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Dedicated to Prof. Qamrul Hasan Ansari on the occasion of his 60^{th} anniversary

The Opial condition in variable exponent sequence spaces $\ell_{p(\cdot)}$ with applications

MOSTAFA BACHAR[1], MOHAMED A. KHAMSI[2,3] and MESSAOUD BOUNKHEL[1]

ABSTRACT. In this work, we show an analogue to the Opial property for the coordinate-wise convergence in the variable exponent sequence space $\ell_{p(\cdot)}$. This property allows us to prove a fixed point theorem for the mappings which are nonexpansive in the modular sense.

1. INTRODUCTION

Orlicz [16] is credited as the first one to introduce the concept of a modular in a vector space in 1931. Of course, the main dominant concept at that time was the norm defined in a vector space. In his 1931 paper, Orlicz considered the space

$$X = \Big\{(x_n) \in \mathbb{R}^{\mathbb{N}};\ \sum_{n=0}^{\infty} |\alpha\, x_n|^n < \infty \ \text{ for some } \alpha > 0\Big\},$$

which was later extended to

$$X = \Big\{(x_n) \in \mathbb{R}^{\mathbb{N}};\ \sum_{n=0}^{\infty} |\alpha\, x_n|^{p(n)} < \infty \ \text{ for some } \alpha > 0\Big\},$$

where $p(n) \geq 1$, for any $n \in \mathbb{N}$. We use the notation $\ell_{p(\cdot)}$ for the vector space X. The geometry and the topological properties of X are well understood and well investigated, see for example [7, 14, 19, 20]. Inspired by the work of Orlicz, Nakano [12, 13] introduced the concept of a modular and modular vector spaces. Moreover, The space $\ell_{p(\cdot)}$ is considered as the precursor of variable exponent spaces [3] (in short VES). In recent years, these spaces are in vogue and saw a major development. Kováčik and Rákosník [8] are among the first to investigate the vector topological properties of VES. It is worth mentioning that the rapid development of the theory of VES is closely related to electrorheological fluids introduced by Rajagopal and Ružička [17, 18]. These fluids are an example of smart materials with major applications in aerospace, mechanical and civil structures.

In this work, we establish a property analogue to the Opial condition for the coordinate-wise convergence in the space $\ell_{p(\cdot)}$. This investigation allowed us to prove a fixed point theorem for nonexpansive mappings in the modular sense.

Since our work deals with metric fixed point theory and modular vector spaces, we recommend the books [4, 6] as a reference to interested readers.

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Corresponding author: Mohamed A. Khamsi; mohamed@utep.edu

2. Notations and definitions

Let us start first by the following definition.

Definition 2.1. [16] The space $\ell_{p(\cdot)}$ is defined by:

$$\ell_{p(\cdot)} = \Big\{(x_n) \in \mathbb{R}^{\mathbb{N}};\ \sum_{n=0}^{\infty} \frac{1}{p(n)}\ |\alpha\, x_n|^{p(n)} < \infty\ \text{ for some } \alpha > 0\Big\},$$

where $p(n) \geq 1$, for any $n \in \mathbb{N}$.

The formal definition of a modular was given by Nakano [12, 13, 14]. The following proposition captures the general approach introduced by Nakano.

Proposition 2.1. [7, 12, 19] *Define the function $\rho : \ell_{p(\cdot)} \to [0, \infty]$ by*

$$\varrho(x) = \varrho((x_n)) = \sum_{n=0}^{\infty} \frac{1}{p(n)}\ |x_n|^{p(n)}.$$

Then ϱ satisfies the following properties:

(1) $\varrho(x) = 0$ if and only if $x = 0$,
(2) $\varrho(-x) = \varrho(x)$,
(3) $\varrho(\alpha x + \beta y) \leq \alpha\varrho(x) + \beta\varrho(y)$, for any $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$,

for any $x, y \in X$. The function ϱ is called a convex modular.

In the following definition, we introduce a kind of modular topology which mimics the classical metric topology.

Definition 2.2. [5]

(a) The sequence $\{x_n\} \subset \ell_{p(\cdot)}$ is said to be ϱ-convergent to $x \in \ell_{p(\cdot)}$ if and only if $\varrho(x_n - x) \to 0$. Note that the ϱ-limit is unique if it exists.
(b) A sequence $\{x_n\} \subset \ell_{p(\cdot)}$ is called ϱ-Cauchy if $\varrho(x_n - x_m) \to 0$ as $n, m \to \infty$.
(c) A nonempty subset $C \subset \ell_{p(\cdot)}$ is called ϱ-bounded if

$$\delta_\varrho(C) = \sup\{\varrho(x - y); x, y \in C\} < \infty.$$

(d) ϱ is said to satisfy the Δ_2-condition if there exists $K \geq 0$ such that $\varrho(2x) \leq K\ \varrho(x)$, for any $x \in \ell_{p(\cdot)}$.

In $\ell_{p(\cdot)}$, it is easy to see that ϱ satisfies the Δ_2-condition if and only if $p_+ = \sup\limits_{n\in\mathbb{N}}\ p(n) < \infty$. For the importance of the Δ_2-condition and its variants, the reader may consult [6, 9, 11]. To a modular ϱ, we associate of what is known as the Luxemburg norm defined by

$$\|x\|_\varrho = \inf\left\{\lambda > 0; \varrho\left(\frac{1}{\lambda}x\right) \leq 1\right\}.$$

Recall that $(\ell_{p(\cdot)}, \|.\|_\varrho)$ is a Banach space. Moreover the ϱ-convergence and the norm convergence are equivalent if and only if ϱ satisfies the Δ_2-condition. In this case, if a sequence is ϱ-Cauchy, then it is ϱ-convergent.

For the geometric and topological properties of $(\ell_{p(\cdot)}, \|.\|_\varrho)$, we recommend the work of Sundaresan [19].

3. OPIAL PROPERTY IN $\ell_{p(\cdot)}$

While investigating an extension of a fixed point theorem obtained by Browder and Petryshyn [1] in a Hilbert space, Opial came up with a property that bears his name [15]. This property has had a tremendous impact on the fixed point property of nonexpansive mappings.

Let $\{x_n\}$ be a sequence in $\ell_{p(\cdot)}$. Set $x_n = (x^n_m)_{m\in\mathbb{N}}$. We will say that $\{x_n\}$ coordinate-wise converges to $x = (x_m)_{m\in\mathbb{N}} \in \ell_{p(\cdot)}$ if and only $\lim\limits_{n\to\infty} x^n_m = x_m$, for any $m \in \mathbb{N}$. Throughout this work, we will write τ-convergence for the coordinate-wise convergence.

Definition 3.3. [2, 15] We will say that $\ell_{p(\cdot)}$ satisfies the τ-Opial property if for any ϱ-uniformly bounded sequence $\{x_n\}$ in $\ell_{p(\cdot)}$ which τ-converges to $x \in \ell_{p(\cdot)}$, we have

$$\limsup_{n\to\infty} \ \varrho(x_n - x) < \limsup_{n\to\infty} \ \varrho(x_n - y),$$

for any $y \in \ell_{p(\cdot)}$ such that $y \neq x$.

Throughout, we will need the following notations:

$$\varrho_m(x) = \sum_{n\leq m} \frac{1}{p(n)} \, |x_n|^{p(n)}, \quad \textit{and} \quad \varrho^c_m(x) = \sum_{n>m} \frac{1}{p(n)} \, |x_n|^{p(n)},$$

for any $m \in \mathbb{N}$ and any $x = (x_n) \in \ell_{p(\cdot)}$.

The following result is crucial in the proof of the main result of this work.

Lemma 3.1. *Assume $p_+ = \sup\limits_{n\in\mathbb{N}} \ p(n) < \infty$. Let $K > 1$ and $\varepsilon > 0$ be such that $\varepsilon\, K < 1$. Then, for any $x, y \in \ell_{p(\cdot)}$, we have*

$$|\varrho(x+y) - \varrho(x)| \leq \varepsilon|\varrho(K\,x) - K\ \varrho(x)| + \varrho(C_\varepsilon\, y),$$

where $C_\varepsilon = \dfrac{1}{\varepsilon(K-1)}$.

Proof. Set $\alpha = 1 - K\,\varepsilon$ and $\beta = (K-1)\,\varepsilon$. Then we have $\alpha + \beta + \varepsilon = 1$ and

$$x + y = \alpha\, x + \varepsilon\, K\, x + \beta\, C_\varepsilon\, y.$$

Since ϱ is convex, we conclude that

$$\varrho(x+y) \leq \alpha\ \varrho(x) + \varepsilon\ \varrho(K\,x) + \beta\ \varrho(C_\varepsilon\, y),$$

which implies

$$\begin{aligned}\varrho(x+y) - \varrho(x) \quad &\leq \quad \varepsilon\,\Big(\varrho(K\,x) - K\ \varrho(x)\Big) + (K-1)\,\varepsilon\ \varrho(C_\varepsilon\, y)\\ &\leq \quad \varepsilon\,\Big(\varrho(K\,x) - K\ \varrho(x)\Big) + \varrho(C_\varepsilon\, y).\end{aligned}$$

If we set

$$a = \frac{1}{1+K\,\varepsilon},\ b = \frac{\varepsilon}{1+K\,\varepsilon},\ \text{and } c = \frac{\varepsilon\,(K-1)}{1+K\,\varepsilon},$$

then $a + b + c = 1$. It is easy to check that

$$x = a\,(x+y) + b\,K\,x + c\,(-C_\varepsilon\, y),$$

which implies by convexity of ϱ

$$\varrho(x) \leq a\ \varrho(x+y) + b\ \varrho(K\,x) + c\ \varrho(-C_\varepsilon\, y).$$

Hence

$$(1+K\,\varepsilon)\varrho(x) \leq \varrho(x+y) + \varepsilon\ \varrho(K\,x) + \varepsilon\,(K-1)\ \varrho(-C_\varepsilon\, y),$$

which implies

$$\begin{aligned}\varrho(x)-\varrho(x+y) &\leq \varepsilon\Big(\varrho(K\,x)-K\,\varrho(x)\Big)+\varepsilon\,(K-1)\,\varrho(C_\varepsilon\,y)\\ &\leq \varepsilon\Big(\varrho(K\,x)-K\,\varrho(x)\Big)+\varrho(C_\varepsilon\,y).\end{aligned}$$

Hence

$$|\varrho(x)-\varrho(x+y)|\leq \varepsilon\Big(\varrho(K\,x)-K\,\varrho(x)\Big)+\varrho(C_\varepsilon\,y).$$

Since $\varrho(K\,x)-K\,\varrho(x)\geq 0$, the conclusion of Lemma 3.1 holds. □

Now we are ready to state the main result of this work.

Theorem 3.1. *Assume $p_+=\sup_{n\in\mathbb{N}} p(n)<\infty$. Let $\{x_n\}$ be ϱ-uniformly bounded in $\ell_{p(\cdot)}$ which τ-converges to 0. We have*

$$\lim_{n\to\infty}\varrho(x_n+x)-\varrho(x_n)=\varrho(x),$$

for any $x\in\ell_{p(\cdot)}$.

Proof. Fix $m\in\mathbb{N}$ and $x\in\ell_{p(\cdot)}$. We have

$$\begin{aligned}\varrho(x_n+x)-\varrho(x_n)-\varrho(x) &= \varrho_m(x_n+x)-\varrho_m(x_n)-\varrho_m(x)\\ &\quad+\varrho_m^c(x_n+x)-\varrho_m^c(x_n)-\varrho_m^c(x),\end{aligned}$$

for any $n\in\mathbb{N}$. Since $\{x_n\}$ τ-converges to 0, we have

$$\lim_{n\to\infty}\varrho_m(x_n+x)-\varrho_m(x_n)-\varrho_m(x)=0.$$

Therefore, we focus on the term $\varrho_m^c(x_n+x)-\varrho_m^c(x_n)-\varrho_m^c(x)$. Since $\{x_n\}$ is ϱ-uniformly bounded, then $M=\sup_{n\in\mathbb{N}}\varrho(x_n)<\infty$. Fix $K>1$. Then $\sup_{n\in\mathbb{N}}\varrho(K\,x_n)\leq K^{p_+}M<\infty$. Let $\varepsilon>0$ such that $\varepsilon\,K<1$. The Lemma 3.1 implies

$$\begin{aligned}|\varrho_m^c(x_n+x)-\varrho_m^c(x_n)-\varrho_m^c(x)| &\leq \varepsilon\,|\varrho_m^c(K\,x_n)-K\,\varrho_m^c(x_n)|+\varrho_m^c(C_\varepsilon\,x)+\varrho_m^c(x)\\ &\leq \varepsilon\,(K^{p_+}+K)M+(C_\varepsilon^{p_+}+1)\,\varrho_m^c(x),\end{aligned}$$

for any $n\in\mathbb{N}$, which implies

$$\limsup_{n\to\infty}|\varrho_m^c(x_n+x)-\varrho_m^c(x_n)-\varrho_m^c(x)|\leq\varepsilon\,(K^{p_+}+K)M+(C_\varepsilon^{p_+}+1)\,\varrho_m^c(x).$$

Since $\lim_{n\to\infty}\varrho_m(x_n+x)-\varrho_m(x_n)-\varrho_m(x)=0$, we conclude

$$\limsup_{n\to\infty}|\varrho(x_n+x)-\varrho(x_n)-\varrho(x)|\leq\varepsilon\,(K^{p_+}+K)M+(C_\varepsilon^{p_+}+1)\,\varrho_m^c(x).$$

Using the Δ_2-condition satisfied by $\ell_{p(\cdot)}$, we have $\lim_{m\to\infty}\varrho_m^c(x)=0$, which implies

$$\limsup_{n\to\infty}|\varrho(x_n+x)-\varrho(x_n)-\varrho(x)|\leq\varepsilon\,(K^{p_+}+K)M.$$

Since ε may be chosen arbitrarily close to 0, we conclude that

$$\lim_{n\to\infty}|\varrho(x_n+x)-\varrho(x_n)-\varrho(x)|=0,$$

which implies $\lim_{n\to\infty}\varrho(x_n+x)-\varrho(x_n)=\varrho(x)$. □

Example 3.1. In this example, we show that if $p_+ = \infty$, then the conclusion of Theorem 3.1 may fail. Indeed, consider the case when $p(n) = n$. Set $x = \left(\frac{1}{m^{1-1/m}}\right)_{m\in\mathbb{N}}$. Then $x \in \ell_{p(\cdot)}$. Moreover, consider the sequence $\{x_n\}$ defined by $x_n = \left(\delta_{nm}\; m^{1/m}\right)_{m\in\mathbb{N}}$ where δ_{nm} is the Kronecker function, i.e., $\delta_{nm} = 0$ if $n \neq m$ and $\delta_{nn} = 1$. Then, $\{x_n\}$ τ-converges to 0. Moreover, we have

$$\begin{aligned} \varrho(x_n + x) - \varrho(x_n) &= \sum_{m\neq n} \frac{1}{m^{m-1}} + \left(\frac{1}{n^{1-1/n}} + n^{1/n}\right)^n - n \\ &= \sum_{m\neq n} \frac{1}{m^{m-1}} + n\left(\frac{1}{n} + 1\right)^n - n \\ &= \varrho(x) + n\left[\left(1 + \frac{1}{n}\right)^n - 1\right] - \frac{1}{n^{n-1}}, \end{aligned}$$

which implies

$$\lim_{n\to\infty}\; \varrho(x_n + x) - \varrho(x_n) \neq \varrho(x),$$

since $\lim\limits_{n\to+\infty} n\left[\left(1+\frac{1}{n}\right)^n - 1\right] - \frac{1}{n^{n-1}} = +\infty$.

The following result is a direct consequence of Theorem 3.1.

Theorem 3.2. *Assume $p_+ = \sup\limits_{n\in\mathbb{N}}\; p(n) < \infty$. Let $\{x_n\}$ be ϱ-uniformly bounded in $\ell_{p(\cdot)}$ which τ-converges to $x \in \ell_{p(\cdot)}$. We have*

$$\limsup_{n\to\infty}\; \varrho(x_n - y) = \limsup_{n\to\infty}\; \varrho(x_n - x) + \varrho(x - y),$$

for any $y \in \ell_{p(\cdot)}$, which implies

$$\limsup_{n\to\infty}\; \varrho(x_n - x) < \limsup_{n\to\infty}\; \varrho(x_n - y)$$

whenever $x \neq y$.

The conclusion of Theorem 3.2 is similar to the property discovered by Opial [15] when the exponent function $p(\cdot)$ is constant and the convergence is for the weak-topology. Almost all the proofs given for this fact are based on the use of the duality-function and its properties. The duality-function is closely related to the norm. Since the norm in $\ell_{p(\cdot)}$ is not directly related to the modular function, the previous approaches are not suitable. In the next section, we give an application of the conclusion of Theorem 3.2.

4. Application

In this section, we prove a fixed point result similar to the one discovered by Opial [15] and others [2, 10].

Definition 4.4. [6] Let $K \subset \ell_{p(\cdot)}$ be a nonempty subset. Let $T : K \to K$ be a map.

(1) T is said to be ϱ-contraction if there exists a constant $k \in [0, 1)$ such that

$$\varrho(T(x) - T(y)) \leq k\; \varrho(x - y), \quad \text{for any } x, y \in K.$$

(2) T is said to be ϱ-nonexpansive whenever

$$\varrho(T(x) - T(y)) \leq \varrho(x - y), \quad \text{for any } x, y \in K.$$

A fixed point of T is any point $x \in K$ such that $T(x) = x$.

Before we state the main result of this section, recall that a nonempty subset K of $\ell_{p(\cdot)}$ is said to be compact for the τ-convergence if any sequence $\{x_n\}$ in K has a subsequence $\{x_{\varphi(n)}\}$ which τ-converges to a point in K.

Theorem 4.3. *Assume $p_+ = \sup_{n\in\mathbb{N}} p(n) < \infty$. Let K be a nonempty ϱ-bounded convex subset of $\ell_{p(\cdot)}$. Assume K is compact for the τ-convergence. Then any ϱ-nonexpansive mapping $T : K \to K$ has a fixed point.*

Proof. Let K be a nonempty ϱ-bounded convex subset of $\ell_{p(\cdot)}$ which is compact for the τ-convergence. Let $T : K \to K$ be a ϱ-nonexpansive mapping. If K is reduced to one point, then we have nothing to prove. We assume, thus that K is not reduced to one point. Fix $\varepsilon \in (0, 1)$ and $c \in K$. Define $T_\varepsilon : K \to K$ by

$$T_\varepsilon(x) = \varepsilon\, c + (1-\varepsilon)\, T(x).$$

Using the convexity of ϱ, we obtain

$$\varrho(T_\varepsilon(x) - T_\varepsilon(y)) = \varrho\Big((1-\varepsilon)\ (T(x) - T(y))\Big) \le (1-\varepsilon)\ \varrho((T(x) - T(y)),$$

for any $x, y \in K$. In other words, T_ε is ϱ-contraction which implies

$$\varrho(T_\varepsilon^n(c) - T_\varepsilon^{n+h}(c)) \le (1-\varepsilon)^n\ \varrho(c - T_\varepsilon^h(c)) \le (1-\varepsilon)^n\ \delta_\varrho(K),$$

for any $n, h \in \mathbb{N}$. Hence $\{T_\varepsilon^n(c)\}$ is ϱ-Cauchy. Using the properties of $\ell_{p(\cdot)}$, we conclude that $\{T_\varepsilon^n(c)\}$ ϱ-converges to some x_ε. Since ϱ-convergence implies the τ-convergence for which K is compact, we conclude that $x_\varepsilon \in K$. We claim that x_ε is a fixed point of T_ε. Indeed, using the inequality

$$\varrho(T_\varepsilon^{n+1}(c) - T_\varepsilon(x_\varepsilon)) \le (1-\varepsilon)\ \varrho(T_\varepsilon^n(c) - x_\varepsilon),$$

for any $n \in \mathbb{N}$, we conclude that $\{T_\varepsilon^n(c)\}$ also ϱ-converges to $T_\varepsilon(x_\varepsilon)$. The uniqueness of the ϱ-limit implies $T_\varepsilon(x_\varepsilon) = x_\varepsilon$. Let $x_n \in K$ be the discovered fixed point of $T_{1/n}$, for $n \ge 1$. Hence

$$T(x_n) = \frac{1}{n}\ c + \left(1 - \frac{1}{n}\right)\ T(x_n) = x_n,$$

which implies

$$\varrho(x_n - T(x_n)) = \varrho\left(\frac{1}{n}\ \Big(c - T(x_n)\Big)\right) \le \frac{1}{n}\ \varrho(c - T(x_n)) \le \frac{\delta_\varrho(K)}{n},$$

for any $n \ge 1$. Since K is compact for the τ-convergence, there exists a subsequence $\{x_{\varphi(n)}\}$ of $\{x_n\}$ which τ-converges to some $x \in K$. We claim that x is a fixed point of T. Indeed, we have

$$\begin{aligned}\varrho(x_{\varphi(n)} - T(x)) &= \varrho\left(\frac{1}{\varphi(n)}\ \Big(c - T(x)\Big) + \left(1 - \frac{1}{\varphi(n)}\right)\ (T(x_{\varphi(n)}) - T(x))\right)\\ &\le \frac{1}{\varphi(n)}\ \varrho(c - T(x)) + \left(1 - \frac{1}{\varphi(n)}\right)\ \varrho(x_{\varphi(n)} - x),\end{aligned}$$

for any $n \ge 1$, which implies

$$\limsup_{n\to\infty}\ \varrho(x_{\varphi(n)} - T(x)) \le \limsup_{n\to\infty}\ \varrho(x_{\varphi(n)} - x).$$

Using Theorem 3.2, we conclude that $T(x) = x$ as claimed which finishes the proof of Theorem 4.3. □

Note that the conclusion of Theorem 4.3 is still valid if we only assume that K is star-shaped. Recall that K is said to be star-shaped if and only if there exists a point $c \in K$ such that $\alpha\, c + \beta\, x \in K$, for any $x \in K$ and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta = 1$.

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[1]DEPARTMENT OF MATHEMATICS
KING SAUD UNIVERSITY
SAUDI ARABIA
Email address: mbachar@ksu.edu.sa
Email address: bounkhel@ksu.edu.sa

[2]DEPARTMENT OF MATHEMATICAL SCIENCES
THE UNIVERSITY OF TEXAS AT EL PASO
EL PASO, TX 79968, U.S.A.
Email address: mohamed@utep.edu

[3]DEPARTMENT OF MATHEMATICS AND STATISTICS
KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN 31261, SAUDI ARABIA