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Dedicated to Prof. Qamrul Hasan Ansari on the occasion of his 60^{th} anniversary

Coincidence point theorems for KC-contraction mappings in JS-metric spaces endowed with a directed graph

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ABSTRACT. We introduce the class of KC-contraction mappings and prove some coincidence point theorems for these contractions in JS-metric spaces endowed with a directed graph. An illustrative example as well as an application to integral equations are also given in order to support our main theoretical results.

1. INTRODUCTION

Fixed point theory is one of the most popular current research fields in mathematics. Researchers have been developed various techniques in order to guarantee the existence of a fixed point for a certain mapping.

There is a wide range of aspects in this theory explored and applied to the real world problems. For instance, Qamrul H. Ansari studied, amongst other topics, the connection between fixed point theory and equilibrium problems, see [2] and [4], to mention but a few. His work contributed not only to the theoretical aspects of those fields but also to various applied sciences like economics, game theory, optimization and variational inequalities, see [1] and [3].

Fixed point theory and its counterparts have been studied on different settings: on metric spaces, cone metric spaces, [12], topological vector spaces, [2] etc. In 2015, Jleli and Samet, see [9], introduced a notion of generalized metric space usually known as a JS-metric space, which contains many classes of topological spaces such as metric spaces, b-metric spaces, dislocated metric spaces and modular spaces. These new setting has been of interest amongst many researchers. For instance, ElKouch and Marhrani [7] extended some fixed point theorems for Kannan and Chatterjea contraction mappings to this more general setting.

On the other hand, some basic fixed point theorems have been extended to the setting of metric spaces endowed with a graph, see Jachymski [10]. These kind of results paves the way for our direction of study.

In this work, we are particularly concerned with establishing coincidence (also common fixed) point theorems for various types of contraction mappings such as

- Kannan contraction or K-*contraction* mappings (Definition 3.8);
- Chatterjea contraction or C-*contraction* mappings (Definition 3.9);
- *KC-contraction* mappings, a generalization of K-contraction and C-contraction mappings (Definition 3.10),

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in JS-metric spaces endowed with a directed graph. Our main results (Theorems 4.1, 4.2 and 4.3) provide the existence of a coincidence point for those types of contraction mappings. We also give an example and an application to a certain integral equation where our results are employed.

2. JS-METRIC SPACES WITH A DIRECTED GRAPH

To begin with, let X be a nonempty set, and let $D : X \times X \to [0, +\infty]$ be a function. For each $x \in X$, we set

$$C(D, X, x) = \{\{x_n\} \subseteq X : \lim_{n\to\infty} D(x_n, x) = 0\}.$$

Definition 2.1. [9] Let X be a nonempty set. A function $D : X \times X \to [0, +\infty]$ is called a **generalized metric** on a set X if it satisfies the following conditions:

(D_1) For any $x, y \in X$, $D(x, y) = 0$ implies $x = y$;
(D_2) For any $x, y \in X$, $D(x, y) = D(y, x)$;
(D_3) There is a constant $C_X > 0$ such that

$$D(x, y) \leq C_X \limsup_{n\to\infty} D(x_n, y)$$

whenever $x, y \in X$ and $\{x_n\} \in C(D, X, x)$.

The pair (X, D) is called a JS**-metric space**.

Here is a brief description regarding convergence and continuity in JS-metric spaces. It turns out that any convergent sequence in a JS-metric space converges to a unique point.

Definition 2.2. [9] Suppose that (X, D) is a JS-metric space, and $\{x_n\}$ is a sequence in X. We say that the sequence $\{x_n\}$ D**-converges** to $x \in X$ if $\{x_n\} \in C(D, X, x)$. Moreover, $\{x_n\}$ is called a D**-Cauchy sequence** if $\lim_{m,n\to\infty} D(x_n, x_m) = 0$. Finally, (X, D) is said to be D**-complete** if each D-Cauchy sequence in X D-converges to some element in X.

Proposition 2.1. [9] *Let (X, D) be a JS-metric space and $\{x_n\}$ a sequence in X. For any $x, y \in X$, if $\{x_n\} \in C(D, X, x) \cap C(D, X, y)$, then $x = y$.*

Definition 2.3. Let (X, D) be a JS-metric space. A function $f : X \to X$ is said to be **continuous at a point** $x_0 \in X$ if $\{x_n\} \in C(D, X, x_0)$ implies $\{fx_n\} \in C(D, X, fx_0)$. In addition, f is said to be **continuous** if it is continuous at each x in X.

Suppose now that (X, D) is a JS-metric space, and Δ denotes the diagonal of $X \times X$. We construct a directed graph $G = (V(G), E(G))$ from X such that the set of vertices $V(G)$ consists of all elements in X and the set of edges $E(G)$ contains the diagonal Δ of $X \times X$. Moreover, we assume further throughout this work that $E(G)$ contains no parallel edges. A JS-metric space (X, D) is said to be *endowed with a directed graph G* if all of the above mentioned properties hold.

Definition 2.4 ([10])**.** Let (X, D) be a JS-metric space endowed with a directed graph G. A function $f : (X, D) \to (X, D)$ is said to be G**-continuous** if for any $x \in X$ such that there exists a sequence $\{x_n\} \in C(D, X, x)$ which $(x_n, x_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$, we have $\{fx_n\} \in C(D, X, fx)$.

A JS-metric space endowed with a directed graph G is generally required to be well-behaved. The following properties will be used later in our main results.

Definition 2.5. Let (X, D) be a JS-metric space endowed with a directed graph G. The triple (X, D, G) is said to have **property** A if for any $x \in X$ and any sequence $\{x_n\}$ in $C(D, X, x)$ such that $(x_n, x_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$, we obtain $(x_n, x) \in E(G)$.

Definition 2.6. Let (X, D) be a JS-metric space endowed with a directed graph G. We say that the set of edges of G, $E(G)$, satisfies the **transitivity property** if for all $x, y, z \in X$,

$$(x, z), (z, y) \in E(G) \Rightarrow (x, y) \in E(G).$$

3. Contraction mappings

This section introduces certain types of contraction mappings where the distance between two points in the image is controlled. More precisely, if f and g are mappings on a JS-metric space (X, D), we try to impose some conditions so that $D(fx, fy)$ is bounded by some terms of the mappings f, g. Our first condition requires the two functions to be related in some manner.

Definition 3.7. Let (X, D) be a JS-metric space endowed with a directed graph G, and let $f, g : X \to X$ be functions. We say that f is (g, D)**-edge preserving w.r.t** G if

$$(gx, gy) \in E(G) \Rightarrow (fx, fy) \in E(G) \text{ and } D(gx, gy) < \infty.$$

The following are variants of contraction mappings in the work of ElKouch and Marhrani, [7].

Definition 3.8. Let (X, D) be a JS-metric space endowed with a directed graph G, and let $f, g : X \to X$ be given functions. Then the pair (f, g) is called a K**-contraction** if

(a) f is (g, D)-edge preserving w.r.t G;
(b) there exists $\lambda \in [0, 1/2)$ such that for all $x, y \in X$ with $(gx, gy) \in E(G)$, we have

$$D(fx, fy) \leq \lambda[D(gx, fx) + D(gy, fy)]. \tag{3.1}$$

Definition 3.9. Let (X, D) be a JS-metric space endowed with a directed graph G, and let $f, g : X \to X$ be given functions. Then the pair (f, g) is called a C**-contraction** if

(a) f is (g, D)-edge preserving w.r.t G;
(b) there exists $\lambda \in [0, 1/2)$ such that for all $x, y \in X$ with $(gx, gy) \in E(G)$, we have

$$D(fx, fy) \leq \lambda[D(gx, fy) + D(gy, fx)]. \tag{3.2}$$

Definition 3.10. Let (X, D) be a JS-metric space endowed with a directed graph G, and let $f, g : X \to X$ be given functions. The pair (f, g) is called a KC**-contraction** if

(a) f is (g, D)*-edge preserving w.r.t* G;
(b) there exists $\lambda \in [0, 1/2)$ such that for all $x, y \in X$ with $(gx, gy) \in E(G)$, we have

$$D(fx, fy) \leq \lambda \max\{D(gx, fx) + D(gy, fy), D(gx, fy) + D(gy, fx)\}. \tag{3.3}$$

It is clear that if (f, g) is either a K-contraction or a C-contraction, then (f, g) is a KC-contraction. Examples of these mappings are easy to find. We give an example (Example 4.1) in the next section along with our results.

4. Existence of a coincidence point

Given two mappings f and g on a JS-metric space (X, D) endowed with a directed graph G, we show that, under some circumstances, a coincidence point of f and g exists.

Recall that $u \in X$ is a coincidence point of $f, g : X \to X$ if $fu = gu$. We denote the **set of all coincidence points of mappings** f **and** g by

$$C(f, g) = \{u \in X : fu = gu\}.$$

A point $u \in X$ is a common fixed point of $f, g : X \to X$ if $fu = gu = u$. We also define the **set of all common fixed points of mappings** f **and** g by

$$Cm(f, g) = \{u \in X : fu = gu = u\}.$$

The next following two lemmas are needed to prove our main results.

Lemma 4.1. *Let (X, D) be a JS-metric space endowed with a directed graph G. Suppose that $f, g : X \to X$ are functions and (f, g) is a KC-contraction. Then any $x, y \in C(f, g)$ satisfy the following properties.*

(a) $D(gx, gx) = 0$.
(b) *If* $(gx, gy) \in E(G)$, *then* $gx = gy$.

Proof. (a) Let $x \in C(f, g)$. Since $(gx, gx) \in \Delta \subseteq E(G)$, we have that $D(gx, gx) < \infty$ and

$$\begin{aligned} D(gx, gx) &= D(fx, fx) \\ &\leq \lambda \max\{D(gx, fx) + D(gx, fx), D(gx, fx) + D(gx, fx)\} \\ &= 2\lambda D(gx, gx), \end{aligned}$$

where $\lambda \in [0, \frac{1}{2})$. Since $2\lambda < 1$, we have that $D(gx, gx) = 0$.

(b) Let $x, y \in C(f, g)$ and $(gx, gy) \in E(G)$. Then $D(gx, gy) < \infty$ and

$$\begin{aligned} D(gx, gy) &= D(fx, fy) \\ &\leq \lambda \max\{D(gx, fx) + D(gy, fy), D(gx, fy) + D(gy, fx)\}, \end{aligned}$$

where $\lambda \in [0, \frac{1}{2})$.

If $\max\{D(gx, fx)+D(gy, fy), D(gx, fy)+D(gy, fx)\} = D(gx, fx)+D(gy, fy)$, we have that

$$\begin{aligned} D(gx, gy) &= D(fx, fy) \\ &\leq \lambda[D(gx, fx) + D(gy, fy)] \\ &= \lambda[D(gx, gx) + D(gy, gy)] \\ &= 0\,. \end{aligned}$$

This means that $gx = gy$.

If $\max\{D(gx, fx)+D(gy, fy), D(gx, fy)+D(gy, fx)\} = D(gx, fy)+D(gy, fx)$, we have that

$$\begin{aligned} D(gx, gy) &= D(fx, fy) \\ &\leq \lambda[D(gx, fy) + D(gy, fx)] \\ &= \lambda[D(gx, gy) + D(gy, gx)] \\ &= 2\lambda D(gx, gy). \end{aligned}$$

Since $2\lambda < 1$, we obtain that $D(gx, gy) = 0$, and so $gx = gy$. □

For any sequence $\{x_n\} \subseteq X$ and $n \in \mathbb{N} \cup \{0\}$, denote

$$\beta(D, f, x_n) = \sup\{D(fx_{n+i}, fx_{n+j}) : i, j \in \mathbb{N}\}.$$

Note that $\beta(D, f, x_n) \geq \beta(D, f, x_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$.

Lemma 4.2. *Let (X, D) be a JS-metric space endowed with a directed graph G, and let $f, g : X \to X$ be functions such that (f, g) is a KC-contraction. Suppose further that*

(a) $f(X) \subseteq g(X)$;
(b) $E(G)$ *satisfies the transitivity property;*
(c) *there exists* $x_0 \in X$ *such that* $(gx_0, fx_0) \in E(G)$, *and if* $\{x_n\}$ *is a sequence in* (X, D) *satisfying* $gx_n = fx_{n-1}$ *for each* $n \in \mathbb{N}$, *then* $\beta(D, f, x_0) < \infty$.

Then we obtain a D-Cauchy sequence $\{gx_n\}$ for some sequence $\{x_n\}$ in (X, D).

Proof. From assumption (c), let $x_0 \in X$ such that $(gx_0, fx_0) \in E(G)$. Since $f(X) \subseteq g(X)$ and $f(x_0) \in X$, it is easy to construct a sequence $\{x_n\}$ in X for which

$$gx_n = fx_{n-1}$$

for all $n \in \mathbb{N}$. Then it follows that $\beta(D, f, x_0) < \infty$. If $gx_{n_0} = gx_{n_0-1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0-1} is a coincidence point of f and g. Therefore, we only consider the case that $gx_n \neq gx_{n-1}$ is satisfied for each $n \in \mathbb{N}$.

Since $(gx_0, gx_1) = (gx_0, fx_0) \in E(G)$ and f is (g, D)-edge preserving w.r.t G, we obtain that $(gx_1, gx_2) = (fx_0, fx_1) \in E(G)$. Continuing this process inductively, we get that

$$(gx_n, gx_{n+1}) \in E(G) \quad \text{for each} \quad n \in \mathbb{N} \cup \{0\}. \tag{4.4}$$

Moreover, since $E(G)$ satisfies the transitivity property, we have that

$$(gx_k, gx_l) \in E(G) \quad \text{for each} \quad k, l \in \mathbb{N} \text{ such that } k < l. \tag{4.5}$$

Next, let $n \in \mathbb{N}$ with $n \geq 2$. Then, for all $i, j \in \mathbb{N}$ such that $i < j$, consider

$$\begin{aligned}
&D(gx_{n+i+1}, gx_{n+j+1}) \\
&= D(fx_{n+i}, fx_{n+j}) \\
&\leq \lambda \max\{D(gx_{n+i}, fx_{n+i}) + D(gx_{n+j}, fx_{n+j}), D(gx_{n+i}, fx_{n+j}) + D(gx_{n+j}, fx_{n+i})\} \\
&\leq 2\lambda\beta(D, f, x_{n-1})
\end{aligned}$$

which implies that

$$\beta(D, f, x_n) \leq 2\lambda\beta(D, f, x_{n-1}).$$

Consequently, we have that

$$\beta(D, f, x_n) \leq (2\lambda)^n \beta(D, f, x_0)$$

and

$$D(gx_n, gx_m) = D(fx_{n-1}, fx_{m-1}) \leq \beta(D, f, x_{n-2}) \leq (2\lambda)^{n-2}\beta(D, f, x_0)$$

for all integer m such that $m > n \geq 2$.

Since $\beta(D, f, x_0) < \infty$ and $2\lambda < 1$,

$$\lim_{n,m\to\infty} D(gx_n, gx_m) = 0.$$

As a conclusion, it is proved that $\{gx_n\}$ is a D-Cauchy sequence in (X, D). □

We now establish our main theorem for K-contractions.

Theorem 4.1. *Let (X, D) be a D-complete JS-metric space endowed with a directed graph G, and let $f, g : X \to X$ be functions such that (f, g) is a K-contraction. Suppose that*

- *(a) $f(X) \subseteq g(X)$, and $(g(X), D)$ is D-complete;*
- *(b) $E(G)$ satisfies the transitivity property;*
- *(c) there exists $x_0 \in X$ such that $(gx_0, fx_0) \in E(G)$, and if $\{x_n\}$ is a sequence in (X, D) satisfying $gx_n = fx_{n-1}$ for each $n \in \mathbb{N}$, then $\beta(D, f, x_0) < \infty$.*

Then there exists $u \in X$ such that the sequence $\{gx_n\}$ defined in Lemma 4.2 D-converges to $gu \in X$. Moreover, if we assume further that

- *(d) $D(fu, gu) < \infty$;*
- *(e) (X, D, G) has property A;*
- *(f) $C\lambda < 1$ whenever there exist $C > 0$ and $\lambda \in [0, 1/2)$ such that*

$$D(fu, gu) \leq C\lambda \limsup_{n\to\infty}[D(fx_{n-1}, fx_n) + D(gu, fu)],$$

then we can conclude that $C(f, g) \neq \emptyset$.

Proof. By Lemma 4.2, there exists a sequence $\{gx_n\}$ which is D-Cauchy in (X, D) such that $(gx_n, gx_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$. In addition, by assumption (a), $(g(X), D)$ is a complete JS-metric space. Thus, there exists $u \in X$ satisfying

$$\lim_{n\to\infty} D(fx_n, gu) = \lim_{n\to\infty} D(gx_n, gu) = 0.$$

Moreover, by property of D, there exists $C_X > 0$ such that

$$D(fu, gu) \leq C_X \limsup_{n\to\infty} D(fu, fx_n).$$

By the fact that (f, g) is a K-contraction and assumption (e), there is $\lambda \in [0, 1/2)$ such that

$$D(fx_n, fu) \leq \lambda[D(gx_n, fx_n) + D(gu, fu)].$$

Moreover, we obtain that

$$D(fu, gu) \leq C_X \lambda \limsup_{n\to\infty} [D(fx_{n-1}, fx_n) + D(gu, fu)] = C_X \lambda D(gu, fu).$$

By assumption (f), we get that $C_X\lambda < 1$. Since $D(fu, gu) < \infty$, it can be concluded that $D(fu, gu) = 0$. This implies that $C(f, g) \neq \emptyset$.

□

Next, we wish to obtain an existence theorem for coincidence points of C-contractions. We start with the following lemma.

Lemma 4.3. [6] *Suppose that λ is a real number with $0 \leq \lambda < 1$, and $\{b_n\}$ is a sequence of positive real numbers with $\lim_{n\to\infty} b_n = 0$. Then, for any sequence of positive real numbers $\{a_n\}$ such that $a_{n+1} \leq \lambda a_n + b_n$ for all $n \in \mathbb{N}$, we have that $\lim_{n\to\infty} a_n = 0$.*

Theorem 4.2. *Let (X, D) be a D-complete JS-metric space endowed with a directed graph G, and let $f, g : X \to X$ be functions such that (f, g) is a C-contraction. Suppose that*

- *(a) $f(X) \subseteq g(X)$, and $(g(X), D)$ is D-complete;*
- *(b) $E(G)$ satisfies the transitivity property;*
- *(c) there exists $x_0 \in X$ such that $(gx_0, fx_0) \in E(G)$, and if $\{x_n\}$ is a sequence in (X, D) satisfying $gx_n = fx_{n-1}$ for each $n \in \mathbb{N}$, then $\beta(D, f, x_0) < \infty$.*

Then there exists $u \in X$ such that the sequence $\{gx_n\}$ defined in Lemma 4.2 D-converges to $gu \in X$. Moreover, if we assume further that

- *(d) $D(fx_0, fu) < \infty$;*
- *(e) (X, D, G) has property A,*

then we can conclude that $C(f, g) \neq \emptyset$.

Proof. Similar as in the proof of Theorem 4.1, there exists a sequence $\{gx_n\}$ which is D-Cauchy in (X, D) such that $(gx_n, gx_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$. Moreover, there exists $u \in X$ satisfying

$$\lim_{n\to\infty} D(fx_n, gu) = \lim_{n\to\infty} D(gx_n, gu) = 0.$$

Since (X, D, G) has property A, $(gx_n, gu) \in E(G)$. By the fact that (f, g) is a C-contraction, there exists $\lambda \in [0, \frac{1}{2})$ such that

$$D(fx_n, fu) \leq \lambda[D(gx_n, fu) + D(gu, fx_n)] \leq \lambda[D(fx_{n-1}, fu)] + D(gu, fx_n). \tag{4.6}$$

Since f is (g, D)-edge preserving w.r.t G, $D(gu, fx_n) = D(gu, gx_{n+1}) < \infty$. Continuing the process in (4.6), we have that $D(fx_n, fu) < \infty$ by assumption (d). By Lemma 4.3, we obtain that

$$\lim_{n\to\infty} D(fx_n, fu) = 0.$$

In conclusion, we have that $gu = fu$. This implies that $C(f, g) \neq \emptyset$. □

Example 4.1. Let $X = [0, 1]$, and let D be a generalized metric such that

$$D(x,y) = \begin{cases} x+y, & x \neq 0 \text{ and } y \neq 0, \\ \frac{x}{2}, & y = 0, \\ \frac{y}{2}, & x = 0. \end{cases}$$

Then (X, D) is D-complete.

Next, suppose that

$$E(G) = \{(x, y) : x \neq 0 \text{ or } y = 0\}.$$

In addition, define self-mappings f and g on X by

$$f(x) = \frac{x}{x+12} \quad \text{and} \quad g(x) = \frac{x}{4}.$$

We will show that $C(f, g) \neq \emptyset$ by using Theorem 4.2.

First, note that $f(X) \subseteq g(X)$ and $g(X)$ is D-complete. Moreover, we have that $x_0 = 0 \in X$ such that $(g0, f0) = (0, 0) \in E(G)$ and $\beta(D, f, 0) < \infty$.

Moreover, since $x_0 = 0$, we have that $D(fx_0, fu) = \frac{u}{2} < \infty$ for any $u \in X$.

Next, we prove the following claims:

Claim 1: f is (g, D)-edge preserving w.r.t G and $E(G)$ satisfies the transitivity property.

Let $x, y, z \in X$. Assume that $(gx, gy) \in E(G)$. Note that $D(gx, gy) < \infty$. Then $gx \neq 0$ or $gy = 0$. That is, $x \neq 0$ or $y = 0$. Thus, $fx \neq 0$ or $fy = 0$. Therefore, $(fx, fy) \in E(G)$, and we can conclude that f is (g, D)-edge preserving w.r.t G.

Next, assume that $(x, z) \in E(G)$ and $(z, y) \in E(G)$. It can be observed that if $z = 0$, then $y = 0$, and if $z \neq 0$, then $x \neq 0$. That is, $x \neq 0$ or $y = 0$. Therefore, $(x, y) \in E(G)$. Hence, $E(G)$ satisfies the transitivity property.

Claim 2: (f, g) is a C-contraction with $\lambda = \frac{1}{3}$.

Suppose that $x, y \in X$. Assume that $(gx, gy) \in E(G)$. Consider the following cases:
Case 1: $gy = 0$. Then $fy = 0$ and we have that

$$\begin{aligned} D(fx, fy) &= D\left(\frac{x}{x+12}, 0\right) = \frac{1}{2}\left(\frac{x}{x+12}\right) \\ &= \frac{1}{3}\left(\frac{x}{2(x+12)} + \frac{x}{2(x+12)} + \frac{x}{2(x+12)}\right) \\ &\leq \frac{1}{3}\left(\frac{x}{2(8)} + \frac{x}{2(8)} + \frac{x}{2(x+12)}\right) \\ &= \lambda[D(gx, fy) + D(gy, fx)]. \end{aligned}$$

Case 2: $gy \neq 0$. Then $gx \neq 0$ and

$$\begin{aligned} D(fx, fy) &= D\left(\frac{x}{x+12}, \frac{y}{y+12}\right) = \frac{x}{x+12} + \frac{y}{y+12} \\ &\leq \frac{1}{3}\left(\frac{x}{4} + \frac{y}{4}\right) \\ &\leq \frac{1}{3}\left(\frac{x}{4} + \frac{y}{y+12} + \frac{y}{4} + \frac{x}{x+12}\right) \\ &= \lambda[D(gx, fy) + D(gy, fx)]. \end{aligned}$$

Therefore, we have Claim 2.

Finally, we have to prove that (X, D, G) has property A. Let $\{x_n\}$ be a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and $\{x_n\} \in C(D, X, c)$ for some $c \in X$. We will show that $(x_n, c) \in E(G)$. By the definition of $E(G)$,

$$x_n \neq 0 \text{ or } x_{n+1} = 0 \text{ for each } n \in N. \tag{4.7}$$

If $x_n \neq 0$ for each $n \in \mathbb{N}$, then we have $(x_n, c) \in E(G)$ for each $n \in \mathbb{N}$. On the other hand, assume that there exists $n_0 \in N$ such that $x_{n_0} = 0$, then, by (4.7), $x_k = 0$ whenever $k \geq n_0$. Now, we will show that $c = 0$. Suppose on the contrary that $c \neq 0$. Observe that

$$D(x_k, c) = D(0, c) = \frac{c}{2} \neq 0 \text{ for all } k \geq n_0$$

which contradicts the fact that $\{x_n\} \in C(D, X, c)$. Hence, $c = 0$ and we receive that $(x_n, c) \in E(G)$. By Theorem 4.2, there exists a coincidence point of f and g.

Finally, we offer a theorem on the existence of a coincidence point and a common fixed point of KC-contractions as follows.

Theorem 4.3. *Let (X, D) be a D-complete JS-metric space endowed with a directed graph G, and let $f, g : X \to X$ be functions such that (f, g) is a KC-contraction. Suppose that*

(a) $f(X) \subseteq g(X)$;
(b) $E(G)$ *satisfies the transitivity property;*
(c) there exists $x_0 \in X$ such that $(gx_0, fx_0) \in E(G)$, and if $\{x_n\}$ is a sequence in (X, D) satisfying $gx_n = fx_{n-1}$ for each $n \in \mathbb{N}$, then $\beta(D, f, x_0) < \infty$.

Then there exists $u \in X$ such that the sequence $\{gx_n\}$ defined in Lemma 4.2 D-converges to $u \in X$. Moreover, if we assume further that

(d) f is G-continuous and g is continuous;
(e) f and g commute,

then we have that $C(f, g) \neq \emptyset$. Moreover, if $(gx, gy) \in E(G)$ for any $x, y \in C(f, g)$, then $Cm(f, g) \neq \emptyset$.

Proof. By Lemma 4.2 and the fact that (X, D) is a D-complete JS-metric space, there exists $u \in X$ such that

$$\lim_{n\to\infty} D(fx_n, u) = \lim_{n\to\infty} D(gx_n, u) = 0.$$

Thus,

$$\{gx_n\}, \{fx_n\} \in C(D, X, u).$$

By the G-continuity of f and the continuity of g on (X, D), we get that

$$\{fgx_n\} \in C(D, X, fu) \quad \text{and} \quad \{gfx_n\} \in C(D, X, gu).$$

Since f and g commute, we have that $\{gfx_n\} \in C(D, X, fu)$. Moreover, from Proposition 2.1, we have that $fu = gu$. Hence, u is a coincidence point of f and g. This means that $u \in C(f, g)$.

Next, we will show the last statement. Let $c = gu = fu$. Since f and g commute, $gc = gfu = fgu = fc$. Thus, $c \in C(f, g)$. By the assumption, we have that $(gu, gc) \in E(G)$. By Lemma 4.1, we conclude that $fc = gc = gu = c$. Hence, $c \in Cm(f, g)$, and the proof is complete. □

5. Application

In our last section, we establish the existence, under some specific conditions, of a solution to the integral equation of the form

$$x(t) = \int_0^T p(t, s, x(s))ds + b(t) \tag{5.8}$$

for $t \in [0, T]$. Here, T is a positive real number, and the details of functions in the equation are given in the theorem below.

There are many results regarding solving integral equations by using fixed point theorems. For example, see [5, 8, 11, 13].

Now, let us assume that $X = C([0,T],\mathbb{R})$ and define

$$D(x,y) = \max_{t\in[0,T]} |x(t)| + \max_{t\in[0,T]} |y(t)|$$

for any $x, y \in C([0,T],\mathbb{R})$. It can be verified that (X, D) is a D-complete JS-metric space. Moreover, we have the following theorem.

Theorem 5.4. *According to* (5.8), *suppose that*

(a) $p : [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function;*

(b) for any $x, y \in \mathbb{R}$, $x \le y$ *implies* $p(t,s,x) \le p(t,s,y)$ *and*

$$|p(t,s,x)| + |p(t,s,y)| \le \frac{1}{KT}\left(|x| + |y|\right),$$

where $s, t \in [0,T]$ *and* K *is a constant such that* $K > 2$*;*

(c) there is $x_0 \in X$ *such that* $x_0(t) \ge \int_0^T p(t,s,x_0(s))ds$*, and if* $\{x_n\}$ *is a sequence in* (X,D) *such that* $x_n(t) = \int_0^T p(t,s,x_{n-1}(s))ds$ *for each* $n \in \mathbb{N}$ *and* $t \in [0,T]$*, then* $\beta(D, \int_0^T p(t,s,x(s))ds, x_0) < \infty$.

Then there is a solution to the integral equation (5.8) *if it is homogeneous.*

Proof. Let us define functions f and g on X so that

$$fx(t) = \int_0^T p(t,s,x(s))ds,$$

and $gx(t) = x(t)$ for any $x \in X$ and $t \in [0,T]$.

Next, let $E(G)$ be defined by

$$E(G) = \{(x,y) : x(t) \ge y(t) \text{ for any } t \in [0,T]\}.$$

It is obvious that $f(X) \subseteq g(X)$, and f and g are continuous functions. In addition, assumption (c) in this theorem induces assumption (c) of Theorem 4.2.

Next, we will show that (f,g) is a C-contraction for some $\lambda \in [0,1/2)$.

To begin, we prove that f is (g,D)-edge preserving w.r.t G. Assume that $(gx,gy) \in E(G)$ and $t \in [0,T]$. Then $gx(t) \ge gy(t)$. In other words, $x(t) \ge y(t)$. Thus, by assumption (b), we have that $p(t,s,x) \ge p(t,s,y)$. This leads to

$$\begin{aligned} fx(t) &= \int_0^T p(t,s,x(s))ds \ge \int_0^T p(t,s,y(s))ds \\ &= fy(t). \end{aligned}$$

Consequently, $(fx,fy) \in E(G)$. Note that $D(gx,gy) = D(x,y) < \infty$ since $x, y \in C([0,T],\mathbb{R})$. Therefore, f is (g,D)-edge preserving w.r.t G. Furthermore, it can be seen that $E(G)$ satisfies the transitivity property.

Finally, we show the following.

Given that $x(t) \ge y(t)$ for all $t \in [0,T]$, we have, by assumption (b), that for any $t \in [0,T]$,

$$\begin{aligned} |fx(t)| + |fy(t)| &\le \int_0^T (|p(t,s,x(s))| + |p(t,s,y(s))|)ds \le \frac{1}{KT}\int_0^T (|x(s)| + |y(s)|)ds \\ &\le \frac{1}{K}\left(\max_{t\in[0,T]} |gx(t)| + \max_{t\in[0,T]} |gy(t)|\right) \\ &\le \frac{1}{K}\left(\max_{t\in[0,T]} |gx(t)| + \max_{t\in[0,T]} |fy(t)| + \max_{t\in[0,T]} |gy(t)| + \max_{t\in[0,T]} |fx(t)|\right). \end{aligned}$$

This implies that (f,g) is a C-contraction for $\lambda = \frac{1}{K}$.

From Theorem 4.2, there exists $u \in X$ such that the sequence $\{gx_n\}$ defined in Lemma 4.2 D-converges to $gu \in X$. Notice that $D(fx_0, fu) < \infty$. Moreover, assumption (e) of Theorem 4.2 is clearly satisfied.

Thus, we can conclude that there is a coincidence point of f and g. Hence, this point is a solution to the integral equation (5.8) if it is homogeneous. □

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REFERENCES

[1] Ansari, Q. H., *Extension of generalized variational-like inequalities*, Southeast Asian Bull. Math., **23** (1999), No. 1, 1–8

[2] Ansari, Q. H., Idzik, A. and Yao, J., *Coincidence and fixed point theorems with applications* (Dedicated to Juliusz Schauder, 1899-1943), Topol. Methods Nonlinear Anal., **15** (2000), No. 1, 191–202

[3] Ansari, Q. H., Rezaie, M. and Zafarani, J., *Generalized vector variational-like inequalities and vector optimization*, J. Global Optim., **53** (2012), No. 2, 271–284

[4] Ansari, Q. H. and Yao, J., *An existence result for the generalized vector equilibrium problem*, Appl. Math. Lett., **12** (1999), No. 8, 53–56

[5] Ahmad, J., Hussain, N., Azam, A. and Arshad, M., *Common fixed point results in complex valued metric space with applications to system of integral equations*, J. Nonlinear Convex Anal., **16** (2015), No. 5, 855–871

[6] Berinde, V., *Iterative approximation of fixed points*, Lecture Notes in Mathematics 2nd Edition, Springer, Berlin (2007)

[7] ElKouch, Y. and Marhrani, E. M., *On some fixed point theorems in generalized metric spaces*, Fixed Point Theory Appl., **23** (2017), 17 pp.

[8] Hussain, N., Azam A., Ahmad, J. and Arshad, M., *Common fixed point results in complex valued metric spaces with application to integral equations*, Filomat, **28** (2014), No. 7, 1363–1380

[9] Jleli, M. and Samet, B., *A generalized metric space and related fixed point theorems*, Fixed Point Theory Appl., **61** (2015), 14 pp.

[10] Jachymski, J., *The contraction principle for mappings on a metric space with a graph*, Proc. Am. Math. Soc., **136** (2008), 1359–1373

[11] Roshan, J. R., Parvaneh, V. and Altun, I., *Some coincidence point results in ordered b-metric spaces and applications in a system of integral equations*, Appl. Math. Comput., **226** (2014), 725–737

[12] Saluja, G. S. and Tripathi, B. P., *Some common fixed point theorems of generalized contractive mappings in cone metric spaces*, Funct. Anal. Approx. Comput., **5** (2013), No. 1, 33–41

[13] Wu, X. and Zhao, L., *Fixed point theorems for generalized α-ψ type contractive mappings in b-metric spaces and applications*, J. Math. Computer Sci., **18** (2018), 49–62

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