, B \in \mathbf{B}(X,Y)$ be relatively regular. Then A is said to be below B under the minus partial order (denoted by $A \leq^- B$) if there exists an inner generalized inverse A^- of A such that $AA^- = BA^-$ and $A^-A = A^-B$.

For $A, B \in \mathbf{B}(X)$ such that A is group invertible, we say that A is below B under the sharp partial order ($A \leq^\# B$) if $A^\# A = A^\# B$ and $AA^\# = BA^\#$.

Let $A, B \in \mathbf{B}(X)^d$. The operator A is below to B under the generalized Drazin pre-order ($A \leq^d B$) if $A^2A^d \leq^\# B^2B^d$. Recall that $A \leq^d B$ if and only if $A^dA = A^dB$ and $AA^d = BA^d$ [15].

Let $A, B \in \mathbf{B}(X,Y)$ and $W \in \mathbf{B}(Y,X)\backslash\{0\}$. If A is Wg-Drazin invertible, then we say that $A \leq^{d,W} B$ if $AW \leq^d BW$ and $WA \leq^d WB$, where $\leq^d$ is considered on $\mathbf{B}(Y)$ and $\mathbf{B}(X)$, respectively. The relation $\leq^{d,W}$ is a pre-order on the set of all Wg–Drazin invertible operators of $\mathbf{B}(X,Y)$ [15]. For more related results see [6, 7, 13].

The G-Drazin inverse of a square matrix was defined in [19]. Coll, Lattanzi, and Thome [4] extended the notion of G-Drazin inverses to rectangular matrices considering a weight matrix. Let $\mathbb{C}^{m\times n}$ denote the set of $m \times n$ complex matrices. If $W \in \mathbb{C}^{n\times m}\backslash\{0\}$, the W-weighted G-Drazin inverse of $A \in \mathbb{C}^{m\times n}$ is a matrix C satisfying the following three equations $WAWCWAW = WAW$, $(AW)^{k+1}CW = (AW)^k$, $WC(WA)^{k+1} = (WA)^k$, where $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$ and $\mathrm{ind}(D)$ is the index of D. If $m = n$ and $W = I$, then C is a G-Drazin inverse of A. A new pre-order, which generalizes the G-Drazin partial order studied in [19] to the rectangular case, was also characterized in [4].

We introduce the definition of weighted G-Drazin inverses of an operator between two Banach spaces and prove that our definition and the above definition of weighted G-Drazin inverses for a rectangular matrix are equivalent in complex matrix case. Several new characterizations of weighted G-Drazin inverses are given and some known results are extended. Also, we define and investigate a new pre-order on the corresponding subset of all operators between two Banach spaces. As consequences of our results, we present definitions of the G-Drazin inverse and the G-Drazin partial order for operators on Banach spaces and give their new characterizations. Thus, the recent results from [4, 19] are extended to more general settings.

2. Weighted G-Drazin inverses

In the beginning of this section, we define the weighted G-Drazin inverse of an operator between two Banach spaces as an extension of the weighted G-Drazin inverse for a rectangular matrix.

Definition 2.1. Let $W \in \mathbf{B}(Y,X)\backslash\{0\}$ and let $A \in \mathbf{B}(X,Y)$ be Wg-Drazin invertible such that WAW is relatively regular. An operator $C \in \mathbf{B}(X,Y)$ is a W-weighted G-Drazin inverse of A if the following equalities hold:

$$WAWCWAW = WAW \qquad \text{and} \qquad WA^{d,W}WAWCW = WCWA^{d,W}WAW.$$

We use $A\{W-GD\}$ to denote the set of all W-weighted G-Drazin inverses of A. Obviously, $A\{W-GD\} \subseteq (WAW)\{1\}$. If AW (or equivalently WA) is quasinilpotent, then $(WAW)\{1\} \subseteq A\{W-GD\}$ and so $A\{W-GD\} = (WAW)\{1\}$.

Now, we present necessary and sufficient conditions for an operator to be a W-weighted G-Drazin inverse of a given operator.

Theorem 2.1. *Let $W \in \mathbf{B}(Y,X)\backslash\{0\}$ and let $A \in \mathbf{B}(X,Y)$ be Wg-Drazin invertible such that WAW is relatively regular. For $C \in \mathbf{B}(X,Y)$, the following statements are equivalent:*

(i) $C \in A\{W-GD\}$;
(ii) $WAWCWAW = WAW$ *and* $W(AW)^dAWCW = WCW(AW)^dAW$;
(iii) $WAWCWAW = WAW$ *and* $(WA)^dWAWCW = WCW(AW)^dAW$;
(iv) $WAWCWAW = WAW$ *and* $(WA)^d(WA)^2WCW = WAW(AW)^d = WCW(AW)^d(AW)^2$;
(v) $WAWCWAW = WAW$ *and* $(WA)^dWAWCW = W(AW)^d = WCW(AW)^dAW$;
(vi) *there exist topological direct sums $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ such that*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}, \qquad C = \begin{bmatrix} (W_1A_1W_1)^{-1} & C_{12} \\ C_{21} & C_2 \end{bmatrix},$$

where A_1 and W_1 are invertible, W_2A_2 and A_2W_2 are quasinilpotent, $C_{12}W_2 = 0$, $W_2C_{21} = 0$, $W_2A_2W_2$ is relatively regular and $C_2 \in (W_2A_2W_2)\{1\}$.

Proof. (i) $\Leftrightarrow$ (ii)-(iii): These equivalences follow by properties of the Wg-Drazin inverse.

(iii) $\Rightarrow$ (iv): Notice that

$$(WA)^d(WA)^2WCW = WA((WA)^dWAWCW) = (WAWCWAW)(AW)^d = WAW(AW)^d$$

and similarly $WAW(AW)^d = WCW(AW)^d(AW)^2$.

(iv) $\Rightarrow$ (v): Multiplying $(WA)^d(WA)^2WCW = WAW(AW)^d$ by $(WA)^d$ from the left side, we get $(WA)^dWAWCW = W(AW)^dAW(AW)^d = W(AW)^d$. In an analogy way, we prove that $W(AW)^d = WCBW(AW)^dAW$.

(v) $\Rightarrow$ (iii): This is clear.

(ii) $\Leftrightarrow$ (vi): By Lemma 1.1, there exist topological direct sums $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ such that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix},$$

where A_1, W_1 invertible, and W_2A_2 and A_2W_2 quasinilpotent in $\mathbf{B}(X_2)$ and $\mathbf{B}(Y_2)$, respectively. Suppose that

$$C = \begin{bmatrix} C_1 & C_{12} \\ C_{21} & C_2 \end{bmatrix}.$$

Since $WAW = \begin{bmatrix} W_1A_1W_1 & 0 \\ 0 & W_2A_2W_2 \end{bmatrix}$ and

$$WAWCWAW = \begin{bmatrix} W_1A_1W_1C_1W_1A_1W_1 & W_1A_1W_1C_{12}W_2A_2W_2 \\ W_2A_2W_2C_{21}W_1A_1W_1 & W_2A_2W_2C_2W_2A_2W_2 \end{bmatrix},$$

then $WAWCWAW = WAW$ if and only if $C_1 = (W_1A_1W_1)^{-1}$, $C_{12}W_2A_2W_2 = 0$, $W_2A_2W_2C_{21} = 0$ and $W_2A_2W_2C_2W_2A_2W_2 = W_2A_2W_2$. By $(AW)^d = \begin{bmatrix} (A_1W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}$, we get

$$W(AW)^d AWCW = \begin{bmatrix} W_1C_1W_1 & W_1C_{12}W_2 \\ 0 & 0 \end{bmatrix}$$

and

$$WCW(AW)^d AW = \begin{bmatrix} W_1C_1W_1 & 0 \\ W_2C_{21}W_1 & 0 \end{bmatrix}.$$

We deduce that $W(AW)^d AWCW = WCW(AW)^d AW$ is equivalent to $C_{12}W_2 = 0$ and $W_2C_{21} = 0$. Therefore, this equivalence holds. □

In the case that A is Wg-Drazin invertible such that WAW is relatively regular, by Theorem 2.1(vi), notice that the W-weighted G-Drazin inverse of A exists and it is not unique.

We show that the Definition 2.1 and [4, Definition 2.1] are equivalent in the complex matrix case. Applying Theorem 2.1, we obtain new characterizations for the weighted G-Drazin inverse in the finite dimensional case.

Corollary 2.1. *Let $W \in \mathbb{C}^{n\times m}\backslash\{0\}$ and $A \in \mathbb{C}^{m\times n}$. For $C \in \mathbb{C}^{m\times n}$, the following statements are equivalent:*

(i) $C \in A\{W - GD\}$;
(ii) $WAWCWAW = WAW$ *and* $W(AW)^D AWCW = WCW(AW)^D AW$;
(iii) $WAWCWAW = WAW$ *and* $(WA)^D WAWCW = WCW(AW)^D AW$;
(iv) $WAWCWAW = WAW$ *and* $(WA)^D(WA)^2WCW = WAW(AW)^D = WCW(AW)^D(AW)^2$;
(v) $WAWCWAW = WAW$ *and* $(WA)^D WAWCW = W(AW)^D = WCW(AW)^D AW$.

Proof. (i) ⇔ (ii): By [4, Theorem 2.2], $C \in A\{W - GD\}$ if and only if $WAWCWAW = WAW$ and $W(AW)^k CW = WCW(AW)^k$, for $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$. Using properties of the Drazin inverse and $WAWCWAW = WAW$, we easily check that $W(AW)^k CW = WCW(AW)^k$ is equivalent to $W(AW)^D AWCW = WCW(AW)^D AW$.

(i) ⇔ (iii)-(v): It follows by Theorem 2.1. □

Lemma 2.2. *Let $W \in \mathbf{B}(Y, X)\backslash\{0\}$ and let $A \in \mathbf{B}(X, Y)$ be Wg-Drazin invertible such that WAW is relatively regular. If $C \in A\{W - GD\}$, then $(I - CWAW)WAW$ and $WAW(I - WAWC)$ are quasinilpotent.*

Proof. Using $WAWCWAW = WAW$, we have that

$$\sigma((I - CWAW)WAW) = \sigma(WAW(I - CWAW)) = \sigma(0) = \{0\},$$

i.e. $(I - CWAW)WAW$ is quasinilpotent. In a same manner, we obtain that $WAW(I - WAWC)$ is quasinilpotent. □

Using corresponding idempotents, we give one more characterization for the W-weighted G-Drazin inverse, which is new in the finite dimensional case too.

Theorem 2.2. *Let $W \in \mathbf{B}(Y, X)\backslash\{0\}$ and let $A \in \mathbf{B}(X, Y)$ be Wg-Drazin invertible such that WAW is relatively regular. The following statements are equivalent:*

(i) $A\{W - GD\} \neq \emptyset$;
(ii) *there exist idempotents $P \in \mathbf{B}(X)$ and $Q \in \mathbf{B}(Y)$ such that*

$$R(P) = R(WAW), \quad N(Q) = N(WAW) \quad \text{and} \quad WA^{d,W}PW = WQA^{d,W}W.$$

In addition, for arbitrary $(WAW)^- \in (WAW)\{1\}$, $Q(WAW)^-P \in A\{W - GD\}$, that is,

$$Q \cdot (WAW)\{1\} \cdot P \subseteq A\{W - GD\}.$$

Proof. (i) ⇒ (ii): Let $C \in A\{W-GD\}$. Denote by $P = WAWC$ and $Q = CWAW$. Because $C \in (WAW)\{1\}$, then $P = P^2$, $Q = Q^2$, $R(P) = R(WAW)$ and $N(Q) = N(WAW)$. Also, we get

$$WA^{d,W}PW = WA^{d,W}WAWCW = WCWA^{d,W}WAW = WQA^{d,W}W.$$

(ii) ⇒ (i): Suppose that $(WAW)^- \in (WAW)\{1\}$ and $C = Q(WAW)^-P$. The assumption $R(P) = R(WAW)$ gives $P = WAW(WAW)^-P$ and $WAW = PWAW$. Since $N(Q) = N(WAW)$, then $R(I-Q) = N(WAW)$ and $N(Q) = N((WAW)^-WAW) = R(I-(WAW)^-WAW)$ which imply $WAW = WAWQ$ and $Q = Q(WAW)^-WAW$. Hence,

$$WAWCWAW = (WAWQ)(WAW)^-(PWAW) = WAW(WAW)^-WAW = WAW$$

and, by $WA^{d,W}PW = WQA^{d,W}W$,

$$\begin{aligned} WA^{d,W}WAWCW &= WA^{d,W}(WAWQ)(WAW)^-PW = WA^{d,W}(WAW(WAW)^-P)W \\ &= WA^{d,W}PW = WQA^{d,W}W = WQ(WAW)^-WAWA^{d,W}W \\ &= WQ(WAW)^-PWAWA^{d,W}W = WCWAWA^{d,W}W, \end{aligned}$$

i.e. $C \in A\{W-GD\}$. □

Also, we prove the following result.

Theorem 2.3. *Let $W \in \mathbf{B}(Y,X)\backslash\{0\}$ and let $A \in \mathbf{B}(X,Y)$ be Wg-Drazin invertible such that WAW is relatively regular. Then*

$$A\{W-GD\} \cdot WAW \cdot A\{W-GD\} \subseteq A\{W-GD\}.$$

Proof. Assume that $C, C' \in A\{W-GD\}$ and $Z = CWAWC'$. We observe that $Z \in A\{W-GD\}$, by

$$WAWZWAW = (WAWCWAW)C'WAW = WAWC'WAW = WAW$$

and

$$\begin{aligned} WA^{d,W}WAWZW &= (WA^{d,W}WAWCW)AWC'W = WCWA(WA^{d,W}WAWC'W) \\ &= WCWAWC'WAWA^{d,W}W = WZWAWA^{d,W}W. \end{aligned}$$

□

3. Weighted G-Drazin pre-order

Firstly, we introduce a new binary relation on $\mathbf{B}(X,Y)$ generalizing the definition of the weighted G-Drazin relation presented in [4] for complex rectangular matrices to the class of bounded linear operators between Banach spaces.

Definition 3.2. Let $W \in \mathbf{B}(Y,X)\backslash\{0\}$, $B \in \mathbf{B}(X,Y)$ and let $A \in \mathbf{B}(X,Y)$ be Wg-Drazin invertible such that WAW is relatively regular. Then we say that A is below to B under the W-weighted GDrazin relation (denoted by $A \leq^{GD,W} B$) if there exist $C_1, C_2 \in A\{W-GD\}$ such that

$$WAWC_1 = WBWC_1 \qquad \text{and} \qquad C_2WAW = C_2WBW.$$

We characterize the relation $\leq^{GD,W}$ in the following theorem, extending some results from [4].

Theorem 3.4. *Let $W \in \mathbf{B}(Y,X)\backslash\{0\}$, $B \in \mathbf{B}(X,Y)$ and let $A \in \mathbf{B}(X,Y)$ be Wg-Drazin invertible such that WAW is relatively regular. Then the following statements are equivalent:*

(i) $A \leq^{GD,W} B$;

(ii) *there exist* $C \in A\{W - GD\}$ *such that*

$$WAWC = WBWC \qquad \text{and} \qquad CWAW = CWBW;$$

(iii) *there exist topological direct sums* $X = X_1 \oplus X_2$ *and* $Y = Y_1 \oplus Y_2$ *such that*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}, \qquad B = \begin{bmatrix} A_1 & B_3 \\ B_4 & B_2 \end{bmatrix},$$

where A_1 *and* W_1 *are invertible,* W_2A_2 *and* A_2W_2 *are quasinilpotent,* $B_3W_2 = 0$, $W_2B_4 = 0$, $W_2A_2W_2$ *is relatively regular and* $W_2A_2W_2 \leq^- W_2B_2W_2$.

In addition, if B *is* Wg*-Drazin invertible such that* WBW *is relatively regular, then* $D \in B\{W - GD\}$ *if and only if*

$$D = \begin{bmatrix} (W_1A_1W_1)^{-1} & D_{12} \\ D_{21} & D_2 \end{bmatrix},$$

where $D_{12}W_2 = 0$, $W_2D_{21} = 0$ *and* $D_2 \in B_2\{W_2 - GD\}$.

Proof. (i) ⇒ (ii): The proof is analogous to that given in [4, Theorem 3.1].

(ii) ⇒ (iii): Assume that there exist $C \in A\{W - GD\}$ such that $WAWC = WBWC$ and $CWAW = CWBW$. By Theorem 2.1(vi), there exist topological direct sums $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ such that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}, \qquad C = \begin{bmatrix} (W_1A_1W_1)^{-1} & C_{12} \\ C_{21} & C_2 \end{bmatrix},$$

where A_1 and W_1 are invertible, W_2A_2 and A_2W_2 are quasinilpotent, $C_{12}W_2 = 0$, $W_2C_{21} = 0$, $W_2A_2W_2$ is relatively regular and $C_2 \in (W_2A_2W_2)\{1\}$. Let

$$B = \begin{bmatrix} B_1 & B_3 \\ B_4 & B_2 \end{bmatrix}.$$

The equalities $WAWC = WBWC$,

$$WAWC = \begin{bmatrix} I & W_1A_1W_1C_{12} \\ 0 & W_2A_2W_2C_2 \end{bmatrix}$$

and

$$WBWC = \begin{bmatrix} W_1B_1(W_1A_1)^{-1} & W_1B_1W_1C_{12} + W_1B_3W_2C_2 \\ W_2B_4(W_1A_1)^{-1} & W_2B_4W_1C_{12} + W_2B_2W_2C_2 \end{bmatrix}$$

imply $B_1 = A_1$, $W_2B_4 = 0$ and $W_2A_2W_2C_2 = W_2B_2W_2C_2$. From $CWAW = CWBW$,

$$CWAW = \begin{bmatrix} I & 0 \\ C_{21}W_1A_1W_1 & C_2W_2A_2W_2 \end{bmatrix}$$

and

$$CWBW = \begin{bmatrix} I & (A_1W_1)^{-1}B_3W_2 \\ C_{21}W_1A_1W_1 & C_{21}W_1B_3W_2 + C_2W_2B_2W_2 \end{bmatrix},$$

we get $B_3W_2 = 0$ and $C_2W_2A_2W_2 = C_2W_2B_2W_2$. So, $W_2A_2W_2 \leq^- W_2B_2W_2$.

(iii) ⇒ (i): By the hypothesis $W_2A_2W_2 \leq^- W_2B_2W_2$, there exists $C_2 \in (W_2A_2W_2)\{1\}$ such that $W_2A_2W_2C_2 = W_2B_2W_2C_2$ and $C_2W_2A_2W_2 = C_2W_2B_2W_2$. Suppose that

$$C = \begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ 0 & C_2 \end{bmatrix}.$$

Applying Theorem 2.1(vi), we deduce that $C \in A\{W - GD\}$. Notice that $WAWC = WBWC$ and $CWAW = CWBW$ which imply $A \leq^{GD,W} B$.

Assume that $D \in B\{W - GD\}$ and

$$D = \begin{bmatrix} D_1 & D_{12} \\ D_{21} & D_2 \end{bmatrix}.$$

Then $WBW = WBWDWBW$ is equivalent to

$$\begin{bmatrix} W_1 A_1 W_1 & 0 \\ 0 & W_2 B_2 W_2 \end{bmatrix} = \begin{bmatrix} W_1 A_1 W_1 D_1 W_1 A_1 W_1 & W_1 A_1 W_1 D_{12} W_2 B_2 W_2 \\ W_2 B_2 W_2 D_{21} W_1 A_1 W_1 & W_2 B_2 W_2 D_2 W_2 B_2 W_2 \end{bmatrix}$$

which yields $D_1 = (W_1 A_1 W_1)^{-1}$, $D_{12} W_2 B_2 W_2 = 0$, $W_2 B_2 W_2 D_{21} = 0$ and $W_2 B_2 W_2 = W_2 B_2 W_2 D_2 W_2 B_2 W_2$. Using [2, Theorem 2.3], we obtain

$$BW = \begin{bmatrix} A_1 W_1 & 0 \\ B_4 W_1 & B_2 W_2 \end{bmatrix} \quad \text{and} \quad (BW)^d = \begin{bmatrix} (A_1 W_1)^{-1} & 0 \\ S & (B_2 W_2)^d \end{bmatrix}, \tag{3.3}$$

where $S = B_4 W_1 (A_1 W_1)^{-2}$ by $(B_2 W_2)^d B_4 = [(B_2 W_2)^d]^2 B_2 W_2 B_4 = 0$. Now, we have that $W_2 S = 0$,

$$W(BW)^d BWDW = \begin{bmatrix} A_1^{-1} & W_1 D_{12} W_2 \\ 0 & W_2 (B_2 W_2)^d B_2 W_2 D_2 W_2 \end{bmatrix}$$

and

$$WDW(BW)^d BW = \begin{bmatrix} A_1^{-1} & 0 \\ W_2 D_{21} W_1 & W_2 D_2 W_2 (B_2 W_2)^d B_2 W_2 \end{bmatrix}.$$

The equality $W(BW)^d BWDW = WDW(BW)^d BW$ gives $D_{12} W_2 = 0$, $W_2 D_{21} = 0$ and $W_2 (B_2 W_2)^d B_2 W_2 D_2 W_2 = W_2 D_2 W_2 (B_2 W_2)^d B_2 W_2$. Hence, $D_2 \in B_2\{W_2 - GD\}$.

If

$$D = \begin{bmatrix} (W_1 A_1 W_1)^{-1} & D_{12} \\ D_{21} & D_2 \end{bmatrix},$$

where $D_{12} W_2 = 0$, $W_2 D_{21} = 0$ and $D_2 \in B_2\{W_2 - GD\}$, by elementary computations, we verify that $D \in B\{W - GD\}$. □

Remark that $A \leq^{GD,W} B$ implies $WAW \leq^- WBW$, because $A\{W - GD\} \subseteq (WAW)\{1\}$.

Corollary 3.2. *Let $W \in \mathbf{B}(Y, X)\backslash\{0\}$ and let $A, B \in \mathbf{B}(X, Y)$ be Wg-Drazin invertible such that WAW and WBW are relatively regular. If $A \leq^{GD,W} B$, then*

$$B\{W - GD\} \subseteq A\{W - GD\}.$$

Proof. The proof is analogous to that given in [4, Corollary 3.2]. □

Before we prove that the W-weighted G-Drazin relation is a pre-order, recall that the W-weighted G-Drazin relation is not antisymmetric (see [4, Example 3.1]).

Theorem 3.5. *Let $W \in \mathbf{B}(Y, X)\backslash\{0\}$. The W-weighted G-Drazin relation is a pre-order on the set $\{A \in \mathbf{B}(X, Y) : A \; is \; Wg - Drazin \; invertible \; such \; that \; WAW \; is \; relatively \; regular\}$.*

Proof. The proof is analogous to that given in [4, Theorem 3.3]. □

We give equivalent conditions for $A \leq^{GD,W} B$ to be satisfied, generalizing those in [4, Theorem 3.4] and adding the new condition (vii).

Theorem 3.6. *Let $W \in \mathbf{B}(Y, X)\backslash\{0\}$, $B \in \mathbf{B}(X, Y)$ and let $A \in \mathbf{B}(X, Y)$ be Wg-Drazin invertible such that WAW is relatively regular. Then the following statements are equivalent:*

(i) *$A \leq^{GD,W} B$;*
(ii) *$WAW \leq^- WBW$, $(AW)^d BW = (AW)^d AW$ and $WB(WA)^d = WA(WA)^d$;*
(iii) *$WAW \leq^- WBW$, $N((AW)^d) \subseteq N((AW)^d BW)$ and $R(WB(WA)^d) \subseteq R((WA)^d)$;*
(iv) *$WAW \leq^- WBW$ and $W(AW)^d BW = WB(WA)^d W$;*
(v) *$WAW \leq^- WBW$ and $WA^{d,W} WBW = WBWA^{d,W} W$;*

(vi) (a) *There exists* $A^{W-GD} \in A\{W-GD\}$ *such that* $WAWA^{W-GD}WBW = WAW = WBWA^{W-GD}WAW$.
(b) *For every* $C \in A\{W-GD\}$, $WCW(AW)^dBW = WB(WA)^dWCW$.

(vii) *There exist idempotents* $P \in \mathbf{B}(X)$ *and* $Q \in \mathbf{B}(Y)$ *such that* $R(P) = R(WAW)$, $N(Q) = N(WAW)$, $WA^{d,W}PW = WQA^{d,W}W$ *and* $PWBW = WAW = WBWQ$.

Proof. (i) ⇒ (ii): Applying Theorem 3.4, there exist topological direct sums $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ such that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}, \qquad B = \begin{bmatrix} A_1 & B_3 \\ B_4 & B_2 \end{bmatrix},$$

where A_1 and W_1 are invertible, W_2A_2 and A_2W_2 are quasinilpotent, $B_3W_2 = 0$, $W_2B_4 = 0$, $W_2A_2W_2$ is relatively regular and $W_2A_2W_2 \leq^- W_2B_2W_2$. Thus, there exists $C_2 \in (W_2A_2W_2)\{1\}$ such that $W_2A_2W_2C_2 = W_2B_2W_2C_2$ and $C_2W_2A_2W_2 = C_2W_2B_2W_2$. Set

$$C = \begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ 0 & C_2 \end{bmatrix}.$$

Then $WAWCWAW = WAW$, $WAWC = WBWC$ and $CWAW = CWBW$ yield $WAW \leq^- WBW$. Furthermore, we obtain

$$(AW)^dBW = \begin{bmatrix} (A_1W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1W_1 & 0 \\ B_4W_1 & B_2W_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = (AW)^dAW$$

and similarly $WB(WA)^d = WA(WA)^d$.

(ii) ⇒ (iii): This implication is clear.

(iii) ⇒ (i): Let A and W be represented as in (1.1). If

$$B = \begin{bmatrix} B_1 & B_3 \\ B_4 & B_2 \end{bmatrix},$$

then

$$(AW)^dBW = \begin{bmatrix} (A_1W_1)^{-1}B_1W_1 & (A_1W_1)^{-1}B_3W_2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad (AW)^dAW = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Because $N((AW)^dAW) = N((AW)^d) \subseteq N((AW)^dBW)$, we have that $B_3W_2 = 0$. Also $R(WB(WA)^d) \subseteq R((WA)^d)$,

$$WB(WA)^d = \begin{bmatrix} W_1B_1(W_1A_1)^{-1} & 0 \\ W_2B_4(W_1A_1)^{-1} & 0 \end{bmatrix} \quad \text{and} \quad (WA)^dAW = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

imply $W_2B_4 = 0$.

The assumption $WAW \leq^- WBW$ implies that there exists $C \in (WAW)\{1\}$ such that $WAWC = WBWC$ and $CWAW = CWBW$. Let

$$C = \begin{bmatrix} C_1 & C_3 \\ C_4 & C_2 \end{bmatrix}.$$

From $WAWCWAW = WAW$, we get $C_1 = (W_1A_1W_1)^{-1}$, $C_3W_2A_2W_2 = 0$, $W_2A_2W_2C_4 = 0$ and $W_2A_2W_2C_2W_2A_2W_2 = W_2A_2W_2$. By $WAWC = WBWC$, we have that $B_1 = A_1$ and $W_2A_2WC_2 = W_2B_2W_2C_2$. Also, $CWAW = CWBW$ gives $C_2W_2A_2W_2 = C_2W_2B_2W_2$. Hence, $W_2A_2W_2 \leq^- W_2B_2W_2$ and, by Theorem 3.4, $A \leq^{GD,W} B$.

(ii) ⇒ (iv): Consequently, by $(AW)^dA = A(WA)^d$.

(iv) ⇒ (ii): Suppose that A and W are given as in (1.1) and

$$B = \begin{bmatrix} B_1 & B_3 \\ B_4 & B_2 \end{bmatrix}.$$

Since

$$\begin{bmatrix} A_1^{-1}B_1W_1 & A_1^{-1}B_3W_2 \\ 0 & 0 \end{bmatrix} = W(AW)^dBW = WB(WA)^dW = \begin{bmatrix} W_1B_1A_1^{-1} & 0 \\ W_2B_4A_1^{-1} & 0 \end{bmatrix},$$

then $B_3W_2 = 0$ and $W_2B_4 = 0$. Using the condition $WAW \leq^- WBW$, the rest follows as in part (iii) ⇒ (i).

(iv) ⇔ (v): This equivalence is obvious.

(ii) ⇒ (vi): Assume that A, W, B and C are represented as in the part (i) ⇒ (ii). By Theorem 2.1, we deduce that $C \in A\{W - GD\}$. Also, we can verify that $WAWCWBW = WAW = WBWCWAW$. Thus, the part (a) is satisfied.

To prove that part (b) holds, we suppose that $C' \in A\{W - GD\}$. Applying Theorem 2.1, we have that

$$C' = \begin{bmatrix} (W_1A_1W_1)^{-1} & C'_{12} \\ C'_{21} & C'_2 \end{bmatrix},$$

where $C'_{12}W_2 = 0$, $W_2C'_{21} = 0$, $W_2A_2W_2$ is relatively regular and $C'_2 \in (W_2A_2W_2)\{1\}$. Then

$$WC'W(AW)^dBW = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} = WB(WA)^dWC'W.$$

(vi) ⇒ (ii): If $A^{W-GD} \in A\{W - GD\}$ such that $WAWA^{W-GD}WBW = WAW = WBWA^{W-GD}WAW$, by Theorem 2.1, there exist topological direct sums $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ such that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}, \qquad A^{W-GD} = \begin{bmatrix} (W_1A_1W_1)^{-1} & C_{12} \\ C_{21} & C_2 \end{bmatrix},$$

where A_1 and W_1 are invertible, W_2A_2 and A_2W_2 are quasinilpotent, $C_{12}W_2 = 0$, $W_2C_{21} = 0$, $W_2A_2W_2$ is relatively regular and $C_2 \in (W_2A_2W_2)\{1\}$. For

$$B = \begin{bmatrix} B_1 & B_3 \\ B_4 & B_2 \end{bmatrix},$$

we get

$$WA^{W-GD}W(AW)^dBW = \begin{bmatrix} A_1^{-1}(A_1W_1)^{-1}B_1W_1 & A_1^{-1}(A_1W_1)^{-1}B_3W_2 \\ 0 & 0 \end{bmatrix}$$

and

$$WB(WA)^dWA^{W-GD}W = \begin{bmatrix} W_1B_1(W_1A_1)^{-1}A_1^{-1} & 0 \\ W_2B_4(W_1A_1)^{-1}A_1^{-1} & 0 \end{bmatrix}.$$

Now, $WA^{W-GD}W(AW)^dBW = WB(WA)^dWA^{W-GD}W$ gives $B_3W_2 = 0$ and $W_2B_4 = 0$. From

$$WAW = \begin{bmatrix} W_1A_1W_1 & 0 \\ 0 & W_2A_2W_2 \end{bmatrix},$$

$$WAWA^{W-GD}WBW = \begin{bmatrix} W_1B_1W_1 & 0 \\ 0 & W_2A_2W_2C_2W_2B_2W_2 \end{bmatrix}$$

and $WAWA^{W-GD}WBW = WAW$, we obtain $B_1 = A_1$ and $W_2A_2W_2C_2W_2B_2W_2 = W_2A_2W_2$. By

$$WBWA^{W-GD}WAW = \begin{bmatrix} W_1A_1W_1 & 0 \\ 0 & W_2B_2W_2C_2W_2A_2W_2 \end{bmatrix}$$

and $WAW = WBWA^{W-GD}WAW$, we have that $W_2B_2W_2C_2W_2A_2W_2 = W_2A_2W_2$. Set $C'_2 = C_2W_2A_2W_2C_2$. Then $C'_2 \in (W_2A_2W_2)\{1\}$, $W_2A_2W_2C'_2 = W_2B_2W_2C'_2$ and $C'_2W_2A_2W_2 = C'_2W_2B_2W_2$. So, $W_2A_2W_2 \leq^- W_2B_2W_2$ and, by Theorem 3.4, $A \leq^{GD,W} B$.

(i) ⇒ (vii): By Theorem 3.4, there exist $C \in A\{W-GD\}$ such that $WAWC = WBWC$ and $CWAW = CWBW$. For $P = WAWC$ and $Q = CWAW$, we obtain $R(P) = R(WAW)$, $N(Q) = N(WAW)$ and $WA^{d,W}PW = WQA^{d,W}W$ as in the proof of Theorem 2.2 (part (i) ⇒ (ii)). We also have

$$WAW = WAW(CWAW) = WAWCWBW = PWBW$$

and in the same way $WAW = WBWQ$.

(vii) ⇒ (i): Suppose that there exist idempotents $P \in \mathbf{B}(X)$ and $Q \in \mathbf{B}(Y)$ such that $R(P) = R(WAW)$, $N(Q) = N(WAW)$, $WA^{d,W}PW = WQA^{d,W}W$ and $PWBW = WAW = WBWQ$. Set $C = Q(WAW)^-P$, for $(WAW)^- \in (WAW)\{1\}$. Using Theorem 2.2, we deduce that $C \in A\{W-GD\}$. Now, by $WAW = WAWQ = PWAW$, we get

$$WBWC = (WBWQ)(WAW)^-P = WAW(WAW)^-P = WAWQ(WAW)^-P = WAWC$$

and analogously $CWBW = CWAW$. So, $A \leq^{GD,W} B$. □

By [4, Example 3.2], we observe that none of relations $\leq^{d,W}$ and $\leq^{GD,W}$ implies other one. In the next result, we prove that $A \leq^{d,W} B$ and $WAW \leq^- WBW$ give $A \leq^{GD,W} B$.

Theorem 3.7. *Let $W \in \mathbf{B}(Y,X)\backslash\{0\}$, $B \in \mathbf{B}(X,Y)$ and let $A \in \mathbf{B}(X,Y)$ be Wg-Drazin invertible such that WAW is relatively regular. If $A \leq^{d,W} B$ and $WAW \leq^- WBW$, then $A \leq^{GD,W} B$.*

Proof. The proof is analogous to that given in [4, Lemma 3.2]. □

The result given in [4, Theorem 3.5] is also valid for operators on Banach spaces.

Theorem 3.8. *Let $W \in \mathbf{B}(Y,X)\backslash\{0\}$ and let $A, B \in \mathbf{B}(X,Y)$ be Wg-Drazin invertible such that WAW is relatively regular. If $A \leq^{GD,W} B$, then $WA^{d,W}W$ is relatively regular and $WA^{d,W}W \leq^- WB^{d,W}W$.*

Proof. Let A, W and B be represented as in Theorem 3.4(iii). Since (3.3) holds, then

$$WB^{d,W}W = W(BW)^d = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & W_2(B_2W_2)^d \end{bmatrix}.$$

We observe that $WA^{d,W}W = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ is relatively regular. Set $U = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$. Now, we have that $U \in (WA^{d,W}W)\{1\}$,

$$WA^{d,W}WU = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = WB^{d,W}WU$$

and

$$UWA^{d,W}W = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = UWB^{d,W}W,$$

that is $WA^{d,W}W \leq^- WB^{d,W}W$. □

Recall that if $A, B \in \mathbf{B}(X,Y)$ are relatively regular, then $A \leq^- B$ if and only if $B - A \leq^- B$. As in [4, Proposition 3.1], the following result holds.

Theorem 3.9. *Let $W \in \mathbf{B}(Y,X)\backslash\{0\}$ and let $A, B \in \mathbf{B}(X,Y)$ be Wg-Drazin invertible such that WAW and WBW are relatively regular. If $B-A$ is Wg-Drazin invertible, $A \leq^{GD,W} B$ and A, W and B are represented as in Theorem 3.4(iii), then the following conditions are equivalent*

(i) $B - A \leq^{GD,W} B$;

(ii) $B_2 - A_2 \leq^{GD,W} B_2$.

Proof. We see that WAW, WBW, $W_2A_2W_2$ and $W_2B_2W_2$ are relatively regular. Using Theorem 3.4 and Theorem 3.6, we deduce that $WAW \leq^- WBW$ and $W_2A_2W_2 \leq^- W_2B_2W_2$ which is equivalent to $W(B-A)W \leq^- WBW$ and $W_2(B_2-A_2)W_2 \leq^- W_2B_2W_2$. Because $B-A$ is Wg-Drazin invertible, then $(B-A)W$ and $W(B-A)$ are generalized Drazin invertible,

$$((B-A)W)^d = \begin{bmatrix} 0 & 0 \\ B_4W_1 & (B_2-A_2)W_2 \end{bmatrix}^d = \begin{bmatrix} 0 & 0 \\ 0 & ((B_2-A_2)W_2)^d \end{bmatrix}$$

and

$$(W(B-A))^d = \begin{bmatrix} 0 & W_1B_3 \\ 0 & W_2(B_2-A_2) \end{bmatrix}^d = \begin{bmatrix} 0 & 0 \\ 0 & (W_2(B_2-A_2))^d \end{bmatrix}.$$

From

$$W((B-A)W)^dBW = \begin{bmatrix} 0 & 0 \\ 0 & W_2((B_2-A_2)W_2)^dB_2W_2 \end{bmatrix}$$

and

$$WB(W(B-A))^dW = \begin{bmatrix} 0 & 0 \\ 0 & W_2B_2(W_2(B_2-A_2))^dW_2 \end{bmatrix},$$

we deduce that $W((B-A)W)^dBW = WB(W(B-A))^dW$ is equivalent to $W_2((B_2-A_2)W_2)^dB_2W_2 = W_2B_2(W_2(B_2-A_2))^dW_2$. Hence, by Theorem 3.6, (i) and (ii) are equivalent. □

4. G-Drazin inverses

If $A \in \mathbf{B}(X)$ and $W = I \in \mathbf{B}(X)$ in results of Section 2 and Section 3, we obtain definitions and characterizations of the G-Drazin inverse and the G-Drazin partial order for operators on Banach space. Thus, we extend recent results from [4, 19] and present some new results.

Definition 4.3. Let $A \in \mathbf{B}(X)$ be generalized Drazin invertible such that A is relatively regular. An operator $C \in \mathbf{B}(X)$ is a G-Drazin inverse of A if the following equalities hold:

$$ACA = A \qquad \textit{and} \qquad A^dAC = CA^dA.$$

Denote by $A\{GD\}$ the set of all G-Drazin inverses of A.

Corollary 4.3. *Let $A \in \mathbf{B}(X)$ be generalized Drazin invertible such that A is relatively regular. For $C \in \mathbf{B}(X)$, the following statements are equivalent:*

(i) $C \in A\{GD\}$;
(ii) $ACA = A$ *and* $A^dA^2C = AA^d = CA^dA^2$;
(iii) $ACA = A$ *and* $A^dAC = A^d = CA^dA$;
(iv) $ACA = A$ *and* $A^dC = CA^d$;
(v) *there exist a topological direct sum* $X = X_1 \oplus X_2$ *such that*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad C = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & C_2 \end{bmatrix},$$

where A_1 is invertible, A_2 is quasinilpotent, A_2 is relatively regular and $C_2 \in A_2\{1\}$.

Proof. We need only to prove that (ii) ⇔ (iv). Because the group inverse is double commutative and $(A^dA^2)^\# = A^d$, we conclude that $A^dA^2C = CA^dA^2$ is equivalent to $A^dC = CA^d$. We observe that $A^dC = CA^d$ and $ACA = A$ give $A^dA^2C = A^2CA^d = A^2CA(A^d)^2 = A^2(A^d)^2 = AA^d$ and also $AA^d = CA^dA^2$. □

Corollary 4.4. *Let $A \in \mathbf{B}(X)$ be generalized Drazin invertible such that A is relatively regular. If $C \in A\{GD\}$, then $(I-CA)A$ and $A(I-AC)$ are quasinilpotent.*

Corollary 4.5. *Let $A \in \mathbf{B}(X)$ be generalized Drazin invertible such that A is relatively regular. The following statements are equivalent:*

(i) $A\{GD\} \neq \emptyset$;

(ii) *there exist idempotents $P \in \mathbf{B}(X)$ and $Q \in \mathbf{B}(X)$ such that*

$$R(P) = R(A), \quad N(Q) = N(A) \quad and \quad A^d P = QA^d.$$

In addition, for arbitrary $A^- \in A\{1\}$, $QA^-P \in A\{GD\}$, that is,

$$Q \cdot A\{1\} \cdot P \subseteq A\{GD\}.$$

Corollary 4.6. *Let $A \in \mathbf{B}(X)$ be generalized Drazin invertible such that A is relatively regular. Then*

$$A\{GD\} \cdot A \cdot A\{GD\} \subseteq A\{GD\}.$$

The definition of the G-Drazin relation is stated now in the Banach space setting.

Definition 4.4. Let $B \in \mathbf{B}(X)$ and let $A \in \mathbf{B}(X)$ be generalized Drazin invertible such that A is relatively regular. Then we say that A is below to B under the G-Drazin relation (denoted by $A \leq^{GD} B$) if there exist $C_1, C_2 \in A\{GD\}$ such that

$$AC_1 = BC_1 \qquad and \qquad C_2A = C_2B.$$

Corollary 4.7. *Let $B \in \mathbf{B}(X)$ and let $A \in \mathbf{B}(X)$ be generalized Drazin invertible such that A is relatively regular. Then the following statements are equivalent:*

(i) $A \leq^{GD} B$;

(ii) *there exist $C \in A\{GD\}$ such that*

$$AC = BC \qquad and \qquad CA = CB;$$

(iii) *there exist topological direct sum $X = X_1 \oplus X_2$ such that*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad B = \begin{bmatrix} A_1 & 0 \\ 0 & B_2 \end{bmatrix},$$

where A_1 is invertible, A_2 is quasinilpotent, A_2 is relatively regular and $A_2 \leq^- B_2$.

In addition, if B is generalized Drazin invertible such that B is relatively regular, then $D \in B\{GD\}$ if and only if

$$D = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & D_2 \end{bmatrix},$$

where $D_2 \in B_2\{GD\}$.

It is interesting to note that the G-Drazin relation is a partial order.

Corollary 4.8. *The G-Drazin relation is a partial order on the set $\{A \in \mathbf{B}(X)^d : A \ is \ relatively \ regular\}$.*

Proof. It is enough to prove that the G-Drazin relation is antisymmetric. Assume that $A, B \in \mathbf{B}(X)^d$ such that A and B are relatively regular, $A \leq^{GD} B$ and $B \leq^{GD} A$. There exists $D \in B\{GD\}$ such that $BD = AD$ and $DB = DA$. Notice that A, B and D can be represented as in Corollary 4.7 and so $A_2 \leq^- B_2$. The equalities $BD = AD$ and $DB = DA$ give $B_2D_2 = A_2D_2$ and $D_2B_2 = D_2A_2$, that is $B_2 \leq^- A_2$. Since $\leq^-$ is antisymmetric, then $A_2 = B_2$. Thus, $A = B$. □

Corollary 4.9. *Let $B \in \mathbf{B}(X)$ and let $A \in \mathbf{B}(X)$ be generalized Drazin invertible such that A is relatively regular. Then the following statements are equivalent:*

(i) $A \leq^{GD} B$;

(ii) $A \leq^- B$, $A^dB = A^dA$ *and* $BA^d = AA^d$;

(iii) $A \leq^- B$, $N(A^d) \subseteq N(A^d B)$ *and* $R(BA^d) \subseteq R(A^d)$*;*
(iv) $A \leq^- B$ *and* $A^d B = BA^d$*;*
(v) $A \leq^- B$ *and* $A \leq^d B$*;*
(vi) (a) *There exists* $A^{GD} \in A\{GD\}$ *such that* $AA^{GD}B = A = BA^{GD}A$*.*
(b) *For every* $C \in A\{GD\}$, $CA^d B = BA^d C$*.*
(vii) *There exist idempotents* $P, Q \in \mathbf{B}(X)$ *such that* $R(P) = R(A)$, $N(Q) = N(A)$, $A^d P = QA^d$ *and* $PB = A = BQ$*.*

Corollary 4.10. *Let* $A, B \in \mathbf{B}(X, Y)$ *be generalized Drazin invertible such that* A *is relatively regular. If* $A \leq^{GD} B$*, then* A^d *is relatively regular and* $A^d \leq^- B^d$*.*

Corollary 4.11. *Let* $A, B \in \mathbf{B}(X)$ *be generalized Drazin invertible such that* A *and* B *are relatively regular. If* $B - A$ *is generalized Drazin invertible,* $A \leq^{GD} B$ *and* A*,* B *are represented as in Corollary 4.7(iii), then the following conditions are equivalent*

(i) $B - A \leq^{GD} B$*;*
(ii) $B_2 - A_2 \leq^{GD} B_2$*.*

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UNIVERSITY OF NIŠ
FACULTY OF SCIENCES AND MATHEMATICS
P.O. BOX 224, 18000 NIŠ, SERBIA
Email address: dijana@pmf.ni.ac.rs