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Dedicated to Prof. Qamrul Hasan Ansari on the occasion of his 60^{th} anniversary

A nonsmooth Stackelberg equilibrium problem via mixed variational inequalities

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ABSTRACT. In this paper, we introduce and study a class of nonsmooth Stackelberg equilibrium problems with differential revenue functions and nonsmooth cost functions. First, we describe the best response set of the follower and the best strategy set of the leader for the considered nonsmooth Stackelberg equilibrium problem via mixed variational inequalities. Then, by using the resolvent operator method and the Banach fixed point theorem, we show the existence and uniqueness of its solution.

1. INTRODUCTION

It is well known that Stackelberg equilibrium problems play an important role in the study of economics, finance, risk management, design of mechanical structures, migration problems, transportation, internet advertising, resources allocation, minimax mathematical programming and decision science (see, for example, [2, 7, 8, 16, 10, 12, 18, 19, 20] and the references therein). Thus, it has caught much attention of a large number of researchers in mathematics, economics and other disciplines. At earlier years, Murphy et al. [14] studied an oligopolistic market equilibrium problem with differential cost functions. After that, Agiza and Elsadany [1] considered a nonlinear duopoly game with heterogeneous players and linear cost functions while Tomasz [5] discussed a heterogeneous duopoly game with differential cost functions. Recently, by employing variational inequality methods and fixed point arguments, Nagy [15] studied the existence and location of solutions for a class of Stackelberg equilibrium problems with two players. Based on this, in 2018, Lu et al. [13] introduced and studied a class of Stackelberg quasi-equilibrium problems with two players in finite dimensional spaces. They showed the existence and location of solutions for the Stackelberg quasi-equilibrium problem by employing the quasi-variational inequality techniques and the fixed point arguments.

It should be noticed that the cost functions of the problems considered in most above mentioned references are assumed to be smooth. However, for the consideration of reality in practice, the cost functions of some problems may not be smooth in general. Therefore, based on this fact, it is necessary and important to study Stackelberg equilibrium problems involving nonsmooth cost functions, which are called nonsmooth Stackelberg equilibrium problems as a generalization of the classic Stackelberg equilibrium problems. In the present paper, we consider the following nonsmooth Stackelberg equilibrium problem

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(NSEP):

$$\begin{aligned}&\min_{x\in\mathbb{R}^n} f(x,y)+\phi(x)\\ &\text{s.t.}\quad y\in\arg\min_{y\in\mathbb{R}^m}\{g(x,y)+\psi(y)\},\end{aligned}$$

where $f(x,y)$ and $g(x,y)$: $\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ are two differentiable functions, while $\phi(x)$: $\mathbb{R}^n\to\mathbb{R}$ and $\psi(y)$: $\mathbb{R}^m\to\mathbb{R}$ are two convex and lower semi-continuous functions which are not necessary differentiable in general.

Following the paper [15] by Nagy, for any given $x\in\mathbb{R}^n$, we define the best response set of the follower in (NSEP) as follows

$$R_{NSE}(x)=\{y^*\in\mathbb{R}^m:[g(x,y)+\psi(y)]-[g(x,y^*)+\psi(y^*)]\ge 0,\ \forall y\in\mathbb{R}^m\}.$$

Assume that $R_{NSE}(x)\neq\emptyset$. Let $r(x)$ be a selector of $R_{NSE}(x)$, that is, $r(x)\in R_{NSE}(x)$ for all $x\in\mathbb{R}^n$. Then, the best strategy set of the leader in (NSEP) can be defined by

$$S_{NSE}=\{x^*\in\mathbb{R}^n:[f(x,r(x))+\phi(x)]-[f(x^*,r(x^*))+\phi(x^*)]\ge 0,\ \forall x\in\mathbb{R}^n\}.$$

In order to study the location of the points in the best response set of the follower, we define a slightly larger set than the best response set of the follower in (NSEP) by means of mixed variational inequality [11]. More precisely, with the C^1 class functions $f(x,y)$ and $g(x,y)$, and the convex functions $\phi(x)$ and $\psi(y)$, we define the Stackelberg mixed variational response set of the follower as follows

$$R_{SMV}(x)=\left\{y^*\in\mathbb{R}^m:\left\langle \frac{\partial g(x,y)}{\partial y}\bigg|_{y=y^*},y-y^*\right\rangle+\psi(y)-\psi(y^*)\ge 0,\forall y\in\mathbb{R}^m\right\},x\in\mathbb{R}^n,$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbb{R}^n$. Assume that $R_{SMV}(x)\neq\emptyset$ and $r(x)$ is a selector of $R_{SMV}(x)$. If $r(x)$ is a C^1 class function, then the Stackelberg mixed variational leader set can be defined by

$$S_{SMV}=\left\{x^*\in\mathbb{R}^n:\left\langle \frac{\partial f(x,r(x))}{\partial x}\bigg|_{x=x^*},x-x^*\right\rangle+\phi(x)-\phi(x^*)\ge 0,\ \forall x\in\mathbb{R}^n\right\}.$$

We would like to remark that (NSEP) was studied by Nagy [15] via variational inequalities and projections in the special case when $m=n=N$, $\phi(x)=I_{K_1}(x)$ and $\psi(y)=I_{K_2}(y)$ with K_1,K_2 being two nonempty closed convex subsets of $\mathbb{R}^N$, and I_{K_1},I_{K_2} being indicator functions of K_1,K_2, respectively. In recent years, various theoretical results, numerical algorithms and applications to practical problems have been studied extensively by many authors for the classical Stackelberg equilibrium problems in the literature (see, for example, [4, 6, 9, 17] and the references therein). As mentioned above, the (NSEP) is a generalization of the Stackelberg equilibrium problem, in which the nonsmooth cost functions are involved. The main purpose of this paper is to study some new existence theorems concerning with solutions of (NSEP) by employing the mixed variational inequality method and the Banach fixed point theorem.

The rest of this paper is organized as follows. In Section 2, with some known facts on the resolvent operator technique for maximal operators, we present some properties for the Stackelberg mixed variational response set and the Stackelberg mixed variational leader set. Then, some suitable conditions are given in Section 3 to ensure the nonemptyness of the Stackelberg mixed variational response set and the Stackelberg mixed variational leader set.

2. PRELIMINARIES

In this section, we present some basic properties concerning with the Stackelberg mixed variational response set and the Stackelberg mixed variational leader set for the nonsmooth Stackelberg equilibrium problem (NSEP).

Proposition 2.1. *Let g be a function of class C^1 and ψ be a proper convex lower semi-continuous functional. Then, we have the following conclusions:*

(a) $R_{NSE}(x) \subseteq R_{SMV}(x)$ *for every* $x \in \mathbb{R}^n$*;*

(b) *if* $g(x, \cdot)$ *is convex for some* $x \in \mathbb{R}^n$*, then* $R_{NSE}(x) = R_{SMV}(x)$*.*

Proof. (a) For any $y^* \in R_{NSE}(x)$, by the definition, one has

$$[g(x,y) + \psi(y)] - [g(x,y^*) + \psi(y^*)] \geq 0, \quad \forall y \in \mathbb{R}^m.$$

This means that y^* is a global minimum point for the function $g(x,\cdot) + \psi(\cdot)$. Since g is a function of class C^1 and ψ is a proper convex lower semi-continous functional, it follows that $0 \in \frac{\partial g(x,y)}{\partial y}\Big|_{y=y^*} + \partial\psi(y^*)$, where $\partial\psi$ denotes the subdifferential of ψ. By the definition of the subdifferential, we have

$$\left\langle \frac{\partial g(x,y)}{\partial y}\Big|_{y=y^*}, y - y^* \right\rangle + \psi(y) - \psi(y^*) \geq 0, \quad \forall y \in \mathbb{R}^m, \tag{2.1}$$

which implies that $y^* \in R_{SMV}(x)$ and thus $R_{NSE}(x) \subseteq R_{SMV}(x)$.

(b) It is sufficient to show that $R_{NSE}(x) \supseteq R_{SMV}(x)$. For any $y^* \in R_{SMV}(x)$, we know that (2.1) holds. Since $g(x,\cdot)$ is convex and of class C^1, it follows that

$$g(x,y) - g(x,y^*) \geq \left\langle \frac{\partial g(x,y)}{\partial y}\Big|_{y=y^*}, y - y^* \right\rangle, \quad \forall y \in \mathbb{R}^m. \tag{2.2}$$

Thus, by (2.1) and (2.2), we have

$$g(x,y) - g(x,y^*) + \psi(y) - \psi(y^*) \geq 0, \quad \forall y \in \mathbb{R}^m$$

and thus $y^* \in R_{NSE}(x)$. This completes the proof of Proposition 2.1. □

Remark 2.1. If $m = n = N$, $\phi(x) = I_{K_1}(x)$ and $\psi(y) = I_{K_2}(y)$ with K_1, K_2 being two nonempty closed convex subsets of $\mathbb{R}^N$, then Proposition 2.1 reduces to Proposition 2.1 of [15].

With similar arguments as the proof for Proposition 2.1, we can show the following result.

Proposition 2.2. *Let f be a function of class C^1 and ϕ be a proper convex lower semi-continuous functional. Assume that $x \mapsto R_{SMV}(x)$ is a single-valued function of class C^1. Then the following conclusions hold:*

(a) $S_{NSE} \subseteq S_{SMV}$*;*

(b) *if* $f(\cdot, R_{SMV}(\cdot))$ *is convex, then* $S_{NSE} = S_{SMV}$*.*

Proof. (a). It follows from Proposition 2.1 (a) that $R_{NSE}(x) \subseteq R_{SMV}(x)$ for all $x \in \mathbb{R}^n$. Since $x \mapsto R_{SMV}(x)$ is single-valued, it follows that $R_{NSE}(x) = R_{SMV}(x)$ and thus

$$r(x) = R_{NSE}(x) = R_{SMV}(x), \quad \forall x \in \mathbb{R}^n.$$

Let $x^* \in S_{NSE}$. Then, by definition, one has

$$[f(x, r(x)) + \phi(x)] - [f(x^*, r(x^*)) + \phi(x^*)] \geq 0, \quad \forall x \in \mathbb{R}^n.$$

This means that x^* is a global minimum point for the function $f(x, r(x)) + \phi(x)$. Since f and $r(x) = R_{SMV}(x)$ are functions of class C^1 and ϕ is a proper convex lower semi-continous functional, it follows that $0 \in \frac{\partial f(x,r(x))}{\partial x}\Big|_{x=x^*} + \partial\phi(x^*)$, where $\partial\phi$ denotes the subdifferential of ϕ. By the definition of the subdifferential, we have

$$\left\langle \frac{\partial f(x, r(x))}{\partial x}\Big|_{x=x^*}, x - x^* \right\rangle + \phi(x) - \phi(x^*) \geq 0, \quad \forall x \in \mathbb{R}^n, \tag{2.3}$$

which implies that $x^* \in S_{SMV}$ and thus $S_{NSE} \subseteq S_{SMV}$.

(b). We only need to show that $S_{SMV} \subseteq S_{NSE}$. For $x^* \in S_{SMV}$, it follows that (2.3) holds. Since $f(\cdot, R_{SMV}(\cdot))$ is convex and of class C^1, we know that $f(\cdot, r(\cdot))$ is convex and of class C^1. Thus,

$$f(x, r(x)) - f(x^*, r(x^*)) \geq \left\langle \frac{\partial f(x, r(x))}{\partial x}\Big|_{x=x^*, y=r(x^*)}, x - x^* \right\rangle, \quad \forall x \in \mathbb{R}^n. \tag{2.4}$$

Now, it follows from (2.3) and (2.4) that

$$[f(x, r(x)) + \phi(x)] - [f(x^*, r(x^*)) + \phi(x^*)] \geq 0, \quad \forall x \in \mathbb{R}^n$$

and thus $x^* \in S_{NSE}$. This completes the proof of Proposition 2.2. □

Remark 2.2. If $m = n = N$, $\phi(x) = I_{K_1}(x)$ and $\psi(y) = I_{K_2}(y)$ with K_1, K_2 being two nonempty closed convex subsets of $\mathbb{R}^N$, then Proposition 2.2 reduces to Proposition 2.3 of [15].

We now recall some know facts on the resolvent operator technique for maximal operators, which can be found in [3].

Let T be a maximal monotone operator from $\mathbb{R}^n$ to $2^{\mathbb{R}^n}$, I be an identity operator from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $\rho > 0$ be a constant. Then it is well known that the resolvent operator of T, defined by $J_T(x) := (I + \rho T)^{-1}(x)$, is a single-valued operator. Moreover, the resolvent operator J_T is nonexpansive, i.e.,

$$\|J_T(x) - J_T(y)\| \leq \|x - y\|, \quad \forall x, y \in \mathbb{R}^n.$$

Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. It is well known that $\partial\varphi$ is maximal monotone and $(I + \rho\partial\varphi)^{-1}$ is single-valued and nonexpansive. Moreover,

$$x = (I + \rho\partial\varphi)^{-1}(z) \iff \langle x - z, y - x\rangle + \rho\varphi(y) - \rho\varphi(x) \geq 0, \quad \forall y \in \mathbb{R}^n. \tag{2.5}$$

We now turn to the follower's Stackelberg mixed variational response set $R_{SMV}(x)$. The element acts as a solution to a parametric mixed variational inequality problem: find $y^* \in \mathbb{R}^m$ such that

$$\left\langle \frac{\partial g(x, y)}{\partial y}\Big|_{y=y^*}, y - y^* \right\rangle + \psi(y) - \psi(y^*) \geq 0, \quad \forall y \in \mathbb{R}^m.$$

For any $\rho > 0$, we define a parametric operator $A^x_\rho : \mathbb{R}^m \to \mathbb{R}^m$ as follows:

$$A^x_\rho(y) = (I + \rho\partial\psi)^{-1}\left[y - \rho\frac{\partial g(x, y)}{\partial y}\right], \quad \forall y \in \mathbb{R}^m.$$

Thus, the fixed point set of the parametric operator A^x_ρ can be defined as follows:

$$FPS^\rho_A(x) = \left\{ y^* : y^* = (I + \rho\partial\psi)^{-1}\left[y^* - \rho\,\frac{\partial g(x, y)}{\partial y}\Big|_{y=y^*}\right]\right\}, \quad x \in \mathbb{R}^n.$$

Now, with the above notations, we are in a position to present the following result.

Proposition 2.3. *Let g be a function of class C^1 and ψ be a proper convex lower semi-continuous functional. Then the following assertions are equivalent:*

(i) $y^* \in R_{SMV}(x)$;
(ii) $y^* \in FPS_A^{\rho}(x)$ *for any* $\rho > 0$;
(iii) $y^* \in FPS_A^{\rho}(x)$ *for some* $\rho > 0$.

Proof. (i) $\Rightarrow$ (ii). If $y^* \in R_{SMV}(x)$, then it is easy to see that

$$\left\langle \left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y^*}, y - y^* \right\rangle + \psi(y) - \psi(y^*) \geq 0.$$

For any $\rho > 0$, letting $z = y^* - \rho \left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y^*}$, it follows from (2.5) that

$$\langle y^* - z, y - y^* \rangle + \rho\psi(y) - \rho\psi(y^*) = \left\langle \rho \left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y^*}, y - y^* \right\rangle + \rho\psi(y) - \rho\psi(y^*) \geq 0.$$

This shows that $y^* = (I + \rho\partial\psi)^{-1}(z)$ and thus

$$y^* = (I + \rho\partial\psi)^{-1} \left[y^* - \rho \left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y^*} \right] \in FPS_A^{\rho}(x).$$

(ii) $\Rightarrow$ (iii) It is obvious.

(iii) $\Rightarrow$ (i). Suppose that $y^* \in FPS_A^{\rho}(x)$ for some $\rho > 0$. Then

$$y^* = (I + \rho\partial\psi)^{-1} \left[y^* - \rho \left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y^*} \right].$$

Letting $z = y^* - \rho \left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y^*}$, we know that $y^* = (I + \rho\partial\psi)^{-1}(z)$. It follows from (2.5) that

$$0 \leq \langle y^* - z, y - y^* \rangle + \rho\psi(y) - \rho\psi(y^*) = \left\langle \rho \left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y^*}, y - y^* \right\rangle + \rho\psi(y) - \rho\psi(y^*)$$

and thus $y^* \in R_{SMV}(x)$. This completes the proof. □

Remark 2.3. If $m = n = N$, $\phi(x) = I_{K_1}(x)$ and $\psi(y) = I_{K_2}(y)$ with K_1, K_2 being two nonempty closed convex subsets of $\mathbb{R}^N$, then Proposition 2.3 reduces to Proposition 2.2 of [15].

3. The Stackelberg mixed variational response/leader set

We begin with the following existence result for the Stackelberg mixed variational response set of the follower.

Theorem 3.1. *Let g be a function of class C^1 and ψ be a proper convex lower semi-continuous functional. Assume that there exist two positive constants κ_g and L_g such that, for any $x \in \mathbb{R}^n$ and $y_1, y_2 \in \mathbb{R}^m$,*

$$\left\langle \left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y_1} - \left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y_2}, y_1 - y_2 \right\rangle \geq \kappa_g \|y_1 - y_2\|^2 \tag{3.1}$$

and

$$\left\| \left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y_1} - \left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y_2} \right\| \leq L_g \|y_1 - y_2\|. \tag{3.2}$$

Then there exists a unique point $y^* \in \mathbb{R}^m$ such that $R_{SMV}(x) = \{y^*\}$. Moreover, both the discrete dynamical system

$$\begin{cases} y^{k+1} = A_\rho^x(y^k), \quad k \geq 0, \\ y^0 \in \mathbb{R}^m \end{cases} \tag{3.3}$$

and the continuous dynamical system

$$\begin{cases} \frac{dy}{dt} = A_\rho^x(y(t)) - y(t), \\ y(0) = y^0 \in \mathbb{R}^m \end{cases} \tag{3.4}$$

converge exponentially to y^* for some appropriate values of ρ.

Proof. For any $y_1, y_2 \in \mathbb{R}^m$, by the nonexpansiveness of the resolvent operator, it follows from (3.1) and (3.2) that

$$\begin{aligned} &\|A_\rho^x(y_1) - A_\rho^x(y_2)\|^2 \\ &= \left\| (I + \rho\partial\psi)^{-1} \left[y_1 - \rho \left. \frac{\partial g(x,y)}{\partial y} \right|_{y=y_1} \right] - (I + \rho\partial\psi)^{-1} \left[y_2 - \rho \left. \frac{\partial g(x,y)}{\partial y} \right|_{y=y_2} \right] \right\|^2 \\ &\leq \|y_1 - y_2\|^2 - 2\rho\kappa_g \|y_1 - y_2\|^2 + \rho^2 L_g^2 \|y_1 - y_2\|^2 \\ &= (1 - 2\rho\kappa_g + \rho^2 L_g^2)\|y_1 - y_2\|^2. \end{aligned}$$

This shows that

$$\|A_\rho^x(y_1) - A_\rho^x(y_2)\| \leq h\|y_1 - y_2\|, \quad \forall y_1, y_2 \in \mathbb{R}^m$$

with $0 < h = \sqrt{1 - 2\rho\kappa_g + \rho^2 L_g^2}$. Fixing $0 < \rho < \frac{2\kappa_g}{L_g^2}$, we know that $0 < h < 1$. Thus, by the Banach fixed point theorem, the operator A_ρ^x admits a unique fixed point $y^* \in \mathbb{R}^m$.

Now we prove that the discrete dynamical system (3.3) converges exponentially to y^*. In fact, since $y^* \in \mathbb{R}^m$ is a fixed point of A_ρ^x, one has $y^* = A_\rho^x(y^*)$. Thus, it follows from (3.3) that

$$\|y^k - y^*\| = \|A_\rho^x(y^{k-1}) - A_\rho^x(y^*)\| \leq h\|y^{k-1} - y^*\| \leq \cdots \leq h^k \|y^0 - y^*\|,$$

which shows that y^k converges exponentially to y^* as $k \to \infty$ due to $0 < h < 1$.

Next we prove that the continuous dynamical system (3.4) converges exponentially to y^*. In fact, the classical ODE shows that the system (3.4) has a solution in $[0, T)$. Suppose that $T < \infty$. We consider the following Lyapunov function

$$L_x(t) = \frac{1}{2}\|y(t) - y^*\|^2 = \frac{1}{2}\|y(t) - A_\rho^x(y^*)\|^2.$$

Then, for a.e. $t \in [0, T)$, one has

$$\begin{aligned} \frac{d}{dt} L_x(t) &= \left\langle y(t) - y^*, \frac{dy}{dt} \right\rangle \\ &= \left\langle y(t) - y^*, A_\rho^x(y(t)) - y(t) \right\rangle \\ &= -\|y(t) - y^*\|^2 + \left\langle y(t) - y^*, A_\rho^x(y(t)) - A_\rho^x(y^*) \right\rangle \\ &\leq -\|y(t) - y^*\|^2 + \|y(t) - y^*\| \|A_\rho^x(y(t)) - A_\rho^x(y^*)\| \\ &\leq (h - 1)\|y(t) - y^*\|^2. \end{aligned}$$

This shows that

$$\frac{d}{dt} L_x(t) \leq 2(h - 1) L_x(t), \quad \forall t \in [0, T).$$

Thus, we have

$$\frac{d}{dt}\left[L_x(t)e^{2(1-h)t}\right] = \left(\frac{d}{dt}L_x(t) + 2(1-h)L_x(t)\right)e^{2(1-h)t} \leq 0.$$

This means that the function $t \mapsto L_x(t)e^{2(1-h)t}$ is non-increasing and so $L_x(t)e^{2(1-h)t} \leq L_x(0)$ for all $t \in [0, T)$. In particular, the orbit $t \mapsto y(t)$ can be extended beyond T, which contradicts the initial assumption. Thus, $T = \infty$. It follows that $L_x(t) \leq L_x(0)e^{-2(1-h)t}$ for every $t \geq 0$ and so $\|y(t) - y^*\| \leq \|y(0) - y^*\|e^{-(1-h)t}$. Therefore, $y(t)$ converges exponentially to y^*. This completes the proof. □

Remark 3.4. If $m = n = N$, $\phi(x) = I_{K_1}(x)$ and $\psi(y) = I_{K_2}(y)$ with K_1, K_2 being two nonempty closed convex subsets of $\mathbb{R}^N$, then Theorem 3.1 reduces to Theorem 3.2 of [15].

With the existence theorem above, we can further prove the following property for the Stackelberg mixed variational response set.

Theorem 3.2. *Assume that all the assumptions in Theorem 3.1 hold. If*

$$\left\|\frac{\partial g(x_1, y)}{\partial y} - \frac{\partial g(x_2, y)}{\partial y}\right\| \leq L'_g\|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbb{R}^n, y \in \mathbb{R}^m, \tag{3.5}$$

where $L'_g > 0$ *is a constant, then*

$$\|R_{SMV}(x_1) - R_{SMV}(x_2)\| \leq C\|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbb{R}^n,$$

where

$$C = \frac{\rho L'_g}{1 - \sqrt{1 - 2\rho\kappa_g + \rho^2 L_g^2}}. \tag{3.6}$$

Proof. From Theorem 3.1, we know that $R_{SMV}(x) = \{y^*\}$. For any $x_1, x_2 \in \mathbb{R}^n$, let $y_i^* = R_{SMV}(x_i)$ with $i = 1, 2$. Then, by the nonexpansiveness of the resolvent operator, it follows from (3.1), (3.2) and (3.5) that

$$\begin{aligned}\|R_{SMV}(x_1) - R_{SMV}(x_2)\| &\leq \left\|(y_1^* - y_2^*) - \rho\left[\left.\frac{\partial g(x_1, y)}{\partial y}\right|_{y=y_1^*} - \left.\frac{\partial g(x_2, y)}{\partial y}\right|_{y=y_2^*}\right]\right\| \\ &\leq \sqrt{1 - 2\rho\kappa_g + \rho^2 L_g^2}\|y_1^* - y_2^*\| + \rho L'_g\|x_1 - x_2\|.\end{aligned}$$

In view of fact that $0 < \sqrt{1 - 2\rho\kappa_g + \rho^2 L_g^2} < 1$ when $0 < \rho < \frac{2\kappa_g}{L_g^2}$, we have

$$\|y_1^* - y_2^*\| \leq \frac{\rho L'_g}{1 - \sqrt{1 - 2\rho\kappa_g + \rho^2 L_g^2}}\|x_1 - x_2\|$$

and so $\|R_{SMV}(x_1) - R_{SMV}(x_2)\| \leq C\|x_1 - x_2\|$. This completes the proof. □

When the mapping $x \mapsto R_{SMV}(x)$ is single-valued, we denote $r(x) = R_{SMV}(x)$. Next we focus on the existence result for the Stackelberg mixed variational leader set. To this end, we need the following assumptions.

Suppose that there exist constants $\kappa_f > 0$ and $L_f > 0$ such that, for any $x_1, x_2 \in \mathbb{R}^n$,

$$\left\langle \left.\frac{\partial f(x, r(x))}{\partial x}\right|_{x=x_1} - \left.\frac{\partial f(x, r(x))}{\partial x}\right|_{x=x_2}, x_1 - x_2\right\rangle \geq \kappa_f\|x_1 - x_2\|^2 \tag{3.7}$$

and

$$\left\| \frac{\partial f(x, r(x))}{\partial x}\Big|_{x=x_1} - \frac{\partial f(x, r(x))}{\partial x}\Big|_{x=x_2} \right\| \leq L_f \|r(x_1) - r(x_2)\|. \tag{3.8}$$

For any $\eta > 0$, we define an operator $B_\eta : \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$B_\eta(x) = (I + \eta\partial\phi)^{-1} \left[x - \eta \frac{\partial f(x, r(x))}{\partial x} \right], \quad x \in \mathbb{R}^n. \tag{3.9}$$

With the above assumptinos, we have the following result on the existence and uniqueness for the Stackelberg mixed variational leader set.

Theorem 3.3. *Assume that all the conditions in Theorem 3.1 hold. If conditions* (3.5), (3.7) *and* (3.8) *are satisfied, then there exists a unique $x^* \in \mathbb{R}^n$ such that $S_{SMV} = \{x^*\}$ and $y^* = r(x^*)$ with $R_{SMV}(x^*) = \{y^*\}$.*

Proof. By the nonexpansiveness of the resolvent operator $(I + \eta\partial\phi)^{-1}$ and the operator B_η defined by (3.9), for any $x_1, x_2 \in \mathbb{R}^n$, we have

$$\|B_\eta(x_1) - B_\eta(x_2)\| \leq \left\| (x_1 - x_2) - \eta \left[\frac{\partial f(x, r(x))}{\partial x}\Big|_{x=x_1} - \frac{\partial f(x, r(x))}{\partial x}\Big|_{x=x_2} \right] \right\|.$$

It follows from (3.7) and (3.8) that

$$\begin{aligned} & \left\| (x_1 - x_2) - \eta \left[\frac{\partial f(x, r(x))}{\partial x}\Big|_{x=x_1} - \frac{\partial f(x, r(x))}{\partial x}\Big|_{x=x_2} \right] \right\|^2 \\ \leq \; & \|x_1 - x_2\|^2 - 2\eta\kappa_f \|x_1 - x_2\|^2 + \eta^2 L_f^2 \|r(x_1) - r(x_2)\|^2. \end{aligned}$$

Since $r(x) = R_{SMV}(x)$, from Theorem 3.2, one has

$$\|B_\eta(x_1) - B_\eta(x_2)\| \leq \hat{h}\|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbb{R}^n,$$

where $\hat{h} = \sqrt{1 - 2\eta\kappa_f + \eta^2 L_f^2 C^2}$. Choose $\eta \in (0, \frac{2\kappa_f}{L_f^2 C^2})$, where C is defined by (3.6). Then it is easy to see that B_η is a contractive mapping and thus the Banach fixed point theorem yields that there exists a unique $x^* \in \mathbb{R}^n$ such that $x^* = B_\eta(x^*)$ which implies that $S_{SMV} = \{x^*\}$. Moreover, by Theorem 3.1, there exists a unique $y^* \in \mathbb{R}^m$ such that $y^* = r(x^*)$ with $R_{SMV}(x^*) = \{y^*\}$. This completes the proof of Theorem 3.3. □

Remark 3.5. Theorems 3.2 and 3.3 are new results which provide some sufficient conditions for ensuring the Lipschitz continuity of R_{SMV} and the nonemptyness of S_{SMV}. We would like to mention that the similar results were first given by Theorems 3.3 and 3.4 of [13].

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