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A descent three-term derivative-free method for signal reconstruction in compressive sensing

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ABSTRACT. Many real-world phenomena in engineering, economics, statistical inference, compressed sensing and machine learning involve finding sparse solutions to under-determined or ill-conditioned equations. Our interest in this paper is to introduce a derivative-free method for recovering sparse signal and blurred image arising in compressed sensing by solving a nonlinear equation involving a monotone operator. The global convergence of the proposed method is established under the assumptions of monotonicity and Lipschitz continuity of the underlying operator. Numerical experiments are performed to illustrate the efficiency of the proposed method in the reconstruction of sparse signals and blurred images.

1. INTRODUCTION

This article considers finding sparse solutions to an under-determined linear system arising from compressed sensing. In general, the sparse signal and blurred image recovery problem is formulated by the inversion of the following observation model:

$$b = Tz + r, \tag{1.1}$$

where $z \in \mathbb{R}^n$, z, r and b are unknown original image/signal, unknown additive random noise and known degraded observation, respectively and $T \in \mathbb{R}^{m\times n}(m << n)$ is a linear operator.

In the literature, there is a growing interest in using the ℓ_1-norm regularization to recover the sparse signal and blurred image z from (1.1). The ℓ_1-norm regularization problem is given by

$$\min_z \left\{ \frac{1}{2}\|Tz - b\|_2^2 + \tau\|z\|_1 \right\}, \tag{1.2}$$

where τ is a nonnegative parameter.

Several authors have developed several iterative methods for approximating the solution of the ℓ_1-norm regularization problem (1.2). Notable algorithms developed in this direction can be found in [13, 14, 16] and references therein. The gradient descent method is one of the notable methods for approximating the solution of the ℓ_1-norm regularization problem. In [15], a gradient projection algorithm was developed to approximate the solution of (1.2). However, the ℓ_1-norm regularization problem was first transformed into convex quadratic program (CQP). Referring to [15], we present a summary of the reformulation of (1.2) into CQP.

Consider any vector $z \in \mathbb{R}^n$, z can be rewritten as follows

$$z = u^a - u^b,\ u^a \geq 0,\ u^b \geq 0,$$

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where $u^a \in \mathbb{R}^n, u^b \in \mathbb{R}^n$ and $(u^a)^i = (z^i)^+$, $(u^b)^i = (-z^i)^+ \ \forall i \in [1,n]$ with $(\cdot)^+ = \max\{0,\cdot\}$. Therefore, the ℓ_1-norm could be represented as $\|z\|_1 = \langle e^n, u^a\rangle + \langle e^n, u^b\rangle$, where e^n is a vector of dimension n with all element one. Thus, the model (1.2) is rewritten as

$$\min_{u^a,u^b}\{\frac{1}{2}\|b - T(u^a - u^b)\|^2 + \tau\langle e^n, u^a\rangle + \tau\langle e^n, u^b\rangle : u^a, u^b \geq 0, \langle u^a, u^b\rangle = 0\}. \tag{1.3}$$

By the definition of u^a and u^b, in order to ensure equivalence between (1.2) and (1.3), the complementary restriction in (1.3) necessarily holds. Therefore, the penalty approach allows the model (1.3) to be relaxed to

$$\min_{u^a,u^b}\{\frac{1}{2}\|b - T(u^a - u^b)\|^2 + \tau\langle e^n, u^a\rangle + \tau\langle e^n, u^b\rangle + a^0\langle u^a, u^b\rangle : u^a, u^b \geq 0\}. \tag{1.4}$$

where a^0 is a sufficiently large penalty coefficient. Clearly, the model (1.4) can be rewritten in a more standard bound CQPP, we can write the above model as follows:

$$\min_x \frac{1}{2}x^T G x + c^T x,\ x \geq 0, \tag{1.5}$$

where $x = \begin{bmatrix} u^a \\ v^b \end{bmatrix}$, $\quad y = \langle T, b\rangle, \quad c = \tau e^{2n} + \begin{bmatrix} -y \\ y \end{bmatrix}$, $G = \begin{bmatrix} T^T T & -T^T T \\ -T^T T & T^T T \end{bmatrix}$.

One of the first derivative-free method and spectral gradient method for approximating the ℓ_1-norm regularization problem (1.2) is the methods proposed by Xiao et al. [37,38] respectively. However, (1.5) was first transformed into a linear variational inequality problem that is similar to a linear complementarity problem. They also noted that $x \in \mathbb{R}^n$ is a solution to the bound-limited quadratic program problem (1.5) if and only if x is a solution to the following nonlinear equation:

$$F(x) := \min\{x, Gx + c\} = 0. \tag{1.6}$$

The function F is vector valued, the "min" is interpreted as componentwise minimum. Thus, (1.6) is equivalent to (1.2). We note that (1.6) is a monotone system of equation [37,38].

Exploiting the simplicity and low storage requirement of the conjugate gradient method [1,2], in recent times, several authors have extended many conjugate gradient algorithms designed to solve unconstrained optimization problems to solve large-scale nonlinear equations (1.6) (see [3–11,17–33,36]). For instance, using the projection scheme of Solodov and Svaiter [35], Xiao and Zhu [38] extended the Hager and Zhang conjugate descent (CG_DESCENT) method to solve (1.6). Besides, their proposed method was applied for sparse signal and image recovery problem arising in compressed sensing.

Motivated and inspired by the presented results and the approximate equivalence between (1.2) and (1.6), in this paper, we propose a descent three-term projection method for approximating (1.2). The proposed method is motivated by the three-term conjugate gradient method (THREECG) for solving unconstrained optimization problems by Andrei [12]. Under suitable conditions, global convergence is established. Numerical experiments are performed to illustrate the efficiency of the proposed method in reconstruction of sparse signals and blurred images.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries, recall the three-term conjugate gradient method for unconstrained optimization and present our algorithm subsequently. Convergence properties of the proposed algorithm are analyzed in Section 3. In Section 4, we demonstrate the efficiency of the proposed algorithm in signal and image recovery problem. Finally, conclusion is given in the last section. Throughout this article, unless otherwise specified, the symbol $\|\cdot\|$ denotes the

Euclidean norm. In addition, we need the definition of the projection map (P_Ω), which is a mapping from $\mathbb{R}^{2n}$ onto the non-empty, closed and convex subset Ω, that is,

$$P_\Omega(z) := \arg\min\{\|z-y\| : \ y \in \Omega\},$$

that has the well-known property of non-expansivity, that is,

$$\|P_\Omega(z) - P_\Omega(y)\| \leq \|z-y\|, \ \forall z, y \in \mathbb{R}^{2n}. \tag{1.7}$$

2. Preliminaries and Algorithm

In this section, we recall the conjugate gradient method for the unconstrained optimization problem

$$\min\{f(z) : z \in \mathbb{R}^n\},$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable whose gradient at point z^k is $g(z^k)$, or g^k for the sake of simplicity. The three term conjugate gradient (THREECG) algorithm of Andrei [12] is a conjugate gradient method which generate a sequence $\{z^k\}$ using the iterative scheme

$$z^{k+1} = z^k + \alpha^k d^k, \ k \geq 0 \tag{2.8}$$

where $\alpha^k > 0$ is the steplength obtained by a suitable line search procedure and d^k is the search direction computed as

$$d^k := \begin{cases} -g^k - \delta^k s^{k-1} - \eta^k \tilde{y}^{k-1}, & \text{if } k > 0, \\ -g^k & \text{if } k = 0, \end{cases} \tag{2.9}$$

where $s^k = z^{k+1} - z^k$ and

$$\delta^k := \left(1 + \frac{\|\tilde{y}^{k-1}\|^2}{\langle \tilde{y}^{k-1}, s^{k-1}\rangle}\right) \frac{\langle s^{k-1}, g^k\rangle}{\langle \tilde{y}^{k-1}, s^{k-1}\rangle} - \frac{\langle \tilde{y}^{k-1}, g^k\rangle}{\langle \tilde{y}^{k-1}, s^{k-1}\rangle}, \ \tilde{y}^{k-1} = g^k - g^{k-1} \tag{2.10}$$

$$\eta^k := \frac{\langle s^{k-1}, g^k\rangle}{\langle \tilde{y}^{k-1}, s^{k-1}\rangle}. \tag{2.11}$$

Based on the THREECG method, (2.8)-(2.11), we propose a projection procedure for solving (1.6) which generates a sequence $\{u^k\}$ such that

$$u^k := z^k + \alpha^k d^k,$$

where the steplength $\alpha^k > 0$ and the search direction is defined as

$$d^0 := -F(z^0), \ d^k := -F(z^k) - \delta^k s^{k-1} - \eta^k y^{k-1}, \ \forall k \geq 0. \tag{2.12}$$

where $s^k = u^k - z^k = \alpha^k d^k$ and the parameter δ^k and η^k are defined as follows

$$\delta^k := \left(1 + \frac{\|y^{k-1}\|^2}{\langle w^{k-1}, s^{k-1}\rangle}\right) \frac{\langle s^{k-1}, F(z^k)\rangle}{\langle w^{k-1}, s^{k-1}\rangle} - \frac{\langle y^{k-1}, F(z^k)\rangle}{\langle w^{k-1}, s^{k-1}\rangle}, \tag{2.13}$$

$$\eta^k := \frac{\langle s^{k-1}, F(z^k)\rangle}{\langle w^{k-1}, s^{k-1}\rangle}, \tag{2.14}$$

and

$$y^{k-1} := F(u^{k-1}) - F(z^{k-1}) + r s^{k-1}, \ s^{k-1} := u^{k-1} - z^{k-1} = \alpha^{k-1} d^{k-1}, r > 0, \tag{2.15}$$

$$w^{k-1} := y^{k-1} + j^{k-1} d^{k-1}, \ j^{k-1} := 1 + \max\left\{0, -\frac{\langle d^{k-1}, y^{k-1}\rangle}{\langle d^{k-1}, d^{k-1}\rangle}\right\}. \tag{2.16}$$

In view of the above, we present the algorithm for the propose method.

Algorithm 2.1.
Initialization. *Choose any arbitrary point* $z^0 \in \Omega$, *the positive constants:* $Tol \in (0,1)$, $\sigma > 0,\ \gamma \in (0,1],\ \rho \in (0,1),\ \kappa \in (0,2)$. *Set* $k = 0$.

Step 0. *Compute* $F(z^k)$. *If* $\|F(z^k)\| \leq Tol$, *stop. Otherwise,* d^k *should be computed as follows*

$$d^0 := -F(z^0),\ d^k := -\theta^k F(z^k) - \delta^k s^{k-1} - \eta^k y^{k-1},\ \forall k \geq 0. \tag{2.17}$$

where δ^k *and* η^k, *are computed using* (2.13) *and* (2.14) *respectively.*

Step 1. *Compute* $u^k = z^k + \alpha^k d^k$ *and let the steplength* $\alpha^k = \sigma\gamma^i$, *be determined by the following line-search*

$$-\langle F(z^k + \alpha^k d^k), d^k\rangle \geq \rho\alpha^k \|F(z^k + \alpha^k d^k)\| \cdot \|d^k\|^2. \tag{2.18}$$

Step 2. *If* $u^k \in \Omega$ *and* $\|F(u^k)\| \leq Tol$, *stop. Otherwise, compute*

$$z^{k+1} = P_\Omega\left[z^k - \kappa\mu^k F(u^k)\right] \tag{2.19}$$

where

$$\mu^k := \frac{\langle F(u^k), (z^k - u^k)\rangle}{\|F(u^k)\|^2}.$$

Step 3. *Set* $k := k + 1$ *and go to step 0.*

In the next section, we investigate the global convergence of Algorithm 2.1.

3. CONVERGENCE ANALYSIS

In this section, we will establish the convergence of the proposed algorithm. We first study the convergence of Algorithm 2.1 under the condition:

(1) The solution set $Sol_{F,\Omega}$ is nonempty.
(2) The operator F is Lipschitz continuous, that is, there exists a constant $L > 0$ such that

$$\|Fz - Fy\| \leq L\|z - y\|, \quad \forall z, y \in \mathbb{R}^n. \tag{3.20}$$

(3) For the problem (1.6), the associated operator F is monotone (see Xiao et al. [37]). That is,

$$\langle Fz - Fy, z - y\rangle \geq 0, \quad \forall z, y \in \mathbb{R}^n. \tag{3.21}$$

Lemma 3.1. *Assume that Suppose that* d_k *is given by* (2.12) *Then, the following result*

$$\langle F(z^k), d^k\rangle \leq -\|F(z^k)\|^2,\ \forall k \geq 0. \tag{3.22}$$

holds for any $k \geq 0$.

Proof. From (2.15) and (2.16), it holds that

$$\langle w^{k-1}, s^{k-1}\rangle \geq \alpha^{k-1}\langle y^{k-1}, d^{k-1}\rangle + \alpha^{k-1}\|d^{k-1}\|^2 - \alpha^{k-1}\langle y^{k-1}, d^{k-1}\rangle = \alpha^{k-1}\|d^{k-1}\|^2. \tag{3.23}$$

Assume that $\alpha^{k-1}\|d^{k-1}\|^2 > 0$, multiplying both sides of (2.12) with $F(z^k)^T$ and using (2.13), (2.14) and (3.23), the the proof of Lemma 3.1 follows. □

Lemma 3.2. *Let the sequences* $\{d^k\}$ *and* $\{z^k\}$ *be generated by Algorithm 2.1, then the line search* (2.18) *is well-defined.*

Proof. By contradiction, suppose there exist $k^0 \geq 0$ such that (2.18) is not satisfied for any nonnegative integer i, that is,

$$-\langle F(z^{k_0} + \sigma\gamma^i d^{k_0}), d^{k_0}\rangle < \rho\sigma\gamma^i \|F(z^{k_0} + \sigma\gamma^i d^{k_0})\|\|d^{k_0}\|^2, \forall i \geq 1.$$

Using the continuity of F and letting $i \to \infty$ yields

$$-\langle F^{k_0}, d^{k_0}\rangle \leq 0$$

which contradicts (3.22). This completes the proof. □

Lemma 3.3. *Suppose F is monotone and Lipchitz continuous on $\mathbb{R}^n$ and the sequence $\{z^k\}$ and $\{u^k\}$ is generated by Algorithm 2.1, then for any solution z^* of* (1.6), *it holds that*

$$\|z^{k+1} - z^*\|^2 \leq \|z^k - z^*\|^2 - \kappa(2-\kappa)\rho^2\|z^k - u^k\|^4, \tag{3.24}$$

and the sequence $\{z^k\}$ and $\{u^k\}$ are bounded. Furthermore, it holds that

$$\lim_{k\to\infty} \|z^k - u^k\| = 0. \tag{3.25}$$

Proof. Let $u^k = z^k + \alpha^k d^k$, it is clear from the line-search (2.18) that,
(3.26)
$$\langle F(u^k), z^k - u^k\rangle = -\alpha^k\langle F(u^k), d^k\rangle \geq \rho(\alpha^k)^2\|F(u^k)\|\|d^k\|^2 = \rho\|F(u^k)\|\|z^k - u^k\|^2.$$

From the monotonicity of F, (3.26) with $z^* \in Sol_{F,\Omega}$, we have

$$\begin{aligned}
\langle F(u^k), z^k - z^*\rangle &= \langle F(u^k),\ z^k + u^k - u^k - z^*\rangle \\
&= \langle F(u^k), u^k - z^*\rangle + \langle F(u^k), z^k - u^k\rangle \\
&\geq \langle F(z^*), u^k - z^*\rangle + \langle F(u^k), z^k - u^k\rangle \\
&= \langle F(u^k), z^k - u^k\rangle \\
&\geq \rho\|F(u^k)\|\|z^k - u^k\|^2. \qquad (3.27)
\end{aligned}$$

From the nonexpansiveness of the operator, it holds that for any $z^* \in Sol_{F,\Omega}$,

$$\begin{aligned}
\|z^{k+1} - z^*\|^2 &= \|P_\Omega[z^k - \kappa\mu^k F(u^k)] - z^*\|^2 \\
&\leq \|z^k - \kappa\mu^k F(u^k) - z^*\|^2 \\
&= \|z^k - z^*\| - 2\kappa\mu^k\langle F(u^k), z^k - z^*\rangle + \kappa^2(\mu^k)^2\|F(u^k)\|^2 \\
&= \|z^k - z^*\|^2 - 2\kappa\mu^k\langle F(u^k), (z^k - z^*)\rangle + \kappa^2(\mu^k)^2\|F(u^k)\|^2 \\
&\leq \|z^k - z^*\|^2 - 2\kappa\mu^k\langle F(u^k), z^k - u^k\rangle + \kappa^2(\mu^k)^2\|F(u^k)\|^2 \\
&= \|z^k - z^*\|^2 - \kappa(2-\kappa)\left(\frac{\langle F(u^k), z^k - u^k\rangle}{\|F(u^k)\|}\right)^2 \\
&\leq \|z^k - z^*\|^2 - \kappa(2-\kappa)\rho^2\|z^k - u^k\|^4 \qquad (3.28)
\end{aligned}$$

From the above, it can be observed that $\{\|z^k - z^*\|\}$ is decreasing as $\kappa \in (0, 2)$. It holds that $\|z^k - z^*\| \leq \|z^0 - z^*\|$. Therefore, by the assumption that F is Lipschitz continuous on $\mathbb{R}^n$, it follows that

$$\|F(z^k)\| = \|F(z^k) - F(z^*)\| \leq L\|z^k - z^*\| \leq L\|z^0 - z^*\|.$$

Now, taking $\varrho = L\|z^0 - z^*\|$, we have

$$\|F(z^k)\| \leq \varrho.$$

Moreover, since F is monotone, by the Cauchy-Schwarz inequality and (2.18), we have

$$\|F(u^k)\|\|z^k - u^k\| \geq \langle F(u^k), z^k - u^k\rangle \geq \rho\|F(u^k)\|\|z^k - u^k\|^2,$$

where the last inequality can be implied from (3.26). Thus, we obtain

$$\rho\|z^k - u^k\| \leq 1.$$

The above implies that $\{u^k\}$ is bounded. By the continuity of F, there exist a constant $H > 0$ such that

$$\|F(u^k)\| \leq H, \ \forall k \geq 0.$$

From (3.28), we have

$$\kappa(2-\kappa)\rho^2\|z^k - u^k\|^4 \leq \|z^k - z^*\|^2 - \|z^{k+1} - z^*\|^2. \tag{3.29}$$

Adding (3.29) for $k \geq 0$, we have

$$\kappa(2-\kappa)\rho^2\sum_{k=0}^{\infty}\|z^k - u^k\|^4 \leq \sum_{k=0}^{\infty}(\|z^k - z^*\|^2 - \|z^{k+1} - z^*\|^2) < \infty. \tag{3.30}$$

We can infer from (3.30) that

$$\lim_{k\to\infty}\|z^k - u^k\| = 0.$$

The proof is completed. □

Lemma 3.4. *Let sequences $\{z^k\}$ and $\{u^k\}$ be generated by Algorithm 2.1. Then, we have*

$$\alpha^k \geq \min\left\{\sigma, \frac{\gamma\|F(z^k)\|^2}{(L+\rho\|F(u_*^k)\|)\|d_k\|^2}\right\} \tag{3.31}$$

where $u_*^k := z^k + \alpha_*^k d^k, \ \alpha_*^k := \gamma^{-1}\alpha^k$.

Proof. From the line search procedure, suppose $\alpha^k \neq \sigma$, then $\alpha_*^k = \gamma^{-1}\alpha^k$ does not satisfy the line search procedure. That is,

$$-\langle(z^k + \alpha_*^k d^k), d^k\rangle < \rho\alpha_*^k\|F(z^k + \alpha_*^k d^k)\| \cdot \|d^k\|^2.$$

By the descent condition (3.22) and the condition 1 (3.20), it holds that

$$\begin{aligned}\|F(z^k)\|^2 &\leq -\langle F(z^k), d^k\rangle \\ &= \langle(F(z^k + \alpha_*^k d^k) - F(z^k)), d^k\rangle - \langle F(z^k + \alpha_*^k d^k), d^k\rangle \\ &\leq \alpha_*^k(L + \rho\|F(u_*^k)\|) \cdot \|d^k\|^2.\end{aligned}$$

Hence, this yield the desired inequality (3.31). □

We next show the global convergence of Algorithm 2.1.

Theorem 3.2. *Consider the iterative method defined by Algorithm 2.1. Suppose conditions 1-2 hold. If $\{z^k\}$ is the sequence generated by Algorithm 2.1, then*

$$\liminf_{k\to\infty}\|F(z^k)\| = 0. \tag{3.32}$$

Proof. Suppose $\liminf_{k\to\infty} \|F(z^k)\| = 0$ does not hold, that is there exist a constant $\omega > 0$ such that

$$\|F(z^k)\| \geq \omega, \ \forall k \geq 0.$$

Connecting the sufficient descent condition (3.22) with the above equation implies that

$$\|d^k\| \geq \|F(z^k)\| \geq \omega, \ \forall k \geq 0.$$

From the definition of y^{k-1}, and by Lipschitz continuity, we have

$$\begin{aligned} \|y^{k-1}\| &= \|F(u^{k-1}) - F(z^{k-1}) + rs^{k-1}\| \\ &\leq (L+r)\|u^{k-1} - z^{k-1}\| \\ &\leq (L+r)\alpha^{k-1}\|d^{k-1}\|. \end{aligned}$$

Utilizing (2.13), (3.23), $\alpha_k \in (0,1)$ and the above inequality, from the proposed direction d^k defined by (2.12), it follows for all $k > 0$,

$$\begin{aligned} \|d^k\| &= \| - \theta^k F(z^k) - \delta^k s^{k-1} - \eta^k y^{k-1}\| \\ &\leq \|F(z^k)\| + |\delta^k|\|s^{k-1}\| + |\eta^k|\|y^{k-1}\| \\ &= \|F(z^k)\| + \left| \left(1 + \frac{\|y^{k-1}\|^2}{\langle w^{k-1}, s^{k-1}\rangle}\right) \frac{\langle s^{k-1}, F(z^k)\rangle}{\langle w^{k-1}, s^{k-1}\rangle} - \frac{\langle y^{k-1}, F(z^k)\rangle}{\langle w^{k-1}, s^{k-1}\rangle} \right| \|s^{k-1}\| \\ &\quad + \left| \frac{\langle s^{k-1}, F(z^k)\rangle}{\langle w^{k-1}, s^{k-1}\rangle} \right| \|y^{k-1}\| \\ &\leq \|F(z^k)\| + \left[\left(1 + \frac{\|y^{k-1}\|^2}{\alpha^{k-1}\|d^{k-1}\|^2}\right) \frac{\|s^{k-1}\|\|F(z^k)\|}{\alpha^{k-1}\|d^{k-1}\|^2} + \frac{\|y^{k-1}\|\|F(z^k)\|}{\alpha^{k-1}\|d^{k-1}\|^2} \right] \|s^{k-1}\| \\ &\quad + \frac{\|s^{k-1}\|\|F(z^k)\|}{\alpha^{k-1}\|d^{k-1}\|^2} \|y^{k-1}\| \\ &= \|F(z^k)\| + \left[\left(1 + \frac{\|y^{k-1}\|^2}{\alpha^{k-1}\|d^{k-1}\|^2}\right) \alpha^{k-1}\|F(z^k)\| + \frac{\|y^{k-1}\|\|F(z^k)\|}{\|d^{k-1}\|} \right] \\ &\quad + \frac{\|F(z^k)\|}{\|d^{k-1}\|} \|y^{k-1}\| \\ &\leq \|F(z^k)\| \left(1 + \left[\Big(1 + (L+r)\Big) + (L+r) \right] + (L+r) \right) \\ &\leq \varrho \Big(1 + [(1 + (L+r)) + (L+r)] + (L+r)\Big) \triangleq \mathcal{J}. \end{aligned}$$

Note, by the Lipschitz continuity of the mapping F, we obtain that the sequences $\{F(z^k)\}$ and $\{F(u^k)\}$ are bounded, which implies that $\|F(z^k + \gamma^{-1}\alpha^k d^k)\|$ and the sequence $\{d^k\}$ are bounded. Thus, there exist $M^* > 0$ such that,

$$\|F(z^k + \gamma^{-1}\alpha^k d^k)\| \leq M^*, \ \forall k \geq 0.$$

On the other hand, it can be implied from Lemma 3.4 and the boundedness of d^k that

$$\begin{aligned} \alpha^k\|d^k\| &\geq \min\left\{\sigma, \frac{\gamma\|F(z^k)\|^2}{(L + \rho\|F(u_*^k)\|)\|d^k\|^2}\right\}\|d^k\| \\ &\geq \min\left\{\sigma\omega, \frac{\gamma\omega^2}{(L + \rho M^*)\mathcal{J}}\right\} > 0, \end{aligned}$$

which contradicts (3.25). Thus, (3.32) holds. □

4. Numerical experiment

This section gives numerical results to illustrate the effectiveness of our proposed algorithm. We first consider two particular applications. In the first, the proposed algorithm applied to recover sparse signals. The second situation we consider is the restoration blurred images using the proposed algorithm. All the codes are written in Matlab R2019b and performed on a HP PC with CPU 2.4GHz, 8.0GB RAM with windows 10 operating system. In what follows, the proposed method (Algorithm 2.1) is referred as **DSTT**.

4.1. **Recovery of Sparse signals.** Consider a sparse signal of length m and n observations distorted with noise. Our interest is to reconstruct the sparse signal from the noisy observations. Based on this, we utilise DSTT in recovering sparse signals. We compare the performance of DSTT with the SGCS [37], CGD [38] and PCG [34] algorithms designed for similar purpose. In the experiment, we consider a signal of size $m = 2^{12}$, $n = 2^{10}$ and the original signal contains 2^6 randomly nonzero elements. The random T is the Gaussian matrix which is generated by command randn(m,n) in Matlab. In this test, the measurement b contains noise, that is, $b = Tz + e_0$, where e_0 is the Gaussian noise distributed normally with mean 0 and variance 10^{-4}. The quality of the recovered signal is assessed by the mean of squared error (MSE) to the original signal z, that is,

$$MSE := \frac{1}{n}\|z - \bar{z}\|^2$$

where $\bar{z}$ is the recovered signal. The proposed algorithm is implemented with the following parameters: $\sigma = 1,\ \gamma = 0.6,\ \kappa = 1,\ \rho = 10^{-4},\ r = 10$. For SGCS, CGD and PCG algorithm, the parameters chosen for their implementation are set as reported in the numerical section of their respective papers. The merit function used is $f(z) = \tau\|z\| + \frac{1}{2}\|b - Az\|^2$

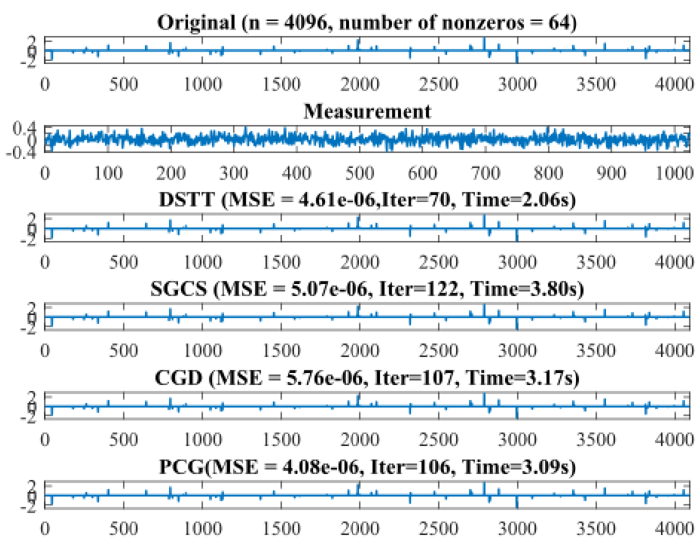

FIGURE 1. Illustration of the sparse signal recovery DSTT, SGCS, CGD and PCG

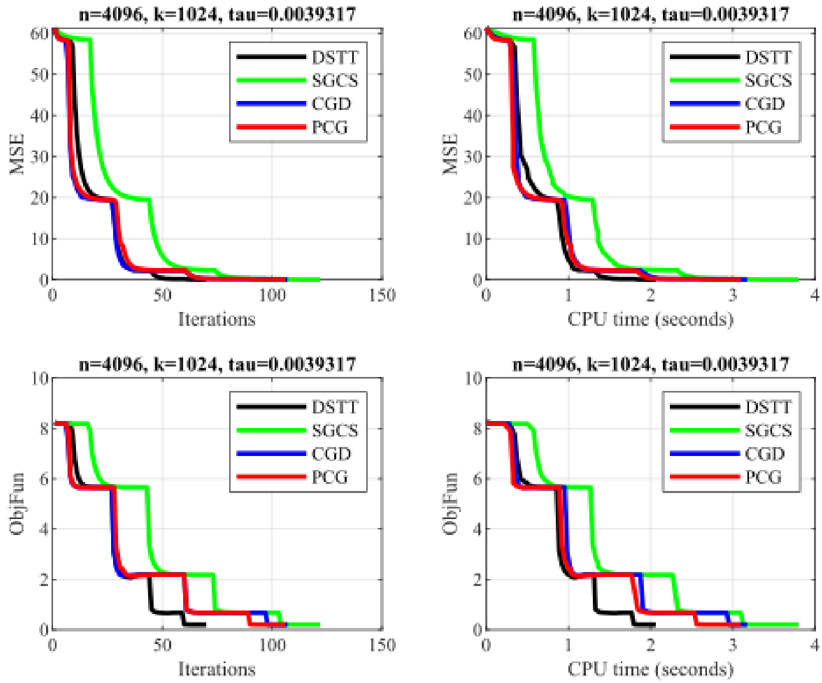

FIGURE 2. Comparison results of DSTT, SGCS, CGD and PCG algorithm. The x-axes represent the number of iterations (top left and bottom left) and the CPU time in seconds (top right and bottom right). The y-axes represent the MSE (top left and top right) and the function values (bottom left and right).

Each code was run from the same initial point, the same continuation method on the parameter $\tau = 0.005\|T^T b\|_\infty$ for fairness. All algorithms are initialized with $z^0 = \langle T, b\rangle$ and terminates when

$$Tol := \frac{|f_k - f_{k-1}|}{|f_{k-1}|} < 10^{-5},$$

where $F(z^k)$ is the function evaluation at z^k.

To assess the performance of the methods, we evaluated the performance profiles of all methods based on a set of metrics such as the number of total iterations, the mean squared error and time in seconds (sec). In view of Figure 1, it is clear that DSTT is the winner in recovering sparse signal.

To further highlight the efficiency of DSTT in recovering sparse signal, we repeat the experiment over nine times. Table 1 contains the number of iterations, the MSE and sec of the reconstructions with respect to the original signal for the repeated experiment. The results presented in the table shows that, in recovering sparse signal, DSTT attains most wins in terms of number of iterations, mean squared error.

TABLE 1. Numerical results as regards to signal recovery

	DSTT			SGCS			CGD			PCG		
	ITER	TIME	MSE	ITER	TIME	MSE	ITER	TIME	MSE	ITER	TIME	MSE
	70	2.06	4.61E-06	122	3.80	5.07E-06	107	3.17	5.76E-06	106	3.09	4.08E-06
	88	2.50	3.13E-06	127	3.36	4.12E-06	140	3.89	3.59E-06	114	3.17	3.31E-06
	84	2.27	4.14E-06	129	3.52	4.74E-06	131	4.05	3.90E-06	111	3.20	3.78E-06
	77	2.11	4.46E-06	129	3.27	5.72E-06	103	2.73	1.15E-05	109	3.20	4.75E-06
	91	2.42	3.87E-06	128	3.31	4.54E-06	110	3.08	5.47E-06	111	3.11	3.60E-06
	97	2.80	2.96E-06	129	3.27	3.47E-06	123	3.27	4.84E-06	110	2.88	2.78E-06
	79	2.17	1.85E-06	132	3.39	2.53E-06	158	4.36	2.33E-06	104	2.77	2.00E-06
	76	2.09	2.49E-06	130	3.44	3.25E-06	143	4.14	2.72E-06	114	3.02	2.63E-06
	82	2.17	2.61E-06	125	3.17	3.52E-06	100	3.22	3.09E-06	94	3.58	5.11E-06
	67	1.72	4.98E-06	125	3.16	5.16E-06	108	3.89	4.22E-06	115	4.33	4.00E-06
Average	81	2.23	3.51E-06	128	3.37	4.21E-06	122	3.58	4.74E-06	109	3.24	3.60E-06

4.2. **Image restoration.** In this subsection, image restoration experiment are presented to demonstrate the performance of DSTT method. Moreover, we compare our method with three different methods, including CGD [38] and SGCS [37]. The test images considered are the benchmark images which includes Tiffany (512×512), Lena (512×512), Barbara (720×576), obtained from `http://hlevkin.com/06testimages.htm`. We employ the signal-to-noise ratio (SNR), peak signal-to-noise ratio (PSNR) and stuctural similarity index (SSIM) metric to comprehensively evaluate the quality of restoration of the methods. In this experiment, the test images were degraded with gausian noise of 10%. The parameters chosen for DSTT are: $\sigma = 0.05,\ \gamma = 0.2,\ \rho = 10^{-4},\ \kappa = 1,\ r = 10$. For CGD and SGCS algorithm, their parameters are chosen as in the respective papers.

FIGURE 3. The original test images: From the left, Tiffany, Lenna and Barbara

From Figure 4, it can be observed that all algorithms were able to restore the images. However, the quality of the restored images by DSTT algorithm is competitive to that of CGD and SGCS methods. This is reflected by almost the same SNR, PSNR and SSIM. Conclusively, the proposed algorithm provides a valid approach to solve image de-blurring problems and its performance is competitive compared with CGD and DCG methods. The numerical results are reported in Table 2 below.

TABLE 2. Numerical result for image restoration

	DSTT			CGD			SGCS		
Images	SNR	PSNR	SSIM	SNR	PSNR	SSIM	SNR	PSNR	SSIM
Tiffany	20.93	22.77	0.9136	20.92	22.76	0.9132	20.89	22.72	0.9122
Lenna	16.68	22.02	0.9122	16.67	22	0.9119	16.62	21.95	0.9109
Barbara	13.61	20.03	0.6265	13.6	20.02	0.6258	13.56	19.98	0.6237

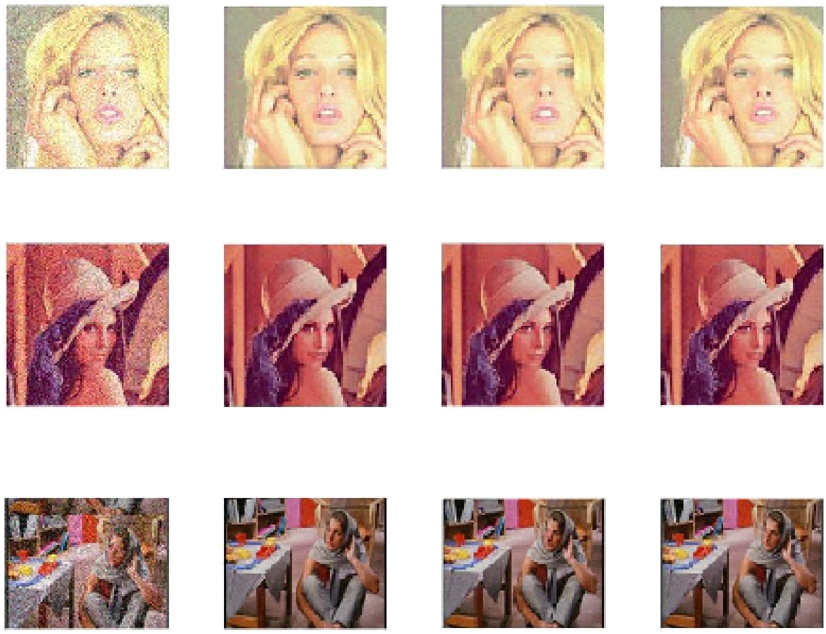

FIGURE 4. The blurred and noisy images (left column), restored images by DSTT(middle left column), restored images by CGD (middle right column) and restored images by SGCS (right column)

CONCLUSION

In this paper, we present a derivative-free conjugate gradient method to recover sparse signal and image arising in compressed sensing by solving a nonlinear equation involving a monotone operator. The search direction of the proposed method is three-term and descent. Furthermore, the global convergence of the proposed method is established under the assumptions of monotonicity and Lipschitz continuity of the underlying operator. Experimental results in reconstruction of sparse signals and blurred images show that our algorithm has a better performance than other related algorithms.

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