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Dedicated to Prof. Qamrul Hasan Ansari on the occasion of his 60^{th} anniversary

A cyclic coordinate-update fixed point algorithm

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ABSTRACT. We prove that a cyclic coordinate fixed point algorithm for nonexpansive mappings when the underlying Hilbert space is decomposed into a Cartesian product of finitely many block spaces is weakly convergent to a fixed point of the mapping under investigation. Our result relaxes a condition imposed on the stepsizes of Theorem 3.4 of Chow, et al [Chow, Y. T., Wu, T. and Yin, W., *Cyclic coordinate-update algorithms for fixed-point problems: analysis and applcations*, SIAM J. Sci. Comput., **39** (2017), No. 4, A1280–A1300].

1. INTRODUCTION

Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Consider the problem of finding a zero of a maximal monotone operator S:

$$Sx = 0, \tag{1.1}$$

where $S : H \to H$ is a maximal monotone operator. Assume S is of the form

$$S = I - T, \tag{1.2}$$

where $T : H \to H$ is a nonexpansive mapping (i.e., $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in H$). Consequently, S is Lipschitzian with Lipschitz constant not bigger than two. We use $\mathrm{zer}(S)$ and $\mathrm{Fix}(T)$ to denote the set of solutions of Eq. (1.1) and the set of fixed points of T, respectively. It is evident that $\mathrm{zer}(S) = \mathrm{Fix}(T) = \{x \in H : Tx = x\}$. We always assume that the solution set $\mathrm{zer}(S)$ (or $\mathrm{Fix}(T)$) is nonempty. Note that in our setting, finding a zero of S is equivalent to finding a fixed point of T. Therefore, the Kransnoselskii-Mann algorithm (KM) [4, 6] is applicable to Eq. (1.1). Recall that KM generates a sequence (x^k) through the iteration scheme:

$$x^{k+1} = (1 - \alpha_k)x^k + \alpha_k T x^k, \quad k = 0, 1, 2, \cdots, \tag{1.3}$$

where the initial guess $x^0 \in H$ is chosen arbitrarily, and $\alpha_k \in [0, 1]$ for all k.

The KM (1.3) has extensively been studied (see [5, 8, 10, 13, 15] and references therein). A basic convergence result of KM (1.3) is given below.

Theorem 1.1. *(cf.* [12]*) Suppose* $\mathrm{Fix}(T) \neq \emptyset$ *and the stepsizes* (α_k) *satisfies the divergence condition:*

$$\sum_{k=0}^{\infty} \alpha_k (1 - \alpha_k) = \infty. \tag{1.4}$$

Then the sequence (x^k) *generated by KM (1.3) converges weakly to a point in* $\mathrm{Fix}(T)$.

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Note that a standard choice of the stepsizes (α_k) that satisfies the divergence condition (1.4) is given by

$$\alpha_k = \frac{1}{k^\tau}, \quad k \geq 1, \text{ with } 0 < \tau \leq 1. \tag{1.5}$$

Chow, et al [1] applied KM (1.3) to find a zero of a maximal monotone mapping $S = I - T$ (with T being nonexpansive) in the case where the underlying space H is decomposed into a Cartesian product of finitely many block spaces:

$$H = H_1 \times H_2 \times \cdots \times H_m \tag{1.6}$$

where $m \geq 1$ is an integer, and H_i is a Hilbert space for each $1 \leq i \leq m$. In this framework, each $x \in H$ is decomposed into $x = (x_1, \cdots, x_m)$, where x_i denotes the ith coordinate of x (we write $(x)_i = x_i$); i.e., the projection of x onto the ith block space H_i.

Basing on KM (1.3), Chow, et al [1] introduced a cyclic coordinate-update algorithm [1, Algorithm 1, page A1283], and proved [1, Theorem 3.4, page A1288] the weak convergence of their Algorithm 1 under the assumption that the stepsizes (α_k) are chosen as

$$\alpha_k = \frac{1}{\sqrt{k}}, \quad k \geq 1. \tag{1.7}$$

The purpose of this paper is to prove that [1, Algorithm 1] remains to be weakly convergent to a solution of Eq. (1.1) if the stepsizes (α_k) are chosen to satisfy the following two conditions:

(α1) $\sum_{k=1}^\infty \alpha_k = \infty$; ($\alpha$2) $\sum_{k=1}^\infty \alpha_k^3 < \infty$.

A particular choice is given by $\alpha_k = \frac{1}{k^\tau}$ for $k \geq 1$ with $\frac{1}{3} < \tau \leq 1$. This includes the choice (1.7) by letting $\tau = \frac{1}{2}$.

2. Preliminaries

The following two lemmas are useful for proving the convergence of our algorithm in this paper.

Lemma 2.1. [11] *Assume (a_k) is a sequence of nonnegative real numbers with the property:*

$$a_{k+1} \leq (1 + r_k)a_k + b_k, \quad k \geq 0,$$

where (r_k) and (b_k) are sequences of nonnegative real numbers such that $\sum_{k=0}^\infty r_k < \infty$ and $\sum_{k=0}^\infty b_k < \infty$. Then (a_k) is bounded and $\lim_{k\to\infty} a_k$ exists.

Lemma 2.2. [5, Lemma 2.5] *Let K be a nonempty subset of a Hilbert space H. Assume (x^k) is a bounded sequence in H with the properties:*

(a) $\lim_{k\to\infty} \|x^k - z\|$ exists for each $z \in K$;
(b) if x' is a weak cluster point of (x^k), then $x' \in K$.

Then the full sequence (x^k) converges weakly to a point in K.

We need the demiclosedness principle of nonexpansive mappings as follows.

Lemma 2.3. [9, 2] *Let C be a closed convex subset of a Hilbert space H and $T : C \to C$ a nonexpansive mapping. Suppose (v^k) is a sequence in C such that $v^k \to v$ weakly and $v^k - Tv^k \to 0$ in norm. Then $v = Tv$.*

2.1. **A cyclic coordinate-update algorithm.** Let H be a real Hilbert space with the decomposition (1.6). Let us consider the equation (1.1), assuming (1.2) and $\mathrm{zer}(S) \neq \emptyset$.

Following [1], we introduce the coordinate mappings (S_i) associated with S as follows: $S_i x := (0, \cdots, 0, (Sx)_i, 0, \cdots, 0), \quad x \in H$. As a result,

$$Sx = \sum_{i=1}^{m} S_i x, \quad \langle S_i x, S_j x \rangle = 0 \ (i \neq j), \quad \|Sx\|^2 = \sum_{i=1}^{m} \|S_i x\|^2$$

for all $x \in H$.

The cyclic coordinate-update algorithm (CCA) introduced in [1, Algorithm 1] is rephrased below:

$$\begin{cases} x^{k,0} = x^k, & (2.8a) \\ x^{k,j} = x^{k,j-1} - \alpha_k S_j(x^{k,j-1}), \quad j = 1, 2, \cdots, m, & (2.8b) \\ x^{k+1} = x^{k,m}. & (2.8c) \end{cases}$$

For $\alpha \in (0,1)$, Chow, et al [1] introduced two operators T^α and E^α defined respectively by

(2.9) $$T^\alpha := I - \alpha S,$$

(2.10) $$E^\alpha := (I - \alpha S_m)(I - \alpha S_{m-1}) \cdots (I - \alpha S_1).$$

Note that T^α is an α-averaged mapping (cf. [3, 14]); indeed, $T^\alpha = (1-\alpha)I + \alpha T$. However, each mapping $I - \alpha S_i$ fails, in general, to be nonexpansive; nevertheless, it is Lipschitzian with Lipschitz constant $L_i \leq 2$ for $1 \leq i \leq m$. Put $L := \max\{L_i : 1 \leq i \leq m\}$.

The following fact is easily proved (see [1, Eq. (2.7), page A1285]):

(2.11) $$\|T^\alpha x - x^*\|^2 \leq \|x - x^*\|^2 - \alpha(1-\alpha)\|Sx\|^2, \quad x \in H, \ x^* \in \mathrm{zer}(S).$$

The CCA (2.8) can also equivalently be reformulated in the form:

(2.12) $$x^{k+1} = E^{\alpha_k} x^k = (I - \alpha_k S_m)(I - \alpha_k S_{m-1}) \cdots (I - \alpha_k S_1) x^k, \quad k = 0, 1, \cdots.$$

The main convergence result of Chow, et al [1] is the following result.

Theorem 2.2. [1, Theorem 3.4] *Assume S is of the form (1.2) with T nonexpansive and $\mathrm{zer}(S) \neq \emptyset$. Assume, in addition, the stepsizes (α_k) satisfy the rule (1.7). Then the sequence (x^k) generated by the CCA (2.8) (or equivalently, (2.12)) converges weakly to a solution of Eq. (1.1).*

3. An improvement of [1, Theorem 3.4]

In this section we will improve [1, Theorem 3.4] by showing the weak convergence of the CCA (2.8) under the much more general, relaxed conditions $(\alpha 1)$ and $(\alpha 2)$ satisfied by the stepsizes (α_k). To this end we need the lemma below.

Lemma 3.4. *Let (α_k) and (β_k) be sequences of nonnegative real numbers. Suppose the following conditions are satisfied:*

- *(i)* $\sum_{k=1}^{\infty} \alpha_k = \infty$;
- *(ii)* $\sum_{k=1}^{\infty} \alpha_k \beta_k < \infty$;
- *(iii)* $|\beta_{k+1} - \beta_k| \leq c\alpha_k$ *for all* $k \geq 1$ *and some constant* $c > 0$.

Then (β_k) converges to zero.

Proof. Let $\mathbb{N}$ denote the set of positive integers. Given $\varepsilon > 0$. We define a subset N_ε of $\mathbb{N}$ by

$$\mathbb{N}_\varepsilon := \left\{ k \in \mathbb{N} : \beta_k < \frac{\varepsilon}{2} \right\}.$$

Set $\mathbb{N}_\varepsilon^c := \mathbb{N} \setminus \mathbb{N}_\varepsilon$.

Since the condition (i) implies that $\liminf_{k\to\infty} \beta_k = 0$, the set $\mathbb{N}_\varepsilon$ is indeed an infinite subset of $\mathbb{N}$. Also we have

$$\sum_{k\in\mathbb{N}_\varepsilon^c} \alpha_k\beta_k \geq \frac{\varepsilon}{2} \sum_{k\in\mathbb{N}_\varepsilon^c} \alpha_k.$$

By the condition (ii) we find that $\sum_{k\in\mathbb{N}_\varepsilon^c} \alpha_k < \infty$. Consequently, there exists a sufficiently large integer k_ε such that

$$\sum_{\substack{k\in\mathbb{N}_\varepsilon^c\\ k\geq k_\varepsilon}} \varepsilon_k < \frac{\varepsilon}{2c}.$$

We now claim that

$$\beta_k < \varepsilon \quad \text{for all } k > k_\varepsilon. \tag{3.13}$$

As a matter of fact, for fixed $k > k_\varepsilon$, if $k \in \mathbb{N}_\varepsilon$, then (3.13) holds trivially and we are done. If $k \in \mathbb{N}_\varepsilon^c$, then, since $\mathbb{N}_\varepsilon$ is infinite, $\mathbb{N}_\varepsilon$ has integers that are bigger than k. Let $n \in \mathbb{N}_\varepsilon$ be the least integer in $\mathbb{N}_\varepsilon$ such that $k < n$. Note that we have $\beta_n < \varepsilon/2$. It follows that (noticing the minimality property of $n \in \mathbb{N}_\varepsilon$)

$$\begin{aligned}\beta_k = \beta_n + (\beta_k - \beta_n) < \frac{\varepsilon}{2} + (\beta_k - \beta_n) = \frac{\varepsilon}{2} + \sum_{i=k}^{n-1}(\beta_i - \beta_{i+1}) \leq \frac{\varepsilon}{2} + c\sum_{i=k}^{n-1}\alpha_i \\ \text{by (iii)} \leq \frac{\varepsilon}{2} + c\sum_{\substack{i\in\mathbb{N}_\varepsilon^c\\ i\geq k_\varepsilon}} \alpha_i < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Consequently, (3.13) holds again. This finishes the proof. □

Now we are in a position to extend [1, Theorem 3.4] to a more general case where the stepsizes (α_k) can be particularly taken to be $k^{-\tau}$ for all $k \geq 1$ with $\tau \in (1/3, 1]$.

Theorem 3.3. *Suppose* $\mathrm{zer}(S) \neq \emptyset$ *and* $I - S$ *is nonexpansive. Assume* (α_k) *satisfies the conditions* $(\alpha1)$ *and* $(\alpha2)$ *in Section 1. Then the sequence* (x^k) *generated by CCA (2.12) (i.e., (2.8)) converges weakly to a point in* $\mathrm{zer}(S)$*.*

Proof. We will use the weak convergence lemma (i.e., Lemma 2.2) to prove the theorem. Namely, we will prove that the iterates (x^k) fulfil the two following conditions:

(C1) $\lim_{k\to\infty} \|x^k - x^*\|$ exists for every $x^* \in \mathrm{zer}(S)$;
(C2) $\omega_w(x^k) \subset \mathrm{zer}(S)$.

We follow the notation and some lines of the proof given in [1] with appropriate modifications and improvements. For $\alpha \in (0, 1)$, put

$$R \equiv R_\alpha := \frac{1}{\alpha}(T^\alpha - E^\alpha).$$

Here T^α and E^α are defined by (2.9) and (2.10), respectively. Below is an estimate given in [1, Lemma 3.1]:

$$\|Rx\| \leq \frac{\alpha Lm}{\sqrt{2}}(1 + \alpha L)^m\|Sx\| \leq \alpha c_m\|Sx\|, \quad x \in H, \tag{3.14}$$

where $c_m = \frac{mL}{\sqrt{2}}(1 + L)^m$. Observing $E^\alpha = T^\alpha - \alpha R$ and using the inequality

$$\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v\rangle, \quad u, v \in H,$$

we get, for $x \in H$ and $x^* \in \mathrm{zer}(S)$,

$$\|E^\alpha x - x^*\|^2 = \|(T^\alpha x - x^*) - \alpha Rx\|^2 \leq \|T^\alpha x - x^*\|^2 - 2\alpha\langle Rx, E^\alpha x - x^*\rangle$$

$$\leq \|T^\alpha x - x^*\|^2 + 2\alpha \|Rx\| \|E^\alpha x - x^*\|.$$

By Young's inequality, we get, for any $\eta > 0$,

$$\|E^\alpha x - x^*\|^2 \leq \|T^\alpha x - x^*\|^2 + \alpha\eta^{-1}\|Rx\|^2 + \alpha\eta\|E^\alpha x - x^*\|^2.$$

It turns out that

$$\|E^\alpha x - x^*\|^2 \leq \frac{1}{1-\alpha\eta}\|T^\alpha x - x^*\|^2 + \frac{\alpha}{\eta(1-\alpha\eta)}\|Rx\|^2. \tag{3.15}$$

Combining (3.14) and (3.15) yields

$$\|E^\alpha x - x^*\|^2 \leq \frac{1}{1-\alpha\eta}\|T^\alpha x - x^*\|^2 + \frac{\alpha^3 c_m^2}{\eta(1-\alpha\eta)}\|Sx\|^2. \tag{3.16}$$

By (2.11) we furthermore derive that

$$\|E^\alpha x - x^*\|^2 \leq \frac{1}{1-\alpha\eta}\left(\|x - x^*\|^2 - \left(\alpha(1-\alpha) - \frac{\alpha^3 c_m^2}{\eta}\right)\|Sx\|^2\right). \tag{3.17}$$

Inserting $x := x^k$, $\alpha := \alpha_k$, $\eta := \eta_k$ into (3.17), and recalling $x^{k+1} = E^{\alpha_k} x^k$, we obtain

$$\|x^{k+1} - x^*\|^2 \leq (1+\xi_k)\left(\|x^k - x^*\|^2 - \left(\alpha_k(1-\alpha_k) - \frac{\alpha_k^3 c_m^2}{\eta_k}\right)\|Sx^k\|^2\right), \tag{3.18}$$

where $\xi_k = \frac{\alpha_k\eta_k}{1-\alpha_k\eta_k}$. Take

$$\eta_k := \frac{2\alpha_k^2 c_m^2}{1-\alpha_k}, \quad k > 1.$$

Then it is easy to find that

$$\xi_k = \frac{2c_m^2\alpha_k^3}{1-\alpha_k-2c_m^2\alpha_k^3}.$$

Since $\alpha_k \to 0$, it is not hard to find from (α2) that $\xi_k = O\left(\alpha_k^3\right)$. Consequently, the series

$$\sum_{k=1}^\infty \xi_k < \infty. \tag{3.19}$$

A consequence of (3.18) is that

$$\|x^{k+1} - x^*\|^2 \leq (1+\xi_k)\|x^k - x^*\|^2. \tag{3.20}$$

By (3.19) and (3.20) and applying Lemma 2.1, we have verified (C1). Returning to (3.18) we immediately get

$$\sum_{k=1}^\infty \alpha_k\|Sx^k\|^2 < \infty. \tag{3.21}$$

Since (x^k) is bounded and S is 2-Lipschitzian, we have a constant $\tilde{c} > 0$ such that $\|x^k\| \leq c$ and $\|Sx^k\| \leq \tilde{c}$ for all k. Set $\beta_k = \|Sx^k\|^2$. It follows that

$$|\beta_{k+1} - \beta_k| = |\|Sx^{k+1}\|^2 - \|Sx^k\|^2| \leq \|Sx^{k+1} - Sx^k\|(\|Sx^{k+1}\| + \|Sx^k\|) \leq 4\tilde{c}\|x^{k+1} - x^k\|.$$

Since $x^{k+1} = E^{\alpha_k}x^k = x^k - \alpha_k(Sx^k + Rx^k)$, it follows from (3.14) that

$$|\beta_{k+1} - \beta_k| \leq 4\tilde{c}\alpha_k(\|Sx^k\| + \|Rx^k\|) \leq 4\tilde{c}\alpha_k(1+\alpha_k c_m)\|Sx^k\| \leq 4\tilde{c}^2\alpha_k(1+c_m) = c\alpha_k, \tag{3.22}$$

where $c = 4\tilde{c}^2(1+c_m)$.

Finally, by (3.21) and (3.22) we can apply Lemma 3.4 to get $\beta_k \to 0$. Alternatively, we get $\|x^k - Tx^k\| = \|Sx^k\| \to 0$. This further enables us to apply Lemma 2.3 to obtain $\omega_w(x^k) \subset \mathrm{Fix}(T) = \mathrm{zer}(S)$. That is, (C2) is proven. This completes the proof. □

Corollary 3.1. *Suppose* $\operatorname{zer}(S) \neq \emptyset$ *and* $I - S$ *is nonexpansive. If the stepsizes* (α_k) *are given by* $\alpha_k = \frac{1}{k^\tau}$ *for all* $k \geq 1$ *and some* $\tau \in (\frac{1}{3}, 1]$*, then the sequence* (x^k) *generated by the CCA (2.12) converges weakly to a point in* $\operatorname{zer}(S)$*.*

Remark 3.1. Corollary 3.1 contains the main convergence result of [1, Theorem 3.4] as a special case (corresponding to the choice $\tau = \frac{1}{2}$).

Remark 3.2. The divergence condition (1.4) guarantees the weak convergence of the Krasnoselskii-Mann algorithm (1.3). Our conditions $(\alpha 1)$ and $(\alpha 2)$ are stronger than the divergence condition (1.4). It is unclear if the CCA (2.8) would converge weakly if the stepsizes (α_k) satisfy the divergence condition (1.4). In particular, we do not know if the CCA (2.8) converges weakly if the stepsizes (α_k) satisfy the two conditions below:

- $\sum_{k=1}^\infty \alpha_k = \infty$, and
- $\sum_{k=1}^\infty \alpha_k^p < \infty$ for any fixed, arbitrarily big positive integer p.

Note that these conditions with $p = 2$ are employed in incremental subgradient methods [7]. Note also that a positive answer to this question implies that the CCA (2.8) generates weakly convergent iterates (x^k), with stepsizes $\alpha_k = \frac{1}{k^\tau}$ for all $k \geq 1$ and $\tau \in (0, 1]$.

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