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Dedicated to Prof. Qamrul Hasan Ansari on the occasion of his 60^{th} anniversary

External and internal stability in set optimization using gamma convergence

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ABSTRACT. The main objective of this paper is to investigate the stability of solution sets of perturbed set optimization problems in the decision space as well as in the image space, by perturbing the objective maps. For a sequence of set-valued maps, a notion of gamma convergence is introduced to establish the external and internal stability in terms of Painlevé-Kuratowski convergence of sequence of solution sets of perturbed problems under certain compactness assumptions and domination properties.

1. INTRODUCTION

In the recent years, the study of set-valued optimization problems has attracted the attention of many authors due to its applications in diverse areas, for example, mathematical finance, game theory etc. see [1, 12] for details. Authors mainly use two criteria to evaluate the optimal solutions of set optimization problems namely, vector criterion and set criterion. The set criterion is more appropriate for the study of set-valued optimization problems [3, 7, 16, 29] as in this approach, the whole image set is compared rather than a point of set.

Stability aspects deal with the analysis of the behavior of solution sets of problems obtained by perturbing the given optimization problem. Different aspects of stability have been investigated in literature such as continuity of solution set maps (see [6, 24, 27, 28]), essential stability (see [25, 26]) and convergence of solution sets (see [4, 5, 9, 11, 17, 18, 19, 20]).

In vector optimization, Huang [9] investigated the Mosco and Painlevé–Kuratowski convergence of sequence of efficient and weak efficient solution sets by perturbing the feasible set and the objective function. Later, Lalitha and Chatterjee [17, 18] studied the Painlevé–Kuratowski convergence of weak efficient, efficient and Henig proper efficient solution sets using notions of continuous convergence and gamma convergence. Further, Li et al. [19] gave the stability results for solution sets by perturbing the ordering cone, the feasible set and the objective function. For set-valued optimization problems, the convergence aspect of stability with respect to vector criterion has been studied in literature. In this direction, Li et al. [20] generalized the stability results given in [17, 18]. Very recently, Gaydu et al. [4], perturbed the objective function and the feasible set and discussed the upper and lower stability of solution sets using gamma convergence of problems.

For set criterion, the convergence aspect of stability in image space is studied in terms of external and internal stability, introduced by Gutiérrez et al. [5]. In set optimization the collection of images of efficient and weak efficient points are family of sets and there

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is not much progress in the development of the theory for the convergence of a sequence of family of sets. To overcome this, Gutiérrez et al. [5] made an attempt to reformulate Painlevé–Kuratowski convergence for set optimization problems. They developed the concepts of external and internal stability by choosing a sequence of sets appropriately from the sequence of the family of sets. In case of external stability, the motive is to establish that every sequence of appropriately chosen solution sets of perturbed problems converge to solution set of original problem, where as in case of internal stability, every solution set of original problem is equivalently estimated by a subsequence of solution sets of perturbed problems. In fact, the external stability is a reformulation of the upper convergence whereas the internal stability is a reformulation of the lower convergence. Gutiérrez et al. [5] perturbed the feasible set and established the external stability of weak minimal solution sets and internal stability of minimal solution sets by employing the assumptions of Hausdorff continuity, compactness and domination properties. Later, Karuna and Lalitha [11], improvised the results given in [5] and discussed the external and internal stability of both minimal and weak minimal solution sets in terms of Hausdorff as well as Painlevé–Kuratowski sense. They also proved the complete Painlevé–Kuratowski convergence of efficient and weak efficient solution sets of perturbed problems.

In the present work, we consider the perturbed problems by perturbing objective maps and establish the stability results both in the decision space and the image space, by using a notion of gamma convergence. Certain compactness assumptions have been used to establish external stability of weak minimal solution sets and an additional strict quasi-convexity is used to establish the external stability for minimal solution sets. However, for the internal stability of minimal solution sets we require the assumption of domination properties as well. We also establish the complete Painlevé–Kuratowski convergence of sequence of weak efficient and efficient solution sets in the decision space.

The rest of the paper is organized as follows. Some basic definitions required in the sequel are recalled in Section 2. In Section 3, we study two types of domination properties and their relations. In Section 4, we introduce a notion of gamma convergence for a sequence of set-valued maps and establish the uniqueness of the limit. In Section 5, we investigate the external stability of weak minimal and minimal solution sets in the image space and the upper Painlevé–Kuratowski convergence of sequence of weak efficient solution sets in the decision space. In Section 6, we prove the internal stability of minimal and weak minimal solution sets in the image space and the lower Painlevé–Kuratowski convergence of sequence of efficient solution sets in the decision space. We give some concluding remarks in Section 7.

2. Preliminaries

Let X and Y be two metric spaces and $\mathcal{P}(Y)$ denote the family of all nonempty subsets of Y. Let $K \subseteq Y$ be a closed convex pointed cone with nonempty interior. We consider the following lower set order relations from [7, 14]. For $M, N \in \mathcal{P}(Y)$,

$$M \preceq^l N \text{ if and only if } N \subseteq M + K,$$

$$M \prec^l N \text{ if and only if } N \subseteq M + \text{int}K,$$

where $\text{int}K$ denotes the interior of K. Also,

$$M \sim^l N \text{ if and only if } M \preceq^l N \text{ and } N \preceq^l M.$$

It can be seen that $M \sim^l N$ if and only if $M + K = N + K$. Clearly, the relation $\sim^l$ is an equivalence relation. For more details, we refer to [7, 15]. We denote the relation $\preceq^l$ ($\prec^l$) by $\leq_K$ ($<_K$), if M and N are singletons.

We now recall the notion of Painlevé–Kuratowski set convergence from [12]. A sequence $\{A_n\}$ of nonempty subsets of X is said to converge to a nonempty subset A of X in Painlevé–Kuratowski sense (denoted by $A_n \xrightarrow{K} A$) if

$$\mathrm{Ls}(A_n) \subseteq A \subseteq \mathrm{Li}(A_n),$$

where the upper limit $\mathrm{Ls}(A_n)$ and the lower limit $\mathrm{Li}(A_n)$ are defined as

$$\mathrm{Ls}(A_n) := \{x \in X : x = \lim_{k\to\infty} x_k,\ x_k \in A_{n_k}, \{n_k\} \text{ is a subsequence of } \{n\}\},$$

$$\mathrm{Li}(A_n) := \{x \in X : x = \lim_{n\to\infty} x_n,\ x_n \in A_n \text{ for sufficiently large } n\},$$

respectively. The inclusion $\mathrm{Ls}(A_n) \subseteq A$ is called the upper Painlevé–Kuratowski convergence (denoted by $A_n \xrightharpoonup{K} A$) and $A \subseteq \mathrm{Li}(A_n)$ is called the lower Painlevé–Kuratowski convergence (denoted by $A_n \xrightharpoondown{K} A$).

In this paper, we consider the set-valued optimization problem

$$\text{(P)} \qquad \text{Min } F(x)$$
$$\text{subject to } x \in S,$$

where $F : S \rightrightarrows Y$ is a set-valued map and S is a nonempty subset of X. We denote the family of image sets of F by $\mathcal{F}$, that is, $\mathcal{F} = \{F(x) : x \in S\}$.

We next recall the notions of l-efficient and l-weak efficient solutions for problem (P).

Definition 2.1. [14, Definition 2.1] A point $\bar{x} \in S$ is said to be

(a) an l-efficient solution of (P) if

$$F(x) \preceq^l F(\bar{x}), x \in S \Rightarrow F(\bar{x}) \preceq^l F(x);$$

(b) an l-weak efficient solution of (P) if there does not exist any $x \in S$ such that $F(x) \prec^l F(\bar{x})$.

We say that $F(\bar{x})$ is an l-minimal (l-weak minimal) solution set of (P), if $\bar{x}$ is an l-efficient (l-weak efficient) solution of (P). We denote the sets of l-efficient and l-weak efficient solutions of (P) by l-Eff($\mathcal{F}$) and l-WEff($\mathcal{F}$), respectively. The sets of l-minimal and l-weak minimal solution sets of (P) are denoted by l-Min($\mathcal{F}$) and l-WMin($\mathcal{F}$), respectively.

For a single-valued map f, we observe that the notions of l-Min($\mathcal{F}$) and l-WMin($\mathcal{F}$) reduce to the notions of minimal and weak minimal points given in [22]. We denote the sets of minimal and weak minimal points by $\mathrm{Min}(f(S))$ and $\mathrm{WMin}(f(S))$, respectively, where $f(S) = \{f(x) : x \in S\}$.

It can be seen that, if $x \in S$ is an l-efficient solution of (P) then it is an l-weak efficient solution of (P) provided $\mathrm{WMin}(F(x)) \neq \emptyset$. However, the converse holds under the following assumption of l-K-strict quasiconvexity.

Definition 2.2. [10, Definition 2.6] Let S be a nonempty convex subset of X and $\bar{x} \in S$. Then F is said to be l-K-strictly quasiconvex at $\bar{x}$ if for any $x \in S$ with $x \neq \bar{x}$, $\lambda \in (0, 1)$

$$F(x) \preceq^l F(\bar{x}) \Rightarrow F(\lambda x + (1-\lambda)\bar{x}) \prec^l F(\bar{x}).$$

The map F is said to be l-K-strictly quasiconvex on S if F is l-K-strictly quasiconvex at every $x \in S$.

The following conclusion is trivial.

Lemma 2.1. *If S is a convex subset of X and F is l-K-strictly quasiconvex on S, then every l-weak efficient solution of (P) is an l-efficient solution of (P).*

Remark 2.1. From Lemma 2.1, it follows that if WMin$(F(x)) \neq \emptyset$ for every $x \in S$, S is a convex subset of X and F is l-K-strictly quasiconvex on S, then l-WEff$(\mathcal{F}) = l$-Eff$(\mathcal{F})$.

We next consider a family of perturbed set optimization problems by perturbing the objective map F by a sequence $\{F_n\}$ of set-valued maps

$$(P_n) \qquad \text{Min } F_n(x)$$
$$\text{subject to } x \in S,$$

where $F_n : S \rightrightarrows Y$. We denote the family of image sets of F_n by $\mathcal{F}_n$.

Throughout this paper, we assume that for each $x \in S$ and $n \in \mathbb{N}$ the sets $F(x)$, $F_n(x)$, l-Eff$(\mathcal{F})$, l-WEff$(\mathcal{F})$, l-Eff$(\mathcal{F}_n)$ and l-WEff$(\mathcal{F}_n)$ are nonempty.

3. Domination properties

In this section, we consider two notions of domination properties and discuss their relations. We first recall the following notion of l-domination property from [8].

Definition 3.3. [8, Definition 4.3] The family of sets $\mathcal{F}$ is said to satisfy l-domination property if for each $x \in S$, there exists $\bar{x} \in l$-Eff$(\mathcal{F})$ such that $F(\bar{x}) \preceq^l F(x)$.

Now, we introduce the following weak domination property.

Definition 3.4. The family of sets $\mathcal{F}$ is said to satisfy l-weak domination property if for each $x \in S$, either $x \in l$-WEff$(\mathcal{F})$ or there exists $\bar{x} \in l$-WEff$(\mathcal{F})$ such that $F(\bar{x}) \prec^l F(x)$.

The next two examples illustrate that l-domination and l-weak domination properties are not related to each other.

Example 3.1. Let $X = \mathbb{R}$, $S = [0,1]$, $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. Let $F : S \rightrightarrows Y$ be a set-valued map defined as

$$F(x) = \begin{cases} (0,1] \times (0,1], & \text{if } 0 \leq x \leq \frac{1}{2}, \\ (0,2] \times (0,2], & \text{if } \frac{1}{2} < x < 1, \\ \{(1,-1)\}, & \text{if } x = 1. \end{cases}$$

Clearly, $\mathcal{F}$ satisfies l-domination property, but not l-weak domination property as $x \notin l$-WEff$(\mathcal{F})$ for $x \in [0,1)$ and there does not exist any $\bar{x} \in l$-WEff$(\mathcal{F})$ such that $F(\bar{x}) \prec^l F(x)$.

Example 3.2. Let $X = \mathbb{R}$, $S = [0,1]$, $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. Let $F : S \rightrightarrows Y$ be a set-valued map defined as

$$F(x) = \begin{cases} [0,x] \times [0,-x], & \text{if } 0 \leq x < 1, \\ \{(-1,1)\}, & \text{if } x = 1. \end{cases}$$

It can be seen that $\mathcal{F}$ satisfies l-weak domination property, but not l-domination property as $x \notin l$-Eff$(\mathcal{F})$ for $x \in [0,1)$ and there does not exist any $\bar{x} \in l$-Eff$(\mathcal{F})$ such that $F(\bar{x}) \preceq^l F(x)$.

The proof of the following theorem, which relates the above defined notions of domination properties under the assumption of l-K-strict quasiconvexity, is trivial.

Theorem 3.1. *For $\mathcal{F} \subseteq \mathcal{P}(Y)$ the following hold.*

(*i*) *If F is l-K-strictly quasiconvex on a convex subset S of X and $\mathcal{F}$ satisfies l-weak domination property, then $\mathcal{F}$ satisfies l-domination property.*

(*ii*) *If $\text{WMin}(F(x)) \neq \emptyset$ for every $x \in S$ and $\mathcal{F}$ satisfies l-domination property, then $\mathcal{F}$ satisfies l-weak domination property.*

4. CONVERGENCE OF SET-VALUED MAPS

In this section, we consider a notion of gamma convergence (denoted by Γ-convergence) for a sequence of set-valued maps. In the sequel, F_n, $F : S \rightrightarrows Y$, $n \in \mathbb{N}$ are set-valued maps and f_n, $f : S \to Y$, $n \in \mathbb{N}$ are vector-valued functions unless specified otherwise.

Definition 4.5. The sequence $\{F_n\}$ is said to Γ-converge to F, denoted by $F_n \overset{\Gamma}{\to} F$, if the following hold.

(a) For every $x \in S$, there exists a sequence $\{x_n\} \subseteq S$, $x_n \to x$ such that $F_n(x_n) \overset{K}{\to} F(x)$.

(b) For every $x \in S$, $\{x_n\} \subseteq S$ with $x_n \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $F_{n_k}(x_{n_k}) \overset{K}{\to} F(x)$.

In finite dimensional spaces, Huang [9] defined the following notion of convergence of a sequence of set-valued maps in terms of convergence of its epigraphs.

Definition 4.6. [9, Definition 2.5] Let $F_n, F : \mathbb{R}^k \rightrightarrows \mathbb{R}^l$, $n \in \mathbb{N}$ be set-valued maps. The sequence $\{F_n\}$ is said to epi-converge to F if, epi$F_n \overset{K}{\to}$ epi$F$, where epi$F := \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^l : y \in F(x) + K\}$.

It can be seen that for $S = \mathbb{R}^k$, $Y = \mathbb{R}^l$, the notion of Γ-convergence in Definition 4.5 reduces to the notion of epi-convergence given in Definition 4.6.

We next deduce the notion of Γ-convergence for vector-valued maps from Definition 4.5.

Definition 4.7. The sequence $\{f_n\}$ is said to Γ-converge to f, denoted by $f_n \overset{\Gamma}{\to} f$, if the following hold.

(a) For every $x \in S$, there exists a sequence $\{x_n\} \subseteq S$, $x_n \to x$ such that $f_n(x_n) \to f(x)$.

(b) For every $x \in S$, $\{x_n\} \subseteq S$ with $x_n \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $f_{n_k}(x_{n_k}) \to f(x)$.

The following result characterizes the gamma convergence of vector-valued maps.

Theorem 4.2. *If $f_n \overset{\Gamma}{\to} f$, $\{x_n\} \subseteq S$ is a sequence with $x_n \to x \in S$, $\varepsilon \in \text{int}K$, then $f_n(x_n) - f(x) + \varepsilon \in K$ for sufficiently large n.*

Proof. On the contrary, assume that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$f_{n_k}(x_{n_k}) - f(x) + \varepsilon \notin K. \tag{4.1}$$

As the sequence $\{f_{n_k}\}$ gamma converges to f , it follows that there exists a further subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $f_{n_{k_l}}(x_{n_{k_l}}) \to f(x)$. Since $f(x) + \text{int}K - \varepsilon$ is an open set containing $f(x)$, it follows that $f_{n_{k_l}}(x) \in f(x) + \text{int}K - \varepsilon$ for sufficiently large l, which contradicts (4.1). □

Remark 4.2. Theorem 4.2 implies that the above notion of gamma convergence for vector-valued maps reduces to a similar notion considered by Oppezzi and Rossi [23, Definition 2.7].

We have the following result for gamma convergent sequences of set-valued maps.

Theorem 4.3. *Every* Γ*-convergent sequence has a unique limit.*

Proof. Let $\{F_n\}$ be a Γ-convergent sequence such that $F_n \xrightarrow{\Gamma} P$ and $F_n \xrightarrow{\Gamma} Q$, where $P, Q : S \rightrightarrows Y$. Let $x \in S$. Since $F_n \xrightarrow{\Gamma} P$, therefore there exists a sequence $\{x_n\} \subseteq S$, $x_n \to x$ such that

$$F_n(x_n) \xrightarrow{K} P(x). \tag{4.2}$$

Also, as $F_n \xrightarrow{\Gamma} Q$ and $x_n \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$F_{n_k}(x_{n_k}) \xrightarrow{K} Q(x). \tag{4.3}$$

It is enough to show that $P(x) = Q(x)$. Let $y \in P(x)$. By (4.2), it follows that $y \in \text{Li}(F_n(x_n))$, that is, there exists $y_n \in F_n(x_n)$ for all n such that $y_n \to y$. Hence, for every subsequence $\{n_{k_l}\}$ of $\{n_k\}$, we have $y_{n_{k_l}} \to y$. Thus from (4.3), it is clear that $y \in \text{Ls}(F_{n_k}(x_{n_k})) \subseteq Q(x)$. For the reverse inclusion, consider $y \in Q(x)$. By (4.3), it follows that $y \in \text{Li}(F_{n_k}(x_{n_k}))$, that is, there exists $y_{n_k} \in F_{n_k}(x_{n_k})$ for all k such that $y_{n_k} \to y$ and so $y \in \text{Ls}(F_n(x_n))$. From (4.2), we obtain that $y \in P(x)$. □

5. External stability

In this section, we discuss the external stability of l-weak minimal solution sets and also give sufficient conditions for the upper Painlevé–Kuratowski convergence of a sequence of l-weak efficient solution sets of the perturbed problems (P_n) to the l-weak efficient solution set of the problem (P). The results are further extended for l-minimal and l-efficient solution sets under l-K-strict quasiconvexity assumption.

In the next theorem, we establish the external stability for l-weak minimal solution set in Painlevé–Kuratowski sense.

Theorem 5.4. *If the following conditions hold:*

(*i*) *S is compact;*

(*ii*) $F_n \xrightarrow{\Gamma} F$;

(*iii*) $\text{cl}(\bigcup_{n\in\mathbb{N}} (\bigcup_{x\in S} (F_n(x))))$ *is compact;*

then for $F_n(x_n) \in l\text{-WMin}(\mathcal{F}_n)$, $n \in \mathbb{N}$, *there exist a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *and* $x \in S$ *such that* $x_{n_k} \to x$, $F_{n_k}(x_{n_k}) \xrightarrow{K} F(x)$ *and* $F(x) \in l\text{-WMin}(\mathcal{F})$.

Proof. Since $\{x_n\} \subseteq S$ and S is compact, therefore $\{x_n\}$ has a convergent subsequence. Without loss of generality, we assume that $x_n \to x$. Since $F_n \xrightarrow{\Gamma} F$ and $x \in S$, therefore there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $F_{n_k}(x_{n_k}) \xrightarrow{K} F(x)$.

We claim that $F(x) \in l\text{-WMin}(\mathcal{F})$. Suppose on the contrary, there exists $u \in S$ such that $F(x) \subseteq F(u) + \text{int}K$. As $F_n \xrightarrow{\Gamma} F$, therefore there exists a sequence $\{u_n\} \subseteq S$, $u_n \to u$ such that $F_n(u_n) \xrightarrow{K} F(u)$. Since $F_{n_k}(x_{n_k}) \in l\text{-WMin}(\mathcal{F}_{n_k})$ for all k, thus $F_{n_k}(x_{n_k}) \nsubseteq F_{n_k}(u_{n_k}) + \text{int}K$. Therefore, there exists $y_{n_k} \in F_{n_k}(x_{n_k})$ for all k such that

$$y_{n_k} \notin F_{n_k}(u_{n_k}) + \text{int}K, \text{ for all } k, \tag{5.4}$$

that is, $y_{n_k} \notin z_{n_k} + \text{int}K$, for any $z_{n_k} \in F_{n_k}(u_{n_k})$. Now, as $\{y_{n_k}\} \subseteq \text{cl}(\bigcup_{n\in\mathbb{N}} (\bigcup_{x\in S} (F_n(x))))$, thus by condition (iii), there exist a subsequence $\{y_{n_{k_l}}\}$ of $\{y_{n_k}\}$ and $y \in Y$ such that $y_{n_{k_l}} \to y$. Clearly, $y \in \text{Ls}(F_{n_k}(x_{n_k}))$. Since $\text{Ls}(F_{n_k}(x_{n_k})) \subseteq F(x) \subseteq F(u) + \text{int}K$, we have $y \in F(u) + \text{int}K$. Therefore, there exists $z \in F(u)$ such that $y - z \in \text{int}K$. Since $F_n(u_n) \xrightarrow{K} F(u)$, it follows that there exists $z_{n_{k_l}} \in F_{n_{k_l}}(u_{n_{k_l}})$ such that $z_{n_{k_l}} \to z$. Clearly,

$y_{n_{k_l}} - z_{n_{k_l}} \to y - z$. As int$K$ is an open set and $y - z \in$ int$K$, we have $y_{n_{k_l}} \in F_{n_{k_l}}(u_{n_{k_l}}) +$ int$K$, for sufficiently large l, which contradicts (5.4). □

Using the set criterion for solution of set-valued optimization problems, Gutiérrez et al. [5] and Karuna and Lalitha [11] perturbed the feasible set and established the similar kind of external stability by employing the notions of Hausdorff continuity and compactness of the objective map. Whereas by using the vector criterion, Li et al. [20] established the upper Painlevé–Kuratowski convergence for weak minimal solution sets in finite dimensional spaces.

Remark 5.3. Even though Γ-convergence of set-valued maps is stronger than epi-convergence in finite dimensional spaces, it can be seen from the following example that the Theorem 5.4 holds only under the stronger assumption of Γ-convergence.

Example 5.3. Let $X = \mathbb{R}$, $S = [-1, 1]$, $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. We consider the maps $F_n, F : S \rightrightarrows Y$, $n \in \mathbb{N}$ defined as

$$F(x) = \begin{cases} [-1, 0] \times [0, 1], & \text{if } -1 \leq x < 0, \\ [0, 1] \times [-1, 0], & \text{if } 0 \leq x \leq 1, \end{cases}$$

$$F_n(x) = \begin{cases} [-1, 0] \times [\frac{1}{n}, 1], & \text{if } -1 \leq x < 0, \\ [0, 1] \times [-1, \frac{1}{n}], & \text{if } 0 \leq x \leq 1. \end{cases}$$

We observe that epi$F_n \xrightarrow{K}$ epi$F$, but $F_n \overset{\Gamma}{\nrightarrow} F$ because for $x_n = -\frac{1}{n}$ and $x = 0$, we have $x_n \to x$, but $F_{n_k}(x_{n_k}) \overset{K}{\nrightarrow} F(x)$ for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Clearly, l-WMin$(\mathcal{F}) = \{F(x) : x \in [-1, 1]\}$ and l-WMin$(\mathcal{F}_n) = \{F_n(x) : x \in [-1, 1]\}$. It can be seen that for $x_n = -\frac{1}{n}$ the conclusion of Theorem 5.4 fails to hold.

The above theorem reduces to the following for vector-valued functions.

Theorem 5.5. *If the following conditions hold:*

(*i*) *S is compact;*

(*ii*) $f_n \xrightarrow{\Gamma} f$;

then for $f_n(x_n) \in$ WMin$(f_n(S))$, $n \in \mathbb{N}$ there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in S$ such that $x_{n_k} \to x$, $f_{n_k}(x_{n_k}) \to f(x)$ and $f(x) \in$ WMin$(f(S))$.

We can conclude the upper convergence of weak minimal solution sets from the above theorem.

Corollary 5.1. *If all the conditions of Theorem 5.5 hold, then* WMin$(f_n(S)) \xrightharpoonup{K}$ WMin$(f(S))$.

Lalitha and Chatterjee [18], established the upper Painlevé–Kuratowski convergence of sequence of weak minimal solution sets of (P_n) to weak minimal solution set of (P) by perturbing both the feasible set and the objective function in finite dimensional spaces. Assumptions considered in [18, Theorem 3.1] with $S_n = S$ for all $n \in \mathbb{N}$, are different from those in Corollary 5.1.

The next example illustrates that the condition of gamma convergence cannot be relaxed in Theorem 5.4.

Example 5.4. Let $X = Y = \mathbb{R}$, $S = [0, 1]$ and $K = \mathbb{R}_+$. We consider the maps $F_n, F : S \rightrightarrows Y$, $n \in \mathbb{N}$ defined as

$$F(x) = \begin{cases} \{1\}, & \text{if } x = 0, \\ [0, 1], & \text{if } 0 < x \leq 1, \end{cases}$$

$$F_n(x) = \begin{cases} [\frac{1}{n}, 1 + \frac{1}{n}], & \text{if } n \text{ is even,} \\ \{1\}, & \text{if } n \text{ is odd.} \end{cases}$$

We observe that $F_n \overset{\Gamma}{\nrightarrow} F$ as for any $x \in S$ and $x_n \to x$, we have $F_n(x_n) \overset{K}{\nrightarrow} F(x)$. Clearly, l-WMin$(\mathcal{F}) = \{F(x) : x \in (0, 1]\}$ and l-WMin$(\mathcal{F}_n) = \{F_n(x) : x \in [0, 1]\}$. We observe that Theorem 5.4 fails to hold for $x_n = \frac{1}{n}$.

The next example illustrates that Theorem 5.4 fails to hold if $\text{cl}(\bigcup\limits_{n\in\mathbb{N}} (\bigcup\limits_{x\in S} (F_n(x))))$ is not compact.

Example 5.5. Let $X = Y = l^2$, $K = \{x \in l^2 : x_1 \geq 0 \text{ and } \sum_{k=2}^{\infty} {x_k}^2 \leq {x_1}^2\}$. Let $e_0 = (0, 0, 0, ...)$, $e_i = (0, 0, .., 0, 1, 0, ...)$ with 1 at the i^{th} place and $S = \{e_0, -e_1\}$. We consider the maps $F_n, F : S \rightrightarrows Y$ defined as

$$F(x) = \begin{cases} \{e_0, -2\sum_{i=1}^{\infty} e_i\}, & \text{if } x = e_0, \\ \{-(e_1 + 2\sum_{i=2}^{\infty} e_i)\}, & \text{if } x = -e_1, \end{cases}$$

$$F_n(x) = \begin{cases} \{e_0, -2\sum_{i=1}^{\infty} e_i\}, & \text{if } x = e_0, \\ \{-e_n, -(e_1 + 2\sum_{i=2}^{\infty} e_i)\}, & \text{if } x = -e_1. \end{cases}$$

We observe that all the conditions of Theorem 5.4 are satisfied but $\text{cl}(\bigcup\limits_{n\in\mathbb{N}} (\bigcup\limits_{x\in S} (F_n(x))))$ is not compact as the sequence $\{-e_n\}$ does not have any convergent subsequence. Clearly, l-WMin$(\mathcal{F}) = \{F(e_0)\}$ and l-WMin$(\mathcal{F}_n) = \{F_n(e_0), F_n(-e_1)\}$. It can be seen that for $x_n = -e_1$, the conclusion of Theorem 5.4 fails to hold.

We next establish the convergence of a sequence of l-weak efficient solution sets in the upper Painlevé–Kuratowski sense.

Corollary 5.2. *If all the conditions of Theorem 5.4 hold, then l-*WEff$(\mathcal{F}_n) \overset{K}{\rightharpoonup}$ *l-*WEff$(\mathcal{F})$.

Proof. Let $x \in$ Ls(l-WEff$(\mathcal{F}_n)$). Thus, there exists $x_{n_k} \in l$-WEff$(\mathcal{F}_{n_k})$ for all k such that $x_{n_k} \to x$. Now, as $F_{n_k}(x_{n_k}) \in l$-WMin$(\mathcal{F}_{n_k})$ for all k, therefore by Theorem 5.4, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $F_{n_{k_l}}(x_{n_{k_l}}) \overset{K}{\to} F(x)$ and $F(x) \in l$-WMin$(\mathcal{F})$. Hence, $x \in l$-WEff$(\mathcal{F})$. □

Remark 5.4. (i) We observe from Example 5.4 that Corollary 5.2 fails to hold if $F_n \overset{\Gamma}{\nrightarrow} F$. In this example, it can be seen that $0 \in$ Ls(l-WEff$(\mathcal{F}_n)$), but $0 \notin l$-WEff$(\mathcal{F})$.

(ii) We refer to Example 5.5 to see that Corollary 5.2 fails to hold if $\text{cl}(\bigcup\limits_{n\in\mathbb{N}} (\bigcup\limits_{x\in S} (F_n(x))))$ is not compact. In this example, $-e_1 \in$ Ls(l-WEff$(\mathcal{F}_n)$), but $-e_1 \notin l$-WEff$(\mathcal{F})$.

An additional assumptions of l-K-strict quasiconvexity is imposed to establish the upper Painlevé–Kuratowski convergence of l-efficient solution sets.

Corollary 5.3. *If the conditions* (i)-(iii) *of Theorem 5.4 hold and*

(*iv*) *S is a convex subset of X and F is l-K-strictly quasiconvex on S;*

(*v*) *for every $x \in S$ and $n \in \mathbb{N}$,* WMin$(F_n(x)) \neq \emptyset$;

*then l-*Eff$(\mathcal{F}_n) \overset{K}{\rightharpoonup}$ *l-*Eff$(\mathcal{F})$.

Proof. Let $x \in$ Ls(l-Eff$(\mathcal{F}_n)$). As WMin$(F_n(x)) \neq \emptyset$, for every $x \in S$ and $n \in \mathbb{N}$, therefore $x \in$ Ls(l-WEff$(\mathcal{F}_n)$). Using Corollary 5.2, we obtain that $x \in l$-WEff$(\mathcal{F})$. As F is l-K-strictly quasiconvex on S, therefore by Lemma 2.1, it follows that $x \in l$-Eff$(\mathcal{F})$. □

The following example illustrates that the assumption of l-K-strict quasiconvexity of F on S cannot be dropped in Corollary 5.3.

Example 5.6. Let $X = \mathbb{R}$, $S = [-1, 1]$, $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. We consider the maps $F_n, F : S \rightrightarrows Y$ defined as $F(x) = [x, 1] \times \{0\}$ and $F_n(x) = [x + \frac{1}{n}, 1] \times \{-\frac{1+x}{n}\}$. It can be seen that F is not l-K-strictly quasiconvex on S, but all other conditions of Corollary 5.3 are satisfied. Clearly, l-Eff$(\mathcal{F}) = \{-1\}$ and l-Eff$(\mathcal{F}_n) = [-1, 1]$. Here the conclusion of Corollary 5.3 fails to hold as $1 \in$ Ls(l-Eff$(\mathcal{F}_n)$), but $1 \notin l$-Eff$(\mathcal{F})$.

We now have the following external stability result for l-minimal solution sets.

Corollary 5.4. *If all the conditions of Corollary 5.3 hold, then for $F_n(x_n) \in l$-Min$(\mathcal{F}_n)$, $n \in \mathbb{N}$, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in S$ such that $x_{n_k} \to x$, $F_{n_k}(x_{n_k}) \overset{K}{\to} F(x)$ and $F(x) \in l$-Min$(\mathcal{F})$.*

Proof. Since for every $x \in S$, WMin$(F_n(x)) \neq \emptyset$, $n \in \mathbb{N}$, it follows that $F_n(x_n) \in l$-WMin$(\mathcal{F}_n)$, $n \in \mathbb{N}$. Thus by Theorem 5.4, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in S$ such that $F_{n_k}(x_{n_k}) \overset{K}{\to} F(x)$ and $F(x) \in l$-WMin$(\mathcal{F})$. Since F is l-K-strictly quasiconvex on S, therefore by Lemma 2.1, we obtain that $F(x) \in l$-Min$(\mathcal{F})$. □

Remark 5.5. From Example 5.6, we see that the assumption of l-K-strict qusiconvexity of F on S cannot be dropped from Corollary 5.4. In this example, the conclusion fails to hold for $x_n = 1$.

6. Internal stability

In this section, we deal with the internal stability of l-minimal solution sets and provide sufficient conditions that ensure the lower Painlevé–Kuratowski convergence of l-efficient solution sets of (P_n). The results have also been proved for l-weak minimal and l-weak efficient solution sets.

In the following result, we establish the internal stability of l-minimal solution sets.

Theorem 6.6. *If the conditions* (i)-(iii) *of Theorem 5.4 hold and*

(iv) $\mathcal{F}_n$, $n \in \mathbb{N}$ satisfies l-domination property on S;

then for $F(x) \in l$-Min$(\mathcal{F})$, there exist a sequence $\{u_n\} \subseteq S$, a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in S$ such that $u_{n_k} \to u$, $F_{n_k}(u_{n_k}) \in l$-Min$(\mathcal{F}_{n_k})$ for all k, $F_{n_k}(u_{n_k}) \overset{K}{\to} F(u)$ and $F(u) \sim^l F(x)$.

Proof. As $x \in S$ and $F_n \overset{\Gamma}{\to} F$, therefore there exists a sequence $\{x_n\} \subseteq S$, $x_n \to x$ such that $F_n(x_n) \overset{K}{\to} F(x)$. Since $\mathcal{F}_n$, $n \in \mathbb{N}$ satisfies l-domination property on S, therefore there exists a sequence $\{u_n\} \subseteq S$ such that $u_n \in l$-Eff$(\mathcal{F}_n)$ and

$$F_n(x_n) \subseteq F_n(u_n) + K. \tag{6.5}$$

Since S is compact, therefore the sequence $\{u_n\}$ has a convergent subsequence. Without loss of generality, we assume that $u_n \to u$. Clearly, $u \in S$. Again as $F_n \overset{\Gamma}{\to} F$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $F_{n_k}(u_{n_k}) \overset{K}{\to} F(u)$. Next, we show that $F(u) \sim^l F(x)$. As $F(x) \in l$-Min$(\mathcal{F})$, it is enough to show that $F(x) \subseteq F(u) + K$.

Let $y \in F(x)$. Since $F(x) \subseteq$ Li$(F_n(x_n))$, it follows that there exists a sequence $y_n \in F_n(x_n)$ for all n such that $y_n \to y$. From (6.5), it follows that there exists $z_{n_k} \in F_{n_k}(u_{n_k})$ for all k such that $y_{n_k} - z_{n_k} \in K$. Since $\{z_{n_k}\}$ is in the compact set cl$(\bigcup_{n\in\mathbb{N}} (\bigcup_{x\in S} (F_n(x))))$, therefore there exist a convergent subsequence $\{z_{n_{k_l}}\}$ of $\{z_{n_k}\}$ and $z \in Y$ such that $z_{n_{k_l}} \to z$. Clearly, $z \in$ Ls$(F_{n_k}(u_{n_k})) \subseteq F(u)$. As $y_{n_{k_l}} - z_{n_{k_l}} \in K$ and K is closed, therefore we have $y - z \in K$, which implies that $F(x) \subseteq F(u) + K$. □

However, Gutiérrez et al. [5] considered the perturbation of feasible set and proved the internal stability of minimal solution sets under the assumption of K-conical closed families of sets. Further, Karuna and Lalitha [11] investigated the internal stability with the weaker assumptions of l-domination property and Hausdorff continuity of the objective map. In finite dimensional spaces Li et al. [20] used the vector criteria for solution of set-valued optimization problem and proved the lower Painlevé–Kuratowski convergence by employing the different assumptions on recession cone and quasiconvexity of set-valued map.

We next recall the domination property for a single-valued map from [21].

Definition 6.8. [21, Definition 4.1]The map f is said to have domination property on S if for each $x \in S$, there exists $\bar{x} \in \text{Eff}(f(S))$ such that $f(\bar{x}) \leq_K f(x)$.

Theorem 6.6 reduces to the following for vector-valued functions.

Theorem 6.7. *If the conditions* (i) and (ii) *of Theorem 5.5 hold and*

(iii) *f_n, $n \in \mathbb{N}$ has domination property on S;*

then for $f(x) \in \text{Min}(f(S))$, there exist a sequence $\{u_n\} \subseteq S$, a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in S$ such that $u_{n_k} \to u$, $f_{n_k}(u_{n_k}) \in \text{Min}(f_{n_k}(S))$ for all k, $f_{n_k}(u_{n_k}) \to f(u)$ and $f(u) = f(x)$.

The following result is immediate from the above theorem.

Corollary 6.5. *If all the conditions of Theorem 6.7 hold, then* $\text{Eff}(f_n(S)) \xrightarrow{K} \text{Eff}(f(S))$.

In finite dimensional spaces Lalitha and Chatterjee [18] considered problems by perturbing both the feasible set and the objective function of a vector optimization problem and established the lower Painlevé–Kuratowski convergence of a sequence of minimal solutions of perturbed problems. However, we observe that in [18, Theorem 3.4] lower Painlevé–Kuratowski convergence is proved under different assumptions.

The following examples justify that the assumption of gamma convergence cannot be dropped in Theorem 6.6.

Example 6.7. Let $X = \mathbb{R}$, $S = [-1, 1]$, $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. We consider the maps $F_n, F : S \rightrightarrows Y$ defined as

$$F(x) = \begin{cases} [0,1] \times \{1\}, & \text{if } -1 \leq x \leq 0, \\ [x, 1-x] \times \{0\}, & \text{if } 0 < x \leq \frac{1}{2}, \\ [1-x, x] \times \{0\}, & \text{if } \frac{1}{2} < x \leq 1, \end{cases}$$

$$F_n(x) = \begin{cases} [0,1] \times \{1\}, & \text{if } -1 \leq x < 0, \\ [x, 1-x] \times \{0\}, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ (1-x, x) \times \{0\}, & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

We observe that $F_n \overset{\Gamma}{\nrightarrow} F$ as $x_n \to x$ for $x = 0$ and $x_n = \frac{1}{n}$, but there does not exist any subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $F_{n_k}(x_{n_k}) \xrightarrow{K} F(x)$. Clearly, l-Min$(\mathcal{F}) = \{F(1)\}$ and l-Min$(\mathcal{F}_n) = \{F_n(0)\}$. It can be seen that for $x = 1$, $F(x) \in l$-Min$(\mathcal{F})$, but the conclusion of Theorem 6.6 fails to hold.

Example 6.8. Let $X = Y = \mathbb{R}$, $S = [0, 1]$ and $K = \mathbb{R}_+$. We consider the maps $F_n, F : S \rightrightarrows Y$, $n \in \mathbb{N}$ defined as

$$F(x) = \begin{cases} [0,1], & \text{if } x = 0, \\ \{1\}, & \text{if } x \neq 0, \end{cases}$$

$$F_n(x) = \begin{cases} [0,1], & \text{if } x = \frac{1}{2}, n \text{ is even,} \\ [\frac{1}{n}, 1], & \text{if } x \neq \frac{1}{2}, 1, n \text{ is even,} \\ \{1\}, & \text{if } x = 1, n \text{ is even,} \\ \{1\}, & \text{if } x \neq 1, n \text{ is odd,} \\ [0,1], & \text{if } x = 1, n \text{ is odd.} \end{cases}$$

We observe that all the conditions of Theorem 6.6 are satisfied but $F_n \overset{\Gamma}{\nrightarrow} F$ as for any $x \in S$ and $x_n \to x$, we have $F_n(x_n) \overset{K}{\nrightarrow} F(x)$. Clearly, l-Min$(\mathcal{F}) = \{F(0)\}$ and

$$l\text{-Min}(\mathcal{F}_n) = \begin{cases} F_n(\frac{1}{2}), & \text{if } n \text{ is even,} \\ F_n(1), & \text{if } n \text{ is odd.} \end{cases}$$

It can be seen that for $x = 0$, the conclusion of Theorem 6.6 fails to hold.

The next example shows that the assumption that $\text{cl}(\bigcup_{n\in\mathbb{N}} (\bigcup_{x\in S} (F_n(x))))$ is compact, is essential in Theorem 6.6.

Example 6.9. Let $X = Y = l^2$, $K = \{x \in l^2 : x_1 \geq 0$ and $\sum_{k=2}^{\infty} {x_k}^2 \leq {x_1}^2\}$. Let $e_0 = (0,0,0,...)$, $e_i = (0,0,..,0,1,0,...)$ with 1 at the i^{th} place and $S = \{e_0, -e_1\}$. We consider the maps $F_n, F : S \rightrightarrows Y$ defined as

$$F(x) = \begin{cases} \{e_1, -(e_1 + 2\sum_{i=2}^{\infty} e_i)\}, & \text{if } x = e_0, \\ \{-(e_1 + 2\sum_{i=2}^{\infty} e_i)\}, & \text{if } x = -e_1, \end{cases}$$

$$F_n(x) = \begin{cases} \{(1 + \frac{1}{n})e_1, -(e_1 + 2\sum_{i=2}^{\infty} e_i)\}, & \text{if } x = e_0, \\ \{\frac{1}{n}e_1 - e_n, -(e_1 + 2\sum_{i=2}^{\infty} e_i)\}, & \text{if } x = -e_1. \end{cases}$$

It can be seen that $\text{cl}(\bigcup_{n\in\mathbb{N}} (\bigcup_{x\in S} (F_n(x))))$ is not compact as $\{\frac{1}{n}e_1 - e_n\}$ does not have any convergent subsequence. Clearly, l-Min$(\mathcal{F}) = \{F(e_0)\}$ and l-Min$(\mathcal{F}_n) = \{F_n(-e_1)\}$. It can be seen that the conclusion of Theorem 6.6 fails to hold for $x = e_0$.

We next establish the lower Painlevé–Kuratowski convergence of a subsequence of l-efficient solution sets.

Corollary 6.6. *If the conditions* (i)-(iv) *of Theorem 6.6 hold and*

(*v*) *for every $x \in l$-Eff$(\mathcal{F})$, $F(x) \nsim^l F(u)$ for any $u \neq x$;*

then there exist a subsequence $\{n_k\}$ of $\{n\}$ such that l-Eff$(\mathcal{F}_{n_k}) \overset{K}{\to} l$-Eff$(\mathcal{F})$.

Proof. Let $x \in l$-Eff$(\mathcal{F})$. Then by Theorem 6.6, it follows that there exist a sequence $\{u_n\} \subseteq S$, a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in S$ such that $u_{n_k} \to u$, $F_{n_k}(u_{n_k}) \in l$-Min$(\mathcal{F}_{n_k})$, $F_{n_k}(u_{n_k}) \overset{K}{\to} F(u)$ and $F(u) \sim^l F(x)$. Using condition (v), we obtain that $u = x$. Thus $u_{n_k} \to x$ and hence l-Eff$(\mathcal{F}) \subseteq$ Li$(l$-Eff$(\mathcal{F}_{n_k}))$. □

The next example illustrates that the condition (v) of Corollary 6.6 is essential.

Example 6.10. Let $X = \mathbb{R}$, $S = [-1,1]$, $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. We consider the maps $F_n, F : S \rightrightarrows Y$, $n \in \mathbb{N}$ defined as $F(x) = [-|x|, 1] \times \{0\}$ and

$$F_n(x) = \begin{cases} (-|x|, 1) \times \{0\}, & \text{if } -1 \leq x \leq 0, \\ [-|x|, 1] \times \{0\}, & \text{if } 0 < x \leq 1. \end{cases}$$

Clearly, l-Eff$(\mathcal{F}) = \{-1, 1\}$ and l-Eff$(\mathcal{F}_n) = \{1\}$. We observe that the condition (v) is not satisfied as $F(-1) \in l$-Min$(\mathcal{F})$ and $F(-1) \sim^l F(1)$. It can be seen that the conclusion of Corollary 6.6 fails to hold as $-1 \in l$-Eff$(\mathcal{F})$ but $-1 \notin$ Li$(l$-Eff$(\mathcal{F}_{n_k}))$ for any subsequence $\{n_k\}$ of $\{n\}$.

Remark 6.6. (i) Similarly, Example 6.7 shows that Corollary 6.6 fails to hold if $F_n \overset{\Gamma}{\nrightarrow} F$. In this example, $1 \in l$-Eff($\mathcal{F}$), but $1 \notin$ Li(l-Eff($\mathcal{F}_{n_k}$)) for any subsequence $\{n_k\}$ of $\{n\}$.

(ii) From Example 6.8, we observe that the assumption of gamma convergence cannot be relaxed in Corollary 6.6. In this example, the result fails to hold for $x = 0$.

(iii) From Example 6.9, we see that $\mathrm{cl}(\bigcup_{n\in\mathbb{N}} (\bigcup_{x\in S} (F_n(x))))$ is not compact and Corollary 6.6 fails to hold as $0 \in l$-Eff($\mathcal{F}$), but $0 \notin$ Li(l-Eff($\mathcal{F}_{n_k}$)) for any subsequence $\{n_k\}$ of $\{n\}$.

Next, we deduce the lower Painlevé–Kuratowski convergence of l-weak efficient solution sets under certain additional assumptions.

Corollary 6.7. *If the conditions* (i)-(v) *of Corollary 6.6 hold and*

(*vi*) *S is a convex subset of X and F is l-K-strictly quasiconvex on S;*

(*vii*) *for every $x \in S$,* WMin$(F_n(x)) \neq \emptyset$, $n \in \mathbb{N}$;

*then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that l-*WEff$(\mathcal{F}_{n_k}) \overset{K}{\rightarrow} l$-WEff$(\mathcal{F})$.

Proof. Let $x \in l$-WEff($\mathcal{F}$). Since F is l-K-strictly quasiconvex on S, therefore by Lemma 2.1, $x \in l$-Eff($\mathcal{F}$). Thus from Corollary 6.6, it follows that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $x \in$ Li(l-Eff($\mathcal{F}_{n_k}$)). Hence, by assumption (vii), we have $x \in$ Li(l-WEff($\mathcal{F}_{n_k}$)). □

Remark 6.7. From Example 6.8, we observe that if $F_n \overset{\Gamma}{\nrightarrow} F$, then the Corollary 6.7 fails to hold. Here l-WEff($\mathcal{F}$)$= \{0\}$ and l-WEff($\mathcal{F}_n$)$= \{\frac{1}{2}\}$, if n is even and l-WEff($\mathcal{F}_n$)$= \{1\}$, if n is odd. Clearly, $0 \in l$-WEff($\mathcal{F}$), but $0 \notin$ Li(l-WEff($\mathcal{F}_{n_k}$)) for any subsequence $\{n_k\}$ of $\{n\}$.

Theorem 6.8. *If the conditions* (i)-(iii) *of Theorem 5.4 hold and*

(*iv*) *$\mathcal{F}_n$, $n \in \mathbb{N}$ satisfies l-weak domination property on S;*

*then for $F(x) \in l$-*WMin$(\mathcal{F})$*, there exists a sequence $\{u_n\} \subseteq S$ such that $F_n(u_n) \in l$-*WMin$(\mathcal{F}_n)$*. Moreover, there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in S$ such that $u_{n_k} \to u$, $F_{n_k}(u_{n_k}) \overset{K}{\rightarrow} F(u)$ and $F(u) \preceq^l F(x)$.*

Proof. Since $x \in S$ and $F_n \overset{\Gamma}{\rightarrow} F$, therefore there exists a sequence $\{x_n\} \subseteq S$ such that $x_n \to x$ and $F_n(x_n) \overset{K}{\rightarrow} F(x)$. If $F_n(x_n) \notin l$-WMin($\mathcal{F}_n$), then by l-weak domination property of $\mathcal{F}_n$, $n \in \mathbb{N}$ on S, there exists a sequence $\{u_n\} \subseteq S$ such that $u_n \in l$-WEff($\mathcal{F}_n$), $n \in \mathbb{N}$ and

$$F_n(x_n) \subseteq F_n(u_n) + \mathrm{int}K. \tag{6.6}$$

As S is compact, it follows that $\{u_n\}$ has a convergent subsequence. Without loss of generality, we assume that $u_n \to u \in S$. Again as $F_n \overset{\Gamma}{\rightarrow} F$, therefore there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $F_{n_k}(u_{n_k}) \overset{K}{\rightarrow} F(u)$.

Now we show that $F(u) \preceq^l F(x)$. Let $y \in F(x)$. Since $F(x) \subseteq$ Li$(F_n(x_n))$, therefore there exists $y_n \in F_n(x_n)$, $n \in \mathbb{N}$ such that $y_n \to y$. From (6.6), it follows that there exists $z_n \in F_n(u_n)$ such that $y_n - z_n \in \mathrm{int}K$. Since $\{z_{n_k}\} \subseteq \mathrm{cl}(\bigcup_{n\in\mathbb{N}} (\bigcup_{x\in S} (F_n(x))))$ which is a compact set, it follows that there exists a subsequence $\{z_{n_{k_l}}\}$ of $\{z_{n_k}\}$ and $z \in Y$ such that $z_{n_{k_l}} \to z$. Clearly, $z \in$ Ls$(F_{n_k}(u_{n_k})) \subseteq F(u)$. As $y_{n_k} - z_{n_k} \in K$, therefore we obtain that $y - z \in K$, which implies that $F(x) \subseteq F(u) + K$. □

The following examples justify that the condition of gamma convergence cannot be dropped from Theorem 6.8.

Example 6.11. Let $X = Y = \mathbb{R}$, $S = [-1, 1]$ and $K = \mathbb{R}_+$. We consider the maps $F_n, F : S \rightrightarrows Y$, $n \in \mathbb{N}$ defined as

$$F(x) = \begin{cases} [0,1], & \text{if } x = -1, \\ \{1\}, & \text{if } x \neq -1, \end{cases} \qquad F_n(x) = \begin{cases} (0,1), & \text{if } x = 0, n \text{ is even}, \\ [\frac{1}{n}, 1], & \text{if } x \neq 0,1, n \text{ is even}, \\ \{1\}, & \text{if } x = 1, n \text{ is even}, \\ \{1\}, & \text{if } x \neq 1, n \text{ is odd}, \\ (0,1), & \text{if } x = 1, n \text{ is odd}. \end{cases}$$

We observe that $F_n \overset{\Gamma}{\nrightarrow} F$ as for every $x \in S$ and $x_n \to x$, we have $F_n(x_n) \overset{K}{\nrightarrow} F(x)$. Clearly, l-WMin$(\mathcal{F}) = \{F(-1)\}$ and

$$l\text{-WMin}(\mathcal{F}_n) = \begin{cases} F_n(0), & \text{if } n \text{ is even}, \\ F_n(1), & \text{if } n \text{ is odd}. \end{cases}$$

It can be seen that for $x = -1$, the conclusion of Theorem 6.8 does not hold.

The next example illustrates that the assumption of l-weak domination property cannot be dropped in Theorem 6.8.

Example 6.12. Let $X = \mathbb{R}$, $S = [0, 1]$, $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. We consider the maps $F_n, F : S \rightrightarrows Y$, $n \in \mathbb{N}$ defined as $F(x) = [0, 1 - x] \times [\min\{x, 1 - x\}, \max\{x, 1 - x\}]$ and

$$F_n(x) = \begin{cases} (0, 1-x) \times (\min\{x, 1-x\}, \max\{x, 1-x\}), & \text{if } x \neq \frac{1}{2}, \\ [0, \frac{1}{2}] \times \{\frac{1}{2}\}, & \text{if } x = \frac{1}{2}. \end{cases}$$

It can be easily seen that $\mathcal{F}_n$, $n \in \mathbb{N}$ does not satisfy l-weak domination property at $x = \frac{1}{2}$. Clearly, l-WMin$(\mathcal{F}) = \{F(x) : x \in [0, 1]\}$ and l-WMin$(\mathcal{F}_n) = \{F_n(\frac{1}{2})\}$. We observe that for $x = 0$, $F(x) \in l$-WMin$(\mathcal{F})$, but the conclusion of Theorem 6.8 fails to hold.

7. Conclusion

In this paper, we consider a sequence of perturbed set optimization problems by perturbing the objective function and discuss the convergence of solution sets, in the sense of Painlevé–Kuratowski convergence of sets. Using certain domination properties and the assumptions of compactness and l-K-strict quasiconvexity, we have studied the external and internal stability of l-weak minimal and l-minimal solution sets in the image space. We have further investigated the Painlevé–Kuratowski convergence of solution sets in the decision space.

In literature, the external and internal stability of solution sets in set optimization have been discussed under the perturbations of feasible set (see [5, 11]). In this paper, we study these stability aspects by considering the perturbation of objective function. It would be worthwhile to extend this study by perturbing the ordering cone in set optimization as has been done for vector problems in [19].

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