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Dedicated to Prof. Juan Nieto on the occasion of his 60^{th} anniversary

Coupled fixed point theorems in quasimetric spaces without mixed monotonicity

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ABSTRACT. In this paper, using the concepts of f-closed set and inverse f-closed set, we will prove some fixed point theorems for graphic contractions in complete quasimetric space. Then, as applications, coupled fixed point theorems in quasimetric spaces without the mixed monotonicity property are obtained.

1. INTRODUCTION

A nice extension of Banach's contraction principle was established by Ran and Reurings (see [28]). They combined, for an operator $f : X \to X$, the contraction condition imposed only for pairs $(x, y) \in X \times X$ of comparable elements (with respect to a partial order relation on X) with a monotonicity assumption. Various extensions of Ran-Reurings theorem were published, see [17], [18], [19], [21], [32].

Another topic of high interest in the last decades is the coupled fixed point theory, later generalized to tripled fixed point theory or to multiple fixed point theory. For coupled fixed point theory, we refer to the seminal papers of D. Guo and V. Lakshmikantham [10], T. Gnana Bhaskar and V. Lakshmikantham [9] and V. Berinde [4]. For related contributions see [2], [5], [7], [6], [14], [11], [22], [23], [24], [26] and the references therein.

In this paper, using the concepts of f-closed set and inverse f-closed set, we will prove some fixed point theorems in complete quasimetric space. Then, as applications, coupled fixed point theorems in quasimetric spaces without any mixed monotonicity property are obtained. Our theorems extend and complement some results given in [25] and [20], where the framework of a complete metric space is considered.

2. FIXED POINTS IN QUASIMETRIC SPACES

In this section, we will focus on fixed point theorems on complete quasimetric spaces. For the sake of completeness we recall the definition of a quasimetric space.

Definition 2.1. ([1, 3, 8]) Let X be a nonempty set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to \mathbb{R}_+$ is said to be a quasimetric (also called, in many papers, b-metric) with constant $s \geq 1$ if the classical axioms of the metric hold, with the following modification of the triangle inequality:

$$d(x, z) \leq s[d(x, y) + d(y, z)], \text{ for all } x, y, z \in X.$$

A pair (X, d) with the above properties is called a quasimetric space.

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For several relevant examples of b-metrics see [3], [13], ...

The following lemma (Lemma 2.2 in see [16]) is essential in our approach. See also [30] and [31].

Lemma 2.1. *Let (X, d) be a b-metric space with constant $s \geq 1$. Then, every sequence $(x_n)_{n\in\mathbb{N}}$ of elements from X for which there exists $\gamma \in (0,1)$ such that*

$$d(x_n, x_{n+1}) \leq \gamma d(x_{n-1}, x_n), \text{ for every } n \in \mathbb{N}^*,$$

is a Cauchy sequence. Moreover, the following estimation holds

$$d(x_{n+1}, x_{n+p}) \leq \frac{\gamma^n S}{1-\gamma} d(x_0, x_1), \text{ for all } n, p \in \mathbb{N},$$

where $S := \sum_{i=1}^{\infty} \gamma^{2i \log_\gamma s + 2^{i-1}}$.

The following abstract notion was defined in [25].

Definition 2.2. Let X be a nonempty set, $\mathbb{P} \subset X^2$ and $f : X \to X$ be an operator. Then, $\mathbb{P}$ is said f-closed if the following implication holds:

$$(z, w) \in \mathbb{P} \text{ implies } (f(z), f(w)) \in \mathbb{P}.$$

Some examples of f-closed sets are presented in [20] and [25].

Example 2.1. 1) Let X be a nonempty set endowed with a partial order $\preceq$ and $f : X \to X$ be an increasing operator. If we define
$\mathbb{P}_1 := \{(x, y) \in X \times X : x \preceq y\}$
$\mathbb{P}_2 := \{(x, y) \in X \times X : y \preceq x\}$.
$\mathbb{P}_3 := \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$,
then $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_3$ are f-closed sets.

2) Let X be a nonempty set and denote by $\Delta := \{(x, x) \in X \times X : x \in X\}$ the diagonal of the Cartesian product $X \times X$. Let $f : X \to X$ be an edges preserving operator, see [12]. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, i.e., $\Delta \subseteq E(G)$. Then $\mathbb{P}_4 := \{(x, y) \in X \times X : (x, y) \in E(G)\}$ is f-closed.

The following lemma gives the relation between an f-closed set and the monotonicity of a given operator.

Lemma 2.2. ([25]) *Let X be a nonempty set, $\mathbb{P} \subset X \times X$ such that $\Delta \subset \mathbb{P}$ and $f : X \to X$ be a given operator. We define*

$$x \preceq y \Leftrightarrow (x, y) \in \mathbb{P}.$$

Then, $\mathbb{P}$ is f-closed if and only if f is increasing with respect to $\preceq$.

If X is a nonempty set and $f : X \to X$, then $Fix(f) := \{x \in X : x = f(x)\}$.

The first main result of this paper is a fixed point theorem for graphic contractions in quasimetric spaces. The result is a generalization of several Ran-Reurings type theorems in the literature.

Theorem 2.1. *Let (X, d) be a complete quasimetric space with constant $s \geq 1$, $\mathbb{P} \subset X^2$ and $f : X \to X$ be an operator. Suppose:*
(i) $\mathbb{P}$ is f-closed;
(ii) there exists $x_0 \in X$ such that $(x_0, f(x_0)) \in \mathbb{P}$;
(iii) $\lim_{n\to\infty} f(f^n(x_0)) = f(\lim_{n\to\infty} f^n(x_0))$;

(iv) there exists $\alpha \in]0,1[$ *such that, for all* $x \in X$ *for which* $(x, f(x)) \in \mathbb{P}$*, we have*

$$d(f(x), f^2(x)) \leq \alpha d(x, f(x)).$$

Then f *has at least one fixed point and the sequence* $x_n := f^n(x_0)$*,* $n \in \mathbb{N}$ *converges to an element* $x^* \in Fix(f)$*. Moreover,*

$$d(f^n(x_0), x^*) \leq \frac{s\alpha^{n-1}S}{1-\alpha} d(x_0, f(x_0)), \text{ for all } n \in \mathbb{N},$$

where $S := \sum_{i=1}^{\infty} \alpha^{2i \log_\alpha s + 2^{i-1}}$.

Proof. Let us denote $x_n := f^n(x_0)$, $n \in \mathbb{N}$. By (ii) and the f-closedness property of $\mathbb{P}$ we obtain that $(x_n, x_{n+1}) \in \mathbb{P}$ for all $n \in \mathbb{N}$. Then, by the graphic contraction condition (iv) we obtain that

$$d(x_n, x_{n+1}) \leq \alpha^n d(x_0, f(x_0)), \text{ for all } n \in \mathbb{N}.$$

By Lemma 2.1 we have that (x_n) is a Cauchy sequence. Using the completeness of the quasimetric space we obtain that (x_n) is convergent in (X, d). Denote by x^* its limit. By (iii), we immediately get that $x^* \in Fix(f)$. By the second conclusion of Lemma 2.1, we obtain that

$$d(x_{n+1}, x_{n+p}) \leq \frac{\alpha^n S}{1-\alpha} d(x_0, f(x_0)), \text{ for all } n, p \in \mathbb{N}.$$

Then, for $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$, we have

$$\frac{1}{s} d(x_{n+1}, x^*) \leq d(x_{n+1}, x_{n+p}) + d(x_{n+p}, x^*) \leq \frac{\alpha^n S}{1-\alpha} d(x_0, f(x_0)) + d(x_{n+p}, x^*).$$

Letting $p \to \infty$ we get that $d(x_{n+1}, x^*) \leq \frac{s\alpha^n S}{1-\alpha} d(x_0, f(x_0))$, for all $n \in \mathbb{N}$. □

We present now a kind of dual definition of an f-closed set.

Definition 2.3. Let X be a nonempty set, $\mathbb{S} \subset X^2$ and $f : X \to X$ be an operator. Then, $\mathbb{S}$ is said inverse f-closed if the following implication holds:

$$(x, y) \in \mathbb{S} \text{ implies } (f(y), f(x)) \in \mathbb{S}.$$

Notice that if $(X, \preceq)$ is a partially ordered set and $f : X \to X$ is an decreasing operator, then the sets $\mathbb{P}_i$ ($i \in \{1, 2, 3\}$) from Example 2.1 are inverse f-closed, while $\mathbb{P}_4$ isn't an inverse f-closed set. Moreover, the following lemma takes place.

Lemma 2.3. ([20]) *Let* X *be a nonempty set,* $\mathbb{S} \subset X \times X$ *such that* $\Delta \subset \mathbb{S}$ *and* $f : X \to X$ *be a given operator. We define*

$$x \preceq y \Leftrightarrow (x, y) \in \mathbb{S}.$$

Then:
(a) $\mathbb{S}$ *has the transitive property if and only if* $\preceq$ *is a preorder on* X*;*
(b) $\mathbb{S}$ *is inverse* f*-closed if and only if* f *is decreasing with respect to* $\preceq$*.*

Proof. (a) follows directly from the definition of $\preceq$. For (b) let us suppose that f is decreasing with respect to $\preceq$. We show that $\mathbb{S}$ is inverse f-closed. Indeed, take $x, y \in X$ with $(x, y) \in \mathbb{S}$. Then $x \preceq y$. By the monotonicity of f we get that $f(y) \preceq f(x)$. Thus $(f(y), f(x)) \in \mathbb{S}$. Hence, $\mathbb{S}$ is inverse f-closed. For the reverse implication, suppose that $\mathbb{S}$ is inverse f-closed and take $x, y \in X$ with $x \preceq y$. Then $(x, y) \in \mathbb{S}$, and by the inverse f-closedness property of $\mathbb{S}$, we have that $(f(y), f(x)) \in \mathbb{S}$. Thus, $f(y) \preceq f(x)$. □

We can prove now a fixed point theorem for a graphic contraction $f : X \to X$ in complete quasimetric spaces in terms of inverse f-closed sets.

Theorem 2.2. *Let (X, d) be a complete quasimetric space with constant $s \geq 1$, $\mathbb{P} \subset X^2$ and $f : X \to X$ be an operator. Suppose that the following conditions are satisfied:*

(1) $\mathbb{P}$ is inverse f-closed;

(2) there exists $x_0 \in X$ such that $(x_0, f(x_0))$ or $(f(x_0), x_0)$ belongs to $\mathbb{P}$;

(3) $\lim\limits_{n\to\infty} f(f^n(x_0)) = f(\lim\limits_{n\to\infty} f^n(x_0))$;

(4) there is $\alpha \in]0,1[$ such that, for $x \in X$ with $\{(x, f(x)) : x \in X\} \cup \{(f(x), x) : x \in X\} \subset \mathbb{P}$, we have that

$$d(f(x), f^2(x)) \leq \alpha d(x, f(x)).$$

Then f has at least one fixed point and the sequence $x_n := f^n(x_0)$, $n \in \mathbb{N}$ converges to an element $x^* \in Fix(f)$. Moreover, we have

$$d(x_n, x^*) \leq \frac{s\alpha^{n-1}S}{1-\alpha} d(x_0, f(x_0)), \text{ for all } n \in \mathbb{N}, n \geq 1,$$

where $S := \sum_{i=1}^{\infty} \alpha^{2i \log_\alpha s + 2^{i-1}}$.

Proof. Denote $x_n := f^n(x_0)$, $n \in \mathbb{N}$. Suppose, for example, that $(x_0, f(x_0)) \in \mathbb{P}$. By the inverse f-closedness property of $\mathbb{P}$ we obtain that (x_2, x_1), $(x_2, x_3), \ldots$, (x_{2n}, x_{2n-1}), $(x_{2n}, x_{2n+1}), \ldots$ are in $\mathbb{P}$, for all $n \in \mathbb{N}^*$. Then, by the graphic contraction condition (4), we obtain that

$$d(x_n, x_{n+1}) \leq \alpha^n d(x_0, f(x_0)), \text{ for all } n \in \mathbb{N}.$$

By Lemma 2.1 we have that (x_n) is a Cauchy sequence. By the completeness of the quasimetric space we obtain that (x_n) is convergent in (X, d). Denote by x^* its limit. By (3), we immediately get that $x^* \in Fix(f)$. The second conclusion can be obtained by a similar approach to that given in the proof of Theorem 2.1. □

By Theorem 2.2 a fixed point theorem for decreasing operators can be obtained.

Theorem 2.3. *Let (X, d) be a complete quasimetric space with constant $s \geq 1$, $\preceq$ be a partial order on X and $f : X \to X$ be an operator. Suppose that the following conditions are satisfied:*

(1) f is decreasing with respect to $\preceq$;

(2) there exists $x_0 \in X$ such that $x_0 \preceq f(x_0)$ or $f(x_0) \preceq x_0$;

(3) $\lim\limits_{n\to\infty} f(f^n(x_0)) = f(\lim\limits_{n\to\infty} f^n(x_0))$;

(4) there is $\alpha \in]0,1[$ such that, for $x \in X$ with $x \preceq f(x)$ or $f(x) \preceq x$, we have that

$$d(f(x), f^2(x)) \leq \alpha d(x, f(x)).$$

Then f has at least one fixed point and the sequence $x_n := f^n(x_0)$, $n \in \mathbb{N}$ converges to an element $x^* \in Fix(f)$. Moreover, we have

$$d(x_n, x^*) \leq \frac{s\alpha^{n-1}S}{1-\alpha} d(x_0, f(x_0)), \text{ for all } n \in \mathbb{N}, n \geq 1,$$

where $S := \sum_{i=1}^{\infty} \alpha^{2i \log_\alpha s + 2^{i-1}}$.

Proof. Since f is decreasing with respect to $\preceq$, the set $\mathbb{P} := \{(x, y) \in X \times X : x \preceq y\}$ is inverse f-closed (by Lemma 2.3). Thus, Theorem 2.3 follows by Theorem 2.2. □

Remark 2.1. For other results for graphic contractions in complete metric spaces see [27].

3. Coupled fixed point theorems in quasimetric spaces

In this section, we will show how the above results can be applied to coupled fixed point theory.

Recall that, if X is a nonempty set and $F : X \times X \to X$ is an operator, then, by definition, a coupled fixed point for F is a pair $(x, y) \in X \times X$ satisfying the system

$$\begin{cases} x = F(x, y) \\ y = F(y, x). \end{cases} \tag{3.1}$$

We denote by $CFix(F)$ the coupled fixed set of F. Notice also that if (x, y) is a solution of the coupled fixed point problem with $x = y$, then x is said to be a fixed point for F.

In the above context we denote

$$F^n(x, y) := F(F^{n-1}(x, y), F^{n-1}(y, x)), n \in \mathbb{N}, n \geq 1,$$

where $F^0(x, y) := x$ and $F^0(y, x) := y$.

The following definition is modeled after Kubti et all. [14].

Definition 3.4. Let (X, d) be a nonempty set and $F : X \times X \to X$ be an operator. A nonempty subset $\mathbb{M}$ of X^4 is said to be F-closed if for all $x, y, u, v \in X$ the following implication holds:

$$(x, y, u, v) \in M \Rightarrow (F(x, y), F(y, x), F(u, v), F(v, u)) \in \mathbb{M}.$$

The following auxiliary result immediately follows by the above definitions.

Lemma 3.4. (see [20]) *Let X be a nonempty set, $F : X \times X \to X$ be a given operator and $\mathbb{M}$ be an F-closed set. Denote $Z := X \times X$. Denote*

$$\mathbb{P} := \{(z, w) \in Z \times Z : \ z = (x, y), \ w = (u, v), \ (x, y, u, v) \in \mathbb{M}\}$$

and define $T_F(x, y) := (F(x, y), F(y, x))$, for all $(x, y) \in Z$.
Then the following implication holds:

$$\mathbb{M} \text{ is } F\text{-closed if and only if } \mathbb{P} \text{ is } T_F\text{-closed.}$$

By Theorem 2.1, we obtain the following very general existence result for the coupled fixed point problem.

Theorem 3.4. *Let (X, d) be a complete quasimetric space with constant $s \geq 1$, $\mathbb{M} \subset X^4$ and $F : X \times X \to X$ be an operator. Suppose:*

(i) $\mathbb{M}$ is F-closed;

(ii) there exists $(x_0, y_0) \in X \times X$ such that $(x_0, y_0, F(x_0, y_0), F(y_0, x_0)) \in \mathbb{M}$;

(iii) the following relations hold:

$(iii)_1$ $\lim\limits_{n\to\infty} F^{n+1}(x_0, y_0) = F(\lim\limits_{n\to\infty} F^n(x_0, y_0), \lim\limits_{n\to\infty} F^n(y_0, x_0))$;

$(iii)_2$ $\lim\limits_{n\to\infty} F^{n+1}(y_0, x_0) = F(\lim\limits_{n\to\infty} F^n(y_0, x_0), \lim\limits_{n\to\infty} F^n(x_0, y_0))$;

(iv) there exists $\alpha \in]0, 1[$ such that, if $(x, y, F(x, y), F(y, x)) \in \mathbb{M}$, then

$$\max\{d(F(x, y), F^2(x, y)), d(F(y, x), F^2(y, x))\} \leq \alpha \max\{d(x, F(x, y)), d(y, F(y, x))\}.$$

Then F has at least one coupled fixed point $(x^*, y^*) \in X \times X$ and the sequences $(F^n(x_0, y_0))_{n\in\mathbb{N}}$ and $(F^n(y_0, x_0))_{n\in\mathbb{N}}$ converge to x^* and y^*, respectively. Moreover, for all $n \in \mathbb{N}$, the following approximations took place

$$\max\{d(F^n(x_0, y_0), x^*), d(F^n(y_0, x_0), y^*)\} \leq K \max\{d(x_0, F(x_0, y_0)), d(y_0, F(y_0, x_0))\},$$

where $K := \frac{s\alpha^{n-1}S}{1-\alpha}$ and $S := \sum\limits_{i=1}^{\infty} \alpha^{2i \log_\alpha s + 2^{i-1}}$.

Proof. We endow $Z := X \times X$ with the quasimetric

$$\widehat{d}((x,y),(u,v)) := \max\{d(x,u), d(y,v)\}, \text{ for all } (x,y),(u,v) \in Z.$$

We consider the operator $T_F : Z \to Z$ given by

$$T_F(x,y) := (F(x,y), F(y,x)), \text{ for all } (x,y) \in Z$$

and define $\mathbb{P} := \{(z,w) \in Z \times Z : \ z = (x,y), \ w = (u,v), \ (x,y,u,v) \in \mathbb{M}\}$.

By our hypotheses, T_F satisfies the following assumptions:
(1) $\mathbb{P}$ is T_F-closed (since $\mathbb{M}$ is F-closed);
(2) there exists $z_0 := (x_0, y_0) \in Z$ such that $(z_0, T_F(z_0)) \in \mathbb{P}$;
(3) $\lim\limits_{n\to\infty} T_F(T_F^n(z_0)) = T_F(\lim\limits_{n\to\infty} T_F^n(z_0))$.
(4) for $\alpha \in]0,1[$ and for all $(z, T_F(z)) \in \mathbb{P}$, we have $\widehat{d}(T_F(z), T_F^2(z)) \leq \alpha\widehat{d}(z, T(z))$.
Since $Fix(T_F) = CFix(F)$, the conclusion follows by Theorem 2.1. □

Remark 3.2. Theorem 3.4 generalizes several results given in [25], where some particular cases of graphic contractions type operators F are considered.

In particular, the following result (which is more appropriate for applications) holds.

Theorem 3.5. *Let (X,d) be a complete quasimetric space with constant $s \geq 1$, $\mathbb{M} \subset X^4$ and $F : X \times X \to X$ be an operator. Suppose:*
(i) $\mathbb{M}$ is F-closed;
(ii) there exists $(x_0, y_0) \in X \times X$ such that $(x_0, y_0, F(x_0,y_0), F(y_0,x_0)) \in \mathbb{M}$;
(iii) the following relations hold:
$(iii)_1$ $\lim\limits_{n\to\infty} F^{n+1}(x_0,y_0) = F(\lim\limits_{n\to\infty} F^n(x_0,y_0), \lim\limits_{n\to\infty} F^n(y_0,x_0))$;
$(iii)_2$ $\lim\limits_{n\to\infty} F^{n+1}(y_0,x_0) = F(\lim\limits_{n\to\infty} F^n(y_0,x_0), \lim\limits_{n\to\infty} F^n(x_0,y_0))$;
(iv) there exists $\alpha \in]0,1[$ such that, if $(x,y,F(x,y),F(y,x)) \in \mathbb{M}$ and $(y,x,F(y,x),F(x,y)) \in \mathbb{M}$, then

$$d(F(x,y), F^2(x,y)) \leq \alpha d(x, F(x,y)).$$

Then F has at least one coupled fixed point $(x^*, y^*) \in X \times X$ and the sequences $(F^n(x_0,y_0))_{n\in\mathbb{N}}$ and $(F^n(y_0,x_0))_{n\in\mathbb{N}}$ converge to x^* and y^*, respectively. Moreover, for all $n \in \mathbb{N}$, the following approximation formula took place:

$$\max\{d(F^n(x_0,y_0), x^*), d(F^n(y_0,x_0), y^*)\} \leq K \max\{d(x_0, F(x_0,y_0)), d(y_0, F(y_0,x_0))\},$$

where $K := \frac{s\alpha^{n-1}S}{1-\alpha}$ *and* $S := \sum\limits_{i=1}^{\infty} \alpha^{2i\log_\alpha s + 2^{i-1}}$.

Remark 3.3. Using Theorem 2.2 and Theorem 2.3, by a similar approach, we can obtain corresponding coupled fixed point theorems in terms of inverse F-closed sets.

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