CARPATHIAN J. MATH. Online version at https://www.semnul.com/carpathian/
38 (2022), No. 3, 523-538 Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

Dedicated to Professor Emeritus Mihail Megan on the occasion of his 75^{th} anniversary

The early developments in fixed point theory on b-metric spaces: a brief survey and some important related aspects

VASILE BERINDE[1,2] AND MĂDĂLINA PĂCURAR[3]

ABSTRACT. A very impressive research work has been devoted in the last two decades to obtaining fixed point theorems in quasimetric spaces (also called b-metric spaces). Some incorrect and incomplete references with respect to the early developments on fixed point theory in b-metric spaces are though perpetually taking over from the existing publications to the new ones.

Starting from this fact, our main aim in this note is threefold:
(1) to briefly survey the early developments in the fixed point theory on quasimetric spaces (b-metric spaces);
(2) to collect some relevant bibliography related to this topic;
(3) to discuss some other aspects of current interest in the fixed point theory on quasimetric spaces (b-metric spaces).

1. INTRODUCTION

In the last few decades, a significant interest in fixed point theory has been directed to transposing classical or recent metric fixed point results from metric spaces to certain generalized metric spaces, see for example Van An et all [10] for a recent survey.

In most of the cases such an approach turned out to be trivial, as the fixed point theorems established in some generalized metric spaces could be obtained from the corresponding ones in the setting of metric spaces by a metrization process, see for example [58], [65], [69], [79], [84], [90], [92], [95].

There are however some generalized metric spaces, like quasimetric spaces (usually called b-metric spaces in the context of fixed point theory), for which this transposing process appears to produce in most of the situations genuine generalizations of the fixed point results from usual metric spaces.

If we want to have a better illustration of the significant interest of researchers working in fixed point theory for transposing existing metric fixed point results from metric spaces to b-metric spaces or to establishing new fixed point results in the latter setting and are searching in MathScinet or zbMATH for the terms "fixed point" and "b-metric space" appearing in the title or in the topic of the publications indexed there, we shall find a very impressive record especially in the last 10 years.

Despite the very extensive research work that has been devoted to obtaining fixed point theorems in b-metric spaces (see, for example, the recent survey [87]), most of these papers are making incorrect and / or incomplete references to the mathematical literature regarding the origins of the main notion involved in the respective research works, i.e., the notion of *quasimetric space* (*b-metric space*), as well as with respect to the early developments in the fixed point theory on quasimetric spaces (b-metric spaces).

Received: 31.10.2021. In revised form: 10.01.2022. Accepted: 24.01.2022
1991 *Mathematics Subject Classification.* 47H10; 54H25; 47H09.
Key words and phrases. *quasimetric space; b-metric space; quasi-norm; quasi-Banach space; fixed point; topology.*
Corresponding author: Vasile Berinde; vasile.berinde@mi.utcluj.ro

Starting from this fact, our main aim in this note is threefold:

(1) to briefly survey the early developments in the fixed point theory on quasimetric spaces (b-metric spaces);
(2) to collect some relevant older and recent bibliography related to this topic;
(3) to also discuss some other aspects of current interest in the fixed point theory on quasimetric spaces (b-metric spaces) and to indicate some further developments.

2. ORIGINS OF QUASIMETRIC SPACES (b-METRIC SPACES)

A fundamental concept in analysis and topology is the notion of *metric space*, which has been introduced by Fréchet [66] (and labeled in this form by Hausdorff [71]), see for example Kantorovich and Akilov [86].

A *metric* (or *distance*) on a nonempty set X is a function $d : X \times X \to \mathbb{R}$ which satisfies the following three conditions:

(1) (*positivity*) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;
(2) (*symmetry*) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(3) (*triangle inequality*) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

If d is a metric on X then the pair (X, d) is called a *metric space*.

Remark 2.1. *Sometimes the metric is defined as a functional with values in* $[0, +\infty)$, *when the axiom of positivity is replaced by the axiom*

(4) *(separability)* $d(x, y) = 0$ *if and only if* $x = y$.

There is a myriad of generalizations of this notion, which were obtained by various alterations of one, two or all three conditions above and / or by dropping out one or two of the axioms of the distance. Amongst these generalizations we can find the notion of *quasimetric space* (commonlly called b-metric space by researchers working in the field of fixed point theory), which is extremely important in analysis, see for example the monograph Mitrea et all [105].

Definition 2.1. *A function* $d : X \times X \to \mathbb{R}$ *is called a* quasimetric *on* X *if it satisfies the positivity and symmetry conditions above and also*

(QM_1) *there exist a constant* $K \geq 1$ *such that*

$$d(x, z) \leq K \left[d(x, y) + d(y, z)\right], \text{ for all } x, y, z \in X. \tag{2.1}$$

If d *is a quasimetric on* X *then the pair* (X, d) *is called a quasimetric space.*

Obviously, in the particular case $K = 1$, a quasimetric d reduces to the usual notion of metric (distance) but, in general, a quasimetric is not a metric.

Remark 2.2. *An equivalent definition of a quasimetric (see for example* [41]*) could be given by replacing condition* (QM_1) *by the following one*

$(QM_1)'$ *there exist a constant* $K_1 \geq 1$ *such that*

$$d(x, z) \leq K_1 \max\{d(x, y), d(y, z)\}, \text{ for all } x, y, z \in X. \tag{2.2}$$

Indeed, in view of the double inequality $a + b \leq 2\max\{a, b\} \leq 2(a + b)$, $(QM_1)'$ *implies* (QM_1) *with* $K = K_1$, *while* (QM_1) *implies* $(QM_1)'$ *with* $K_1 = 2K$.

It is rather difficult to trace back to the origins of the notion of *quasimetric space* and one of the reasons for having such a situation appears to be the simple fact that over the time various distinct metric structures were termed as *quasimetric spaces*.

For example, in a paper from 1931, Wilson [145] called "quasimetric" the generalized metric obtained from a metric by dropping out the symmetry property, see also Cobzaş [45]. For another concept bearing the same name, see Hitzler [72].

So, throughout the present note we shall consider only the quasimetric spaces introduced by Definition 2.1 and obtained from metric spaces by replacing the triangle inequality with the relaxed triangle inequality (2.1) (called quasi-triangle inequality in [123]).

We shall call such a space a *quasimetric space* but will also accept the term *b-metric space* which is accustomed in fixed point theory. Note also that some other names appear in literature to designate a quasimetric space (b-metric space). For example, Khamsi and Hussain [91] used recently the term "metric type space".

One can trace back to the origins of the concept of *quasi-norm* which is closely interrelated to that of quasimetric.

Definition 2.2. *Let X be a real or complex linear space. A map $\|\cdot\| : X \to [0, \infty)$ is a quasi-norm if it satisfies the following conditions:*

(QN_0) $\|x\| = 0$ *implies* $x = 0_X$;
(QN_1) $\|x + y\| \leq Q\,[\|x\| + \|y\|]$, *for all* $x, y \in X$, *where* $Q \geq 1$ *does not depend on* x, y;
(QN_2) $\|\lambda x\| = |\lambda| \cdot \|x\|$, *for all* $x \in X$ *and* $\lambda \in \mathbb{K}$ *(either* $\mathbb{C}$ *or* $\mathbb{R}$*).*

According to Pietsch [123], the concept of quasi-norm has been introduced by Hyers [75] in 1938, under the name "pseudo-norm" (with (QN_1) formulated differently but in an equivalent way) and, in the above form, by Bourgin [36] in 1943, who actually proposed the label "quasi-norm".

Therefore, it is very natural to speculate that the concept of quasimetric (space) appeared in literature approximately in the same period as the quasi-norm. By searching for its origins in MathScinet we succeeded to identify only a paper from 1970 due to Coifman and Guzmán [47] which appears to be the first one that explicitly defines and uses the concept of a quasimetric in the sense of Definition 2.1.

In fact, Coifman and Guzmán [47] used the name *"distance" function* and introduced it in order to construct the so called homogenous spaces and study singular integrals on such spaces. Note also that the priority of Coifman and Guzmán is mentioned by Aimar et all [2], Cobzaş and Czerwik [46].

Anyway, if we are now turning our attention to the way in which researchers working in fixed point theory refer to the early developments of the concept of quasimetric space (b-metric space), it would be noted that the opinions are very different, and apparently, inaccurate.

Indeed, in some papers it is considered that this concept has been introduced by Bourbaki [35] in 1974 (see for example [87]), or that it has been introduced by Bakhtin [17] in 1989 (see for example [26], [49], [106] etc.), or by Czerwik [51] in 1993 (see [11], [3], [68], [87], [93], [136] etc.) or even by Czerwik [52] in 1998 (see [3], [14], [87], [136], [139] etc.).

In the absence of any other available reference, we shall place the origins of quasimetric spaces either simultaneously with that of quasi-norm, that is, in 1938, or in Coifman and Guzmán's paper [47] from 1970, where it is explicitly defined.

In fact, as in the case of the pair norm - metric, the notion of quasi-norm and that of quasimetric are closely interrelated, in the sense that any quasi-norm $\|\cdot\|$ on a linear space E induces a quasimetric d defined by

$$d(x, y) = \|x - y\|, x, y \in E.$$

Similarly to the way a Banach space is defined, one defines the concept of quasi-Banach space as a complete linear quasi-normed space, see Mitrea et all [105] for more details on the importance of quasi-Banach spaces. According to Pietsch [123], the first example of a quasi-Banach space has been given by Tychonoff [143] in 1935, e.g., the space $l_{\frac{1}{2}}$.

This could be also considered as the first example of quasimetric space. Indeed, on the space

$$l_{\frac{1}{2}} = \left\{ x = (x_1, x_2, \dots, x_n, \dots) : \sum_{i=1}^{\infty} \sqrt{|x_i|} < \infty \right\}$$

let us consider the functional $\rho : l_{\frac{1}{2}} \times l_{\frac{1}{2}} \to [0, \infty)$, defined by

$$\rho(x, y) = \left(\sum_{i=1}^{\infty} \sqrt{|x_i - y_i|} \right)^2, \tag{2.3}$$

where $x = (x_1, x_2, \dots, x_n, \dots), y = (y_1, y_2, \dots, y_n, \dots)$.

Then ρ is a quasimetric on $l_{\frac{1}{2}}$ (induced by the corresponding quasi-norm) which satisfies the quasi-triangle inequality (2.1) with $K = 2$, i.e.,

$$\rho(x, z) \leq 2 \left[\rho(x, y) + \rho(y, z) \right], x, y, z \in l_{\frac{1}{2}}.$$

A more general example can be given if we consider $p \in (0, 1)$ instead of $\frac{1}{2}$ in the considerations above, see Example 1.3 [87] (which is taken from [35]).

Remark 2.3. *We note in passing the following important fact: when we introduced the equivalent relaxed triangle conditions* (QM_1) *and* $(QM_1)'$*, we assumed* $K \geq 1$ *and* $K_1 \geq 1$*, respectively, like all the sources we consulted* ([47], [17], [11],...).

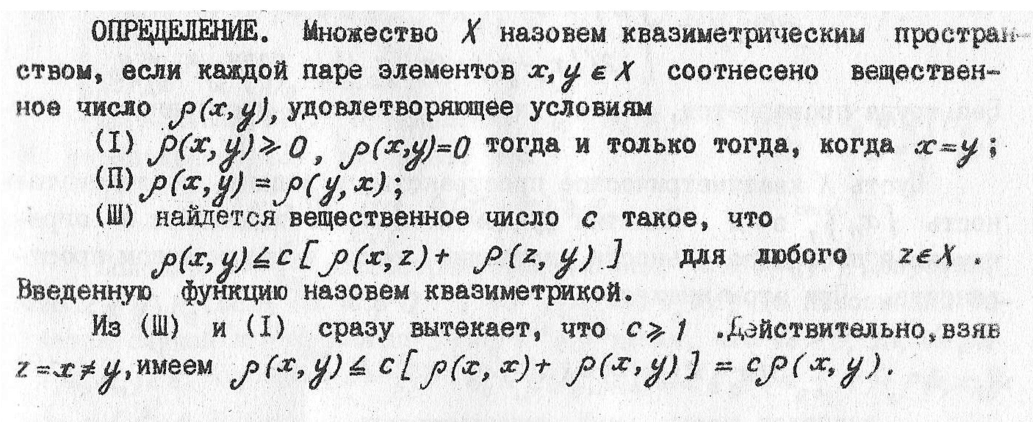
But, in the paper by Vulpe et all [144]*, see also Figure 1, where the constant* K *is denoted by* c*, it is shown that, in the definition of a quasimetric it is enough to assume only that* $c \geq 0$*, and the fact that we have always* $c \geq 1$ *is a consequence of the axioms of the quasimetric.*

Indeed, just take $z = x \neq y$ *in* (2.1) *to get:*

$$\rho(x, y) \leq c \cdot [\rho(x, z) + \rho(z, y)] = c \cdot \rho(x, y)$$

from which we obtain $c \geq 1$*, in view of the fact that* $x \neq y \iff \rho(x, y) > 0$.

The same argument applies when inequality (2.2) *is considered.*



ОПРЕДЕЛЕНИЕ. Множество X назовем квазиметрическим пространством, если каждой паре элементов $x, y \in X$ соотнесено вещественное число $\rho(x,y)$, удовлетворяющее условиям

(I) $\rho(x,y) \geqslant 0$, $\rho(x,y)=0$ тогда и только тогда, когда $x=y$;

(П) $\rho(x,y) = \rho(y,x)$;

(Ш) найдется вещественное число c такое, что

$\rho(x,y) \leq c[\rho(x,z) + \rho(z,y)]$ для любого $z \in X$.

Введенную функцию назовем квазиметрикой.

Из (Ш) и (I) сразу вытекает, что $c \geqslant 1$. Действительно, взяв $z = x \neq y$, имеем $\rho(x,y) \leq c[\rho(x,x) + \rho(x,y)] = c\rho(x,y)$.

FIGURE 1. Excerpt from the paper by Vulpe et all [144], page 15, where the notion of quasimetric is defined (but without any reference to existing literature)

3. On the topology of quasimetric spaces

A quasimetric space is a natural setting in a part of harmonic analysis related to the theory of spaces of homogeneous type. For instance, in Coifman and Weiss [48] it is indicated how to construct a quasimetric space on a topological space on which there is defined a Borel measure (and, therefore, how to construct a space of homogeneous type).

The class of quasimetric spaces contains the family of all quasi-Banach spaces, which in turn includes a multitude of function spaces that are of fundamental importance in analysis: Lebesgue spaces, weak Lebesgue spaces, Lorentz spaces, Hardy spaces, weak Hardy spaces, Lorentz-based Hardy spaces, Besov spaces, Triebel-Lizorkin spaces, and weighted versions of these spaces (among many others), see Mitrea et all [105].

Since we are interested mainly to survey the early developments in the fixed point theory on quasimetric spaces, we summarize in this section some important topological properties of the quasimetric spaces from this point of view. For more details on this topic, the reader may consult [11], [23], [46], [87], [91], [139],...

We note that the label *b-metric space* will be often avoided in this section in favour of that of *quasimetric space*, because there is also a significantly different notion with the same name, i.e., B-metric space, with capital "B" coming from Boolean, see Ellis and Sprinkle [62].

In a quasimetric space one introduces the concepts of convergent and Cauchy sequences similarly to the case of usual metric spaces.

Definition 3.3. *Let (X, d) be a a quasimetric space. A sequence $\{x_n\}_{n\geq 0}$ in X is called:*

(1) convergent *if and only if there exists $x \in X$ such that for any $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon)$ we have $d(x_n, x) < \varepsilon$, i.e., $\lim\limits_{n\to\infty} d(x_n, x) = 0$. In this case, we write $\lim\limits_{n\to\infty} x_n = x$.*

(2) Cauchy *if and only if for any $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon)$ we have $d(x_n, x_m) < \varepsilon$, i.e., $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$.*

Lemma 3.1. *Let (X, d) be a quasimetric space. Then*
(i) Any convergent sequence has a unique limit; *(ii) Any convergent sequence is Cauchy.*

The reverse of (ii) in Lemma 3.1 is generally not true.

Definition 3.4. *A quasimetric space (X, d) is said to be complete if every Cauchy sequence $\{x_n\}_{n\geq 0}$ in X is convergent.*

In a quasimetric space one commonly uses the topology induced by its convergence and therefore most – but not all – of the topological concepts and results from metric spaces can be transposed to quasimetric spaces.

For example, in contrast to the case of a metric space, a quasimetric is not a continuous function of its arguments (see Example 3.1) and the open ball $B(x_0; r) := \{x \in X : d(x_0, x) < r\}$ in a quasimetric space (X, d) is not necessarily an open set, while the closed ball $\overline{B}(x_0; r) := \{x \in X : d(x_0, x) \leq r\}$ is not necessarily a closed set.

Example 3.1 (Vulpe et all [144]). *Consider $X = \mathbb{R}^2$ and $d : X \times X \to \mathbb{R}_+$ defined for $x = (x_1, x_2), y = (y_1, y_2)$ by*

$$d(x,y) = \begin{cases} |x_1 - x_2|, \text{ if } y_1 = y_2 \\ 2\left(|x_1 - x_2| + |y_1 - y_2|\right), \text{ if } y_1 \neq y_2. \end{cases} \tag{3.4}$$

Then (X, d) is a quasimetric space with constant $K = 2$, see [144]. *If we take*

$$x_n = \left(1, \frac{1}{n}\right);\ x = (1, 0);\ y_n = (0, 0);\ y = (0, 0),$$

then

$$x_n \to x \text{ as } n \to \infty;\ y_n \to y \text{ as } n \to \infty,$$

but

$$d(x_n, y_n) = 2 + \frac{2}{n} \to 2 \neq 1 = d(x, y),$$

which shows that, in contrast to the case of a standard metric, the quasimetric is not a continuous function of its arguments.

With the aim of providing more information regarding the topology of quasimetric spaces, we end this section by presenting a few results about the metrization of quasimetric spaces, which are mainly taken from An, Tuyen and Dung [11] and Cobzaş and Czerwik [46].

Let X be a topological space. A subset D of X is called *sequentially open* if each sequence $\{x_n\}$ in X converging to a point x in D is *eventually* in D, that is, there exists n_0 such that $x_n \in D$ for all $n \geq n_0$. A subset F of X is called *sequentially closed* if no sequence in F converges to a point not in F.

X is called a *sequential space* if each sequentially open subset of X is open or, equivalently, each sequentially closed subset of X is closed. We denote by τ the sequential topology on the quasimetric (X, d).

One can define another topology on a quasimetric space (X, d), see Khamsi and Hussain [90] (Definition 8 and Proposition 1) as follows.

A subset A of X is called *open* if for any $a \in A$, there exists $\varepsilon > 0$ such that $B(a, r) \subset A$, where, as in the case of a metric,

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

Then the family of all open subsets of X in the above sense, denoted by τ_d, is also a topology on X.

Another method for generating a topology on a quasimetric space (X, d) is to use a neighborhood system. Indeed, the family $\mathcal{B}$ of all finite intersections of the family

$$\mathcal{C} = \{B(x, r) : x \in X, r > 0\}$$

is a base of a certain topology τ^d on X.

Proposition 3.1 ([11], Proposition 3.3). *Let (X, d) be a quasimetric space. Then*

(1) $\tau = \tau_d$;
(2) $\tau_d \subset \tau^d$.

Remark 3.4. *The inclusion $\tau_d \subset \tau^d$ is in general strict, as shown by the next example.*

Example 3.2 ([11], Example 3.9). *Let $X = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right\}$ and $d : X \times X \to [0, \infty)$ defined by $d(x, y) = 0$, if $x = y$; $d(x, y) = 1$, if $x \neq y$, $x, y \in \{0, 1\}$; $d(x, y) = |x - y|$, if $x \neq y$, $x, y \in \{0\} \cap \left\{\frac{1}{2n} : n = 1, 2, \dots\right\}$; and $d(x, y) = 4$, otherwise.*

Then we have

(1) d is a quasimetric on X with $K = 8$;
(2) d is not a metric on X;
(3) d is not continuous in each variable;
(4) $B(1, 2)$ is not open in τ but it is open in τ^d.

Two (quasi-)metrics ρ_1 and ρ_2 on X are said to be *equivalent* if

$$c^{-1}\rho_2(x, y) \leq \rho_1(x, y) \leq c\rho_2(x, y),\ x, y \in X,$$

for some constant $c > 0$ independent of x, y.

A particular class of quasimetric spaces, i.e., the one obtained by taking $K = 2$ in condition (QM_1) from Definition 2.1, is very important from the point of view of their metrizability:

$$(3.5) \qquad (QM_1a) \quad d(x,z) \le 2\left[d(x,y) + d(y,z)\right], \text{ for all } x,y,z \in X.$$

In 1937, Frink [67] has shown (see also Schroeder [132]) that, for such a quasimetric space (X,d), there exists a metric ρ on X equivalent to d. More specifically, one has

$$(3.6) \qquad \rho(x,y) \le d(x,y) \le 4\rho(x,y),\ x,y \in X.$$

In other words, any quasimetric space with $K = 2$ is metrizable.

This result has been recently extended by Schroeder [132], who has shown that if (X,d) is a quasimetric space with $K \le 2$, then there exists a metric ρ on X such that

$$(3.7) \qquad \rho(x,y) \le d(x,y) \le 2K\rho(x,y),\ x,y \in X.$$

For more details on the metrizability of quasimetric spaces, we refer to [2], [41], [45], [46], [60], [61], [119],...

4. Early developments in fixed point theory on quasimetric spaces

The most important metric fixed point theorem, that plays a crucial role in nonlinear analysis, is the well known *Picard-Banach contraction mapping principle*, first introduced by Banach in [18], in the case of what we call nowadays a Banach space, and then extended to complete metric spaces by Caccioppoli [38].

For a long period of time – almost one century – Banach's theorem has been and continues to be a source of inspiration and intensive research, from a theoretical point of view but also from the point of view of its diverse and important applicative capabilities.

The main efforts of scholars working in metric fixed point theory have been directed to extending the Picard-Banach contraction mapping principle in various ways, of which we mention only the following three:

- by weakening the Banach contraction condition;
- by extending the structure of the metric space (X,d);
- by combining the first two directions.

In what follows we are interested only in the second direction, i.e., when one considers a quasimetric space instead of the usual metric space.

After extensive searches in zbMATH and Mathematical Reviews, it appears that the first fixed point theorem in a quasimetric space has been established in 1981 by Vulpe et all [144], who transposed the Picard-Banach contraction mapping principle from metric spaces to the framework of a quasimetric space, in the sense of Definition 2.1.

Their result reads (see also Figure 2) as follows.

Theorem 4.1 ([144], Theorem 2.1). *Let (X,ρ) be a quasimetric space with the constant K and let A be a mapping of X onto X such that there exists a number $\alpha > 0$ for which*

$$(4.8) \qquad \rho(Ax,Ay) \le \alpha\rho(x,y), \text{ for each } x,y \in X.$$

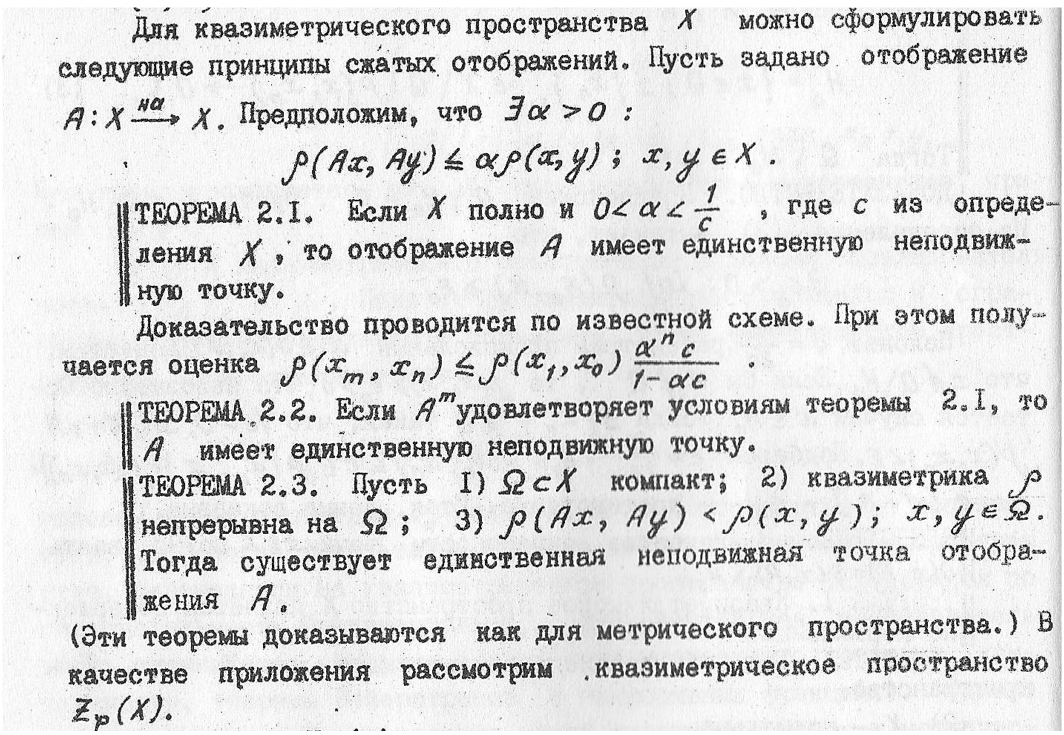
If the space (X,ρ) is complete and $0 < \alpha < \dfrac{1}{K}$, then the mapping A has a unique fixed point.

A similar result to Theorem 4.1, even with some of the examples from [144], has appeared in 1989 in another publication from the former Soviet Union (written in Russian, too) and authored by Bakhtin [17], who appears to not have been aware of the preceding publication [144].

This could be explained by means of at least three reasons that we often meet in many other contexts: 1) the main aim of the paper by Vulpe et all [144] has been to investigate the topological structure of quasimetric spaces, while the fixed point results have been given as applications of those results; 2) the title itself of the paper [144] did not indicate any involvement of fixed point results and 3) the paper [144] appeared in a collection of papers [Investigations in functional analysis and differential equations, Zbl 459.0014] and not in a serial publication (journal).

However, the paper [144] has been reviewed in both zbMATH (Zbl 0502.54032) – this review is describing in detail the fixed point results established by the authors – and Mathematical Reviews (MR0630902).

The next developments in this area appear to be dated in 1993 and are due to the first author [20], who worked in "quasimetric spaces" and established a fixed point theorem for φ-contractions in this setting, and to Czerwik [51], who coined the term "b-metric space", nowadays with an almost general use in fixed point theory.



Для квазиметрического пространства X можно сформулировать следующие принципы сжатых отображений. Пусть задано отображение $A : X \xrightarrow{на} X$. Предположим, что $\exists \alpha > 0$:

$$\rho(Ax, Ay) \leq \alpha \rho(x, y); \; x, y \in X.$$

ТЕОРЕМА 2.1. Если X полно и $0 < \alpha < \frac{1}{c}$, где c из определения X, то отображение A имеет единственную неподвижную точку.

Доказательство проводится по известной схеме. При этом получается оценка $\rho(x_m, x_n) \leq \rho(x_1, x_0) \frac{\alpha^n c}{1 - \alpha c}$.

ТЕОРЕМА 2.2. Если A^m удовлетворяет условиям теоремы 2.I, то A имеет единственную неподвижную точку.

ТЕОРЕМА 2.3. Пусть I) $\Omega \subset X$ компакт; 2) квазиметрика ρ непрерывна на Ω; 3) $\rho(Ax, Ay) < \rho(x, y)$; $x, y \in \Omega$. Тогда существует единственная неподвижная точка отображения A.

(Эти теоремы доказываются как для метрического пространства.) В качестве приложения рассмотрим .квазиметрическое пространство $Z_p(X)$.

FIGURE 2. Excerpt from the paper by Vulpe et all [144], page 18.

In connection with the papers by Czerwik [51], [52], it should be mentioned that the idea of a general quasimetric space was "in the air" in the Polish mathematical community of that time, as shown by the paper of Swiatkowski [140] from 1995, where the author studied the completeness and compactness of a quasi metric space (called there b-metric space), by using the quasi-triangle inequality $(QM_1)'$.

In 1993, Berinde [20] extended the contraction mapping principle on quasimetric spaces, due to Bakhtin [17], to the class of φ-contractions, see also Berinde [21] and [22].

Theorem 1 in [20] essentially states that, if (X, d) is a complete quasimetric space and $f : X \to X$ is a φ-contraction on X, i.e., there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$

such that

$$d(f(x), f(y)) \leq \varphi(d(x, y)), x, y \in X,$$

then f has a unique fixed point in X, provided that there exist $x_0 \in X$ for which the Picard iteration $\{f^n(x_0)\}$ is bounded.

Recall that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a *comparison function*, if it satisfies

(i) φ is increasing;

(ii) $\varphi^n(t) \to 0$, as $n \to \infty$, for each $t \in \mathbb{R}_+$.

By additionally assuming that φ is a (c)-comparison function, Theorem 2 in [20] provides a method for approximating the unique fixed point of a φ-contraction in quasimetric spaces and also offers an error estimate.

Note that the assumption of boundedness of the orbit in [17] and [20] may be dropped, as shown by the second author [118], see also Kirk and Shahzad [93].

In the same year, 1993, Czerwik [51] obtained a similar result to Theorem 1 in [20] but by working in quasimetric spaces (X, d) with $K = 2$, which he called b-metric spaces.

His result can be stated as follows.

Theorem 4.2 ([51], Theorem 1). *Let (X, d) be a complete b-metric space (with $K = 2$) and let $T : X \to X$ satisfy*

$$d[T(x), T(y)] \leq \varphi[d(x, y)],\ x, y \in X, \tag{4.9}$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function such that

$$\lim_{n\to\infty} \varphi^n(t) = 0, \text{ for each fixed } t > 0.$$

Then T has exactly one fixed point u and

$$\lim_{n\to\infty} d[T^n(x), u] = 0,$$

for each $x \in X$.

In 1998, Czerwick [52] also established a fixed point theorem for multivalued mappings in arbitrary b-metric spaces (this time without assuming $K = 2$).

It appears that after the papers by Vulpe et all [144] (1981), Bakhtin [17] (1989), Berinde [20] and Czerwick [51] (1993), Berinde [21] (1996) and [22] (1997) and Czerwick [52] (1998), the interest for working in such a setting declined for a while.

The next paper in this series appears to be due to Singh et all [136] (2005), where the authors presented a fixed-point theorem for generalized set-valued contractions on a quasimetric space X, which actually marked the start of a new wave of interest for working in quasimetric spaces.

Starting from 2008, with the papers by Boriceanu [24]-[28], Olatinwo [111], Olatinwo and Imoru [110], Păcurar [117], [118] and Sing and Prasad [137], one records a real explosion of the research work on this topic (not all were included at References of this paper).

To give a better idea on the phenomenon, we just indicate de dynamics of the publications citing [17], [20], [51] and [52], according to zbMATH and MathScinet (if one would use Google Scholar, the figures will be significantly more impressive).

1. zbMATH

- Number of papers citing Bakhtin [17]: 251 papers [2022 (2 papers); 2021 (58); 2020 (49); 2019 (28); 2018 (41); 2017 (45); 2016 (8); 2015 (11); 2014 (2); 2013 (6); 2010 (1)]
- Number of papers citing Czerwik [51]: 420 papers [2022 (5 papers); 2021 (86); 2020 (69); 2019 (53); 2018 (59); 2017 (69); 2016 (20); 2015 (25); 2014 (18); 2013 (11); 2012 (2); 2011 (2); 2008 (1)]

- Number of papers citing Czerwik [52]: 221 papers [2022 (2 papers); 2021 (34); 2020 (19); 2019 (22); 2018 (31); 2017 (40); 2016 (14); 2015 (22); 2014 (15); 2013 (15); 2012 (2); 2011 (2); 2010 (1); 2008 (2)]
- Number of papers citing Berinde [20]: 52 papers [2022 (1 paper); 2021 (5); 2020 (8); 2019 (3); 2018 (2); 2017 (12); 2016 (1); 2015 (7); 2014 (7); 2013 (3); 2012 (1); 2011 (1); 2010 (1)]

2. MathScinet

- Number of papers citing Bakhtin [17]: 132 papers [2022 (2 papers); 2021 (25); 2020 (26); 2019 (13); 2018 (18); 2017 (17); 2016 (8); 2015 (9); 2014 (5); 2013 (7); 2011 (1); 2010 (1)]
- Number of papers citing Czerwik [51]: 184 papers [2022 (2 papers); 2021 (36); 2020 (32); 2019 (22); 2018 (25); 2017 (21); 2016 (15); 2015 (11); 2014 (14); 2013 (4); 2011 (1); 2008 (1)]
- Number of papers citing Czerwik [52]: 104 papers [2022 (1 paper); 2021 (13); 2020 (6); 2019 (10); 2018 (15); 2017 (17); 2016 (11); 2015 (10); 2014 (11); 2013 (6); 2011 (1); 2010 (1); 2008 (2)]
- Number of papers citing Berinde [20]: 40 papers [2021 (4 papers); 2020 (6); 2019 (1); 2018 (3); 2017 (7); 2016 (6); 2015 (5); 2014 (3); 2013 (3); 2011 (1); 2010 (1)]

5. MORE GENERAL FIXED POINT RESULTS IN b-METRIC SPACES

Most of the fixed point results established for contractive mappings in quasimetric spaces assume a certain relation between the contraction coefficient and the constant K from the quasi-triangle inequality (2.1).

For example, in the case of the fixed point theorem established by Vulpe et all [144], the contraction coefficient α (see also Figure 2) is required to satisfy the condition

$$\alpha < \frac{1}{K}. \tag{5.10}$$

Note that in the case $K = 1$, when the quasimetric space is in fact a metric space, one recovers the classical Banach's condition $\alpha < 1$.

The next Lemma (see for example Miculescu and Mihail [103] and Suzuki [139]) opens the possibility of obtaining more general fixed point results than Theorem 4.1.

Lemma 5.2 ([104]). *Every sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements from a quasimetric space (X, d), having the property that there exists $\alpha \in [0, 1)$ such that*

$$d(x_{n+1}, x_n) \leq \alpha d(x_n, x_{n-1}), n \in \mathbb{N}, \tag{5.11}$$

is Cauchy.

As one can see, the condition (5.10) in Theorem 4.1 can be dropped without affecting the conclusion. Thus, the following general result holds.

Theorem 5.3. *Let (X, ρ) be a complete quasimetric space and let $T : X \to X$ be a mapping such that there exists a number $\alpha \in (0, 1)$ for which*

$$\rho(Ax, Ay) \leq \alpha\rho(x, y), \text{ for each } x, y \in X. \tag{5.12}$$

Then the mapping A has a unique fixed point.

Therefore, as many fixed point results existing in literature are tributary to a superfluous condition like (5.10), it would be useful to investigate the possibility of obtaining more general fixed point results by using Lemma 5.2.

We also suggest the following

Open problem: Identify all fixed point results in quasimetric spaces (b-metric spaces) that could be obtained by means of an appropriate metrization technique (and the corresponding results in the setting of metric spaces).

6. Conclusions

1. Working in a quasimetric space (b-metric space) is not trivial from a topological point of view, so the interest for establishing general fixed point theorems in this setting in the last 15 years appears as well motivated in many respects.

2. The concept of quasimetric space (b-metric space) is much older than it is commonly considered in the papers on fixed point theory in quasimetric spaces, where it is attributed either to Bakhtin [17] or to Czerwik [51], [52].

3. According to the current bibliographical knowledge, the first fixed point theorem (the contraction mapping principle in quasimetric spaces) appears to be due to Vulpe et all [144].

4. Researchers working in fixed point theory are commonly focused on the fixed point results themselves and not particularly on the topological properties of quasimetric spaces (b-metric spaces). It would be of a real benefit in the future to integrate in such kind of fixed point results more topological features, as in the paper by Vulpe et all [144].

5. Such topological insights, see for example Dung and Hang [60], [61], Radenović et all [125], are warning for a careful approach to some fixed point results in quasimetric spaces and their applications (particularly, those related to integral equations), which can be obtained by using the corresponding metric spaces tools.

6. In order to avoid reference errors when considering a certain notion or result, one might better say ”for the concept or for the result ... we refer to” instead of ”it has been introduced by ”, especially when the chronology of the notions / results is not certain.

As a final conclusion we would like to stress on the following fact: in mathematics (and in other subjects, too) it often happens that one notion or result is not attributed to the author who really introduced it, but to the one who has made it more visible to the scientific community.

Therefore, Arnold’s principle *If a notion bears a personal name, then this name is not the name of the discoverer*

could be reformulated in this context as

If a notion or a result is attributed to a mathematician, then it is possible that this is not the true discoverer of it.

Acknowledgments

We would like to thank Dr. Florin Damian from Institutul de Matematică şi Informatică ”Vladimir Andrunachievici”, Academia de Ştiinţe a Moldovei, for providing a scanned copy of [Vulpe, I. M.; Ostrajkh, D.; Khojman, F. *The topological structure of a quasimetric space*. (Russian) Investigations in functional analysis and differential equations, Math. Sci., Interuniv. Work Collect., Kishinev, 1981, 14–19 (1981)], which was an important source of documentation for the present paper.

References

[1] Afshari, H.; Karapınar, E. A solution of the fractional differential equations in the setting of b-metric space. *Carpathian Math. Publ.* **13** (2021), no. 3, 764–774.

[2] Aimar, H.; Iaffei, B.; Nitti, L. On the Macías-Segovia metrization of quasi-metric spaces. *Rev. Un. Mat. Argentina* **41** (1998), no. 2, 67–75.

[3] Akkouchi, M. A common fixed point theorem for expansive mappings under strict implicit conditions on b-metric spaces. *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.* **50** (2011), no. 1, 5–15.

[4] Allahyari, R.; Arab, R.; Shole Haghighi, A. Fixed points of admissible almost contractive type mappings on b-metric spaces with an application to quadratic integral equations. *J. Inequal. Appl.* **2015**, 2015:32, 18 pp.

[5] Aleksić, S.; Mitrović, Z. D.; Radenović, S. On some recent fixed point results for single and multi-valued mappings in b-metric spaces. *Fasc. Math.* **61** (2018), 5–16.

[6] Allahyari, R.; Arab, R.; Shole Haghighi, A. A generalization on weak contractions in partially ordered b-metric spaces and its application to quadratic integral equations. *J. Inequal. Appl.* **2014**, 2014:355, 15 pp.

[7] Ali, M. U.; Kamran, T.; Postolache, M. Solution of Volterra integral inclusion in b-metric spaces via new fixed point theorem. *Nonlinear Anal. Model. Control* **22** (2017), no. 1, 17–30.

[8] Ali, M. U.; Aydi, H.; Alansari, M. New generalizations of set valued interpolative Hardy-Rogers type contractions in b-metric spaces. *J. Funct. Spaces* **2021**, Art. ID 6641342, 8 pp.

[9] Alzaid, S. S.; Fulga, A.; Alshammari, F. S. Discussion on Geraghty type hybrid contractions. *J. Funct. Spaces* **2020**, Art. ID 6614291, 8 pp.

[10] An, T. V.; Van Dung, N.; Kadelburg, Z.; Radenović, S. Various generalizations of metric spaces and fixed point theorems. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **109** (2015), no. 1, 175–198.

[11] An, T. V.; Tuyen, L. Q.; Dung, N. V. Stone-type theorem on b-metric spaces and applications. *Topology Appl.* **185/186** (2015), 50–64.

[12] Ansari, A. H.; Chandok, S.; Ionescu, C. Fixed point theorems on b-metric spaces for weak contractions with auxiliary functions. *J. Inequal. Appl.* **2014**, 2014:429, 17 pp.

[13] Asadi, M.; Rhoades, B. E.; Soleimani, H. Some notes on the paper "The equivalence of cone metric spaces and metric spaces" [MR2743780]. *Fixed Point Theory Appl.* **2012**, 2012:87, 4 pp.

[14] Aydi, H.; Bota, M.-F.; Karapınar, E.; Mitrović, S. A fixed point theorem for set-valued quasi-contractions in b-metric spaces. *Fixed Point Theory Appl.* 2012, 2012:88, 8 pp.

[15] Aydi, H.; Bota, M.-F.; Karapınar, E.; Moradi, S. A common fixed point for weak ϕ-contractions on b-metric spaces. *Fixed Point Theory* **13** (2012), no. 2, 337–346.

[16] Babu Gutti, V. R.; Pathina, S. K. Common fixed points of almost generalized $(\alpha, \psi, \varphi, F)$-contraction type mapping in b-metric spaces. *J. Int. Math. Virtual Inst.* **9** (2019), 123–137.

[17] Bakhtin, I. A. Contracting mapping principle in an almost metric space. (Russian) *Funkts. Anal.* **30** (1989), 26–37.

[18] Banach, S., Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales, *Fund Math.* **3** (1922), 133–181.

[19] Beg, I.; Mani, G.; Gnanaprakasam, A. J. Fixed point of orthogonal F-Suzuki contraction mapping on O-complete b-metric spaces with applications. *J. Funct. Spaces* **2021**, Art. ID 6692112, 12 pp.

[20] Berinde, V. Generalized contractions in quasimetric spaces. *Seminar on Fixed Point Theory*, 3–9, Preprint, 93-3, "Babeş-Bolyai" Univ., Cluj-Napoca, 1993.

[21] Berinde, V. Sequences of operators and fixed points in quasimetric spaces. *Studia Univ. Babeş-Bolyai Math.* **41** (1996), no. 4, 23–27.

[22] Berinde, V. *Generalzed contractions and applications* (in Romanian). Editura Cub Press 22, Baia Mare, Romania, 1997.

[23] Berinde, V. ; Choban, M. Generalized distances and their associate metrics. Impact on fixed point theory. *Creat. Math. Inform.* **22** (2013), no. 1, 23–32.

[24] Boriceanu, M. Fixed point theory for multivalued contractions on a set with two b-metrics. *Creat. Math. Inform.* **17** (2008), no. 3, 326–332.

[25] Boriceanu, M. Fixed point theory for multivalued generalized contraction on a set with two b-metrics. *Stud. Univ. Babeş-Bolyai Math.* **54** (2009), no. 3, 3–14.

[26] Boriceanu, M. Fixed point theory on spaces with vector-valued b-metrics. *Demonstratio Math.* **42** (2009), no. 4, 825–835.

[27] Boriceanu, M. Strict fixed point theorems for multivalued operators in b-metric spaces. *Int. J. Mod. Math.* **4** (2009), no. 3, 285–301.

[28] Boriceanu, M.; Bota, M.; Petruşel, A. Multivalued fractals in b-metric spaces. *Cent. Eur. J. Math.* **8** (2010), No. 2, 367–377.

[29] Boriceanu, M.; Petrućel, A.; Rus, I. A. Fixed point theorems for some multivalued generalized contractions in b-metric spaces. *Int. J. Math. Stat.* **6** (2010), S10, 65–76.

[30] Bota, M.-F.; Chifu, C.; Karapınar, E. Fixed point theorems for generalized $(\alpha_* - \Psi)$-Ćirić-type contractive multivalued operators in b-metric spaces. *J. Nonlinear Sci. Appl.* **9** (2016), no. 3, 1165–1177.

[31] Bota, M.-F.; Karapınar, E. A note on "Some results on multi-valued weakly Jungck mappings in b-metric space". *Cent. Eur. J. Math.* **11** (2013), no. 9, 1711–1712.

[32] Bota, M.-F.; Karapınar, E.; Erdal; Mleşniţe, O. Ulam-Hyers stability results for fixed point problems via α-ψ-contractive mapping in (b)-metric space. *Abstr. Appl. Anal.* **2013**, Art. ID 825293, 6 pp.

[33] Bota, M.-F.; Ilea, V. Fixed point theorems for nonself operators in b-metric spaces. *Fixed Point Theory* **16** (2015), no. 2, 225–232.

[34] Bota, M.; Molnár, A.; Varga, C. On Ekeland's variational principle in b-metric spaces. *Fixed Point Theory* **12** (2011), no. 1, 21–28.

[35] Bourbaki, N. *Topologie Générale*, Herman, Paris, 1974.

[36] Bourgin, D. G. Linear topological spaces. *Amer. J. Math.* **65** (1943), 637–659.

[37] Brzdek, J. Comments on fixed point results in classes of function with values in a b-metric space. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **116** (2022), no. 1, Paper No. 35, 17 pp.

[38] Caccioppoli, R., Un teorema generale sull'esistenza di elementi uniti in una transformazione funzionale, *Rend Accad dei Lincei.* **11** (1930), 794–799.

[39] Chandok, S. Arbitrary binary relations, contraction mappings, and b-metric spaces. Translated from Ukraïn. Mat. Zh. 72 (2020), no. 4, 565–574 *Ukrainian Math. J.* **72** (2020), no. 4, 651–662.

[40] Choban, M. M. On the geometry of b-distances and the fixed points of mappings. *Carpathian J. Math.* **36** (2020), no. 2, 241–257.

[41] Chrzaszcz, K.; Jachymski, J.; Turoboş, F. Two refinements of Frink's metrization theorem and fixed point results for Lipschitzian mappings on quasimetric spaces. *Aequationes Math.* **93** (2019), no. 1, 277–297.

[42] Cicchese, M. Questioni di completezza e contrazioni in spazi metrici generalizzati. (Italian) *Boll. Un. Mat. Ital. A (5)* **13A** (1976), no. 1, 175–179.

[43] Ćirić, L.; Abbas, M.; Rajović, M.; Ali, B. Suzuki type fixed point theorems for generalized multi-valued mappings on a set endowed with two b-metrics. *Appl. Math. Comput.* **219** (2012), no. 4, 1712–1723.

[44] Ćirić, L.; Parvaneh, V.; Hussain, N. Fixed point results for weakly α-admissible pairs. *Filomat* **30** (2016), no. 14, 3697–3713.

[45] Cobzaş, Ş. Fixed points and completeness in metric and generalized metric spaces. (Russian); translated from Fundam. Prikl. Mat. 22 (2018), no. 1, 127–215 *J. Math. Sci. (N.Y.)* **250** (2020), no. 3, 475–535.

[46] Cobzaş, Ş.; Czerwik, S. The completion of generalized b-metric spaces and fixed points. *Fixed Point Theory* **21** (2020), no. 1, 133–150.

[47] Coifman, R. R.; de Guzmán, M. Singular integrals and multipliers on homogeneous spaces. *Rev. Un. Mat. Argentina* **25** (1970/71), 137–143.

[48] Coifman, R. R.; Weiss, G. Extensions of Hardy spaces and their use in analysis. *Bull. Amer. Math. Soc.* **83** (1977), no. 4, 569–645.

[49] Cosentino, M.; Salimi, P.; Vetro, P. Fixed point results on metric-type spaces. *Acta Math. Sci. Ser. B (Engl. Ed.)* **34** (2014), no. 4, 1237–1253.

[50] Cosentino, M.; Jleli, M.; Samet, B.; Vetro, C. Solvability of integrodifferential problems via fixed point theory in b-metric spaces. *Fixed Point Theory Appl.* **2015**, 2015:70, 15 pp.

[51] Czerwik, S. Contraction mappings in b-metric spaces. *Acta Math. Inform. Univ. Ostraviensis* **1** (1993), 5–11.

[52] Czerwik, S. Nonlinear set-valued contraction mappings in b-metric spaces. *Atti Sem. Mat. Fis. Univ. Modena* **46** (1998), no. 2, 263–276.

[53] Czerwik, S. On b-metric spaces and Brower and Schauder fixed point principles. in *Approximation theory and analytic inequalities*, 71–86, Springer, Cham, 2021,

[54] Czerwik, S.; Dlutek, K.; Singh, S. L. Round-off stability of iteration procedures for set-valued operators in b-metric spaces. *J. Natur. Phys. Sci.* **15** (2001), no. 1-2, 1–8.

[55] Demma, M.; Vetro, P. Picard sequence and fixed point results on b-metric spaces. *J. Funct. Spaces* **2015**, Art. ID 189861, 6 pp.

[56] Demma, M.; Saadati, R.; Vetro, P. Fixed point results on b-metric space via Picard sequences and b-simulation functions. *Iran. J. Math. Sci. Inform.* **11** (2016), no. 1, 123–136, 156.

[57] Du, W.-S. A note on cone metric fixed point theory and its equivalence. *Nonlinear Anal.* **72** (2010), no. 5, 2259–2261.

[58] Du, W.-S.; Karapınar, E. A note on cone b-metric and its related results: generalizations or equivalence? *Fixed Point Theory Appl.* **2013**, 2013:210, 7 pp.

[59] Dukić, D.; Kadelburg, Z.; Radenović, S. Fixed points of Geraghty-type mappings in various generalized metric spaces. *Abstr. Appl. Anal.* **2011**, Art. ID 561245, 13 pp.

[60] Dung, N. V.; Hang, V. T. L. On relaxations of contraction constants and Caristi's theorem in b-metric spaces. *J. Fixed Point Theory Appl.* **18** (2016), no. 2, 267–284.

[61] Dung, N. V.; Hang, V. T. L. Remarks on cyclic contractions in b-metric spaces and applications to integral equations. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **111** (2017), no. 1, 247–255.

[62] Ellis, D.; Sprinkle, H. D. Topology of B-metric spaces. *Compositio Math.* **12** (1956), 250–262.

[63] Errai, Y.; Marhrani, El M.; Aamri, M. Fixed point theorems for $(\alpha, k, \theta, \phi)$-contraction multivalued mappings in b-metric space. *Abstr. Appl. Anal.* **2021**, Art. ID 9959374, 8 pp.

[64] Farkas, C.; Molnár, A. É.; Nagy, S.. A generalized variational principle in b-metric spaces. *Matematiche (Catania)* **69** (2014), no. 2, 205–221.

[65] Feng, Y.; Mao, W. The equivalence of cone metric spaces and metric spaces. *Fixed Point Theory* **11** (2010), no. 2, 259–263.

[66] Fréchet, M. Sur quelques points du calcul fonctionnel. *Rendic. Circ. Mat. Palermo* **22** (1906) 1–74.

[67] Frink, A. H. Distance functions and the metrization problem. *Bull. Amer. Math. Soc.* **43** (1937), no. 2, 133–142.

[68] Gholamian, N.; Khanehgir, M. Fixed points of generalized Meir-Keeler contraction mappings in b-metric-like spaces. *Fixed Point Theory Appl.* **2016**, Paper No. 34, 19 pp.

[69] Haghi, R. H.; Rezapour, Sh.; Shahzad, N. Be careful on partial metric fixed point results. *Topology Appl.* **160** (2013), no. 3, 450–454.

[70] Hamaizia, T.; Murthy, P. P. Z-contraction condition involving simulation function in b-metric space under fixed points considerations. *Math. Morav.* **25** (2021), no. 2, 43–52.

[71] Hausdorff, F., *Grundzüge der Mengenlehre*, Viet. Leipzig, 1914.

[72] Hitzler, P., *Generalized Metrics and Topology in Logic Programming Semantics*. PhD thesis, Department of Mathematics, National University of Ireland, University College Cork, 2001.

[73] Hussain, N.; Dorić, D.; Kadelburg, Z.; Radenović, S. Suzuki-type fixed point results in metric type spaces. *Fixed Point Theory Appl.* **2012**, 2012:126, 12 pp.

[74] Hussain, N.; Roshan, J. R.; Parvaneh, V.; Latif, A. A unification of G-metric, partial metric, and b-metric spaces. *Abstr. Appl. Anal.* **2014**, Art. ID 180698, 14 pp.

[75] Hyers, D. H. A note on linear topological spaces. *Bull. Amer. Math. Soc.* **44** (1938) pp. 76–80.

[76] Imdad, M.; Asim, M.; Gubran, R. Common fixed point theorems for g-generalized contractive mappings in b-metric spaces. *Indian J. Math.* **60** (2018), no. 1, 85–105.

[77] Iqbal, M.; Batool, A.; Ege, O.; de la Sen, M. Fixed point of almost contraction in b-metric spaces. *J. Math.* **2020**, Art. ID 3218134, 6 pp.

[78] Javaher, A.; Sedghi, S.; Altun, I. Fixed point results on b-metric space via b-simulation functions. *Thai J. Math.* **19** (2021), no. 4, 1575–1584.

[79] Jleli, M.; Samet, Bessem. Remarks on G-metric spaces and fixed point theorems. *Fixed Point Theory Appl.* **2012**, 2012:210, 7 pp.

[80] Jleli, M.; Samet, B.; Vetro, C.; Vetro, F. Fixed points for multivalued mappings in b-metric spaces. *Abstr. Appl. Anal.* **2015**, Art. ID 718074, 7 pp.

[81] Jovanović, M.; Kadelburg, Z.; Radenović, S. Common fixed point results in metric-type spaces. *Fixed Point Theory Appl.* **2010**, Art. ID 978121, 15 pp.

[82] Kadelburg, Z.; Radenović, S. Meir-Keeler-type conditions in abstract metric spaces. *Appl. Math. Lett.* **24** (2011), no. 8, 1411–1414.

[83] Kadelburg, Z.; Paunović, L.; Radenović, S. A note on fixed point theorems for weakly T-Kannan and weakly T-Chatterjea contractions in b-metric spaces. *Gulf J. Math.* **3** (2015), no. 3, 57–67.

[84] Kadelburg, Z.; Radenović, S.; Rakočević, V. A note on the equivalence of some metric and cone metric fixed point results. *Appl. Math. Lett.* **24** (2011), no. 3, 370–374.

[85] Kamran, T.; Postolache, M.; Ali, M. U.; Kiran, Q. Feng and Liu type F-contraction in b-metric spaces with an application to integral equations. *J. Math. Anal.* **7** (2016), no. 5, 18–27.

[86] Kantorovich, L. V.; Akilov, G. P. *Functional analysis*. Translated from the Russian by Howard L. Silcock. Second edition. Pergamon Press, Oxford-Elmsford, N.Y., 1982.

[87] Karapınar, E. A short survey on the recent fixed point results on b-metric spaces. *Constr. Math. Anal.* **1** (2018), no. 1, 15–44.

[88] Karapınar, E.; Alsulami, H. H.; Noorwali, M. Some extensions for Geragthy type contractive mappings. *J. Inequal. Appl.* **2015**, 2015:303, 22 pp.

[89] Karapınar, E.; Czerwik, S.; Aydi, H. (α, ψ)-Meir-Keeler contraction mappings in generalized b-metric spaces. *J. Funct. Spaces* **2018**, Art. ID 3264620, 4 pp.

[90] Khamsi, M. A. Remarks on cone metric spaces and fixed point theorems of contractive mappings. *Fixed Point Theory Appl.* **2010**, Art. ID 315398, 7 pp.

[91] Khamsi, M. A.; Hussain, N. KKM mappings in metric type spaces. *Nonlinear Anal.* **73** (2010), no. 9, 3123–3129.

[92] Khani, M.; Pourmahdian, M. On the metrizability of cone metric spaces. *Topology Appl.* **158** (2011), no. 2, 190–193.

[93] Kirk, W. A.; Shahzad, N. *Fixed point theory in distance spaces*. Springer, Cham, 2014.

[94] Kirk, W. A.; Shahzad, N. Fixed points and Cauchy sequences in semimetric spaces. *J. Fixed Point Theory Appl.* **17** (2015), no. 3, 541–555.

[95] Kumam, P.; Nguyen Van Dung; Vo Thi Le Hang. Some equivalences between cone b-metric spaces and b-metric spaces. *Abstr. Appl. Anal.* **2013**, Art. ID 573740, 8 pp.

[96] Kumam, P.; Sintunavarat, W. The existence of fixed point theorems for partial q-set-valued quasi-contractions in b-metric spaces and related results. *Fixed Point Theory Appl.* **2014**, 2014:226, 20 pp.

[97] Kumam, P.; Sintunavarat, W.; Sedghi, S.; Shobkolaei, N. Common fixed point of two R-weakly commuting mappings in b-metric spaces. *J. Funct. Spaces* **2015**, Art. ID 350840, 5 pp.

[98] Kutbi, M. A.; Karapınar, E.; Ahmad, J.; Azam, A. Some fixed point results for multi-valued mappings in b-metric spaces. *J. Inequal. Appl.* **2014**, 2014:126, 11 pp.

[99] Latif, A.; Parvaneh, V.; Salimi, P.; Al-Mazrooei, A. E. Various Suzuki type theorems in b-metric spaces. *J. Nonlinear Sci. Appl.* **8** (2015), no. 4, 363–377.

[100] Lukács, A.; Kajántó, S. Fixed point theorems for various types of F-contractions in complete b-metric spaces. *Fixed Point Theory* **19** (2018), no. 1, 321–333.

[101] Mani, G.; Gnanaprakasam, A. J.; Park, C.; Yun, S. Orthogonal F-contractions on O-complete b-metric space. *AIMS Math.* **6** (2021), no. 8, 8315–8330.

[102] Miculescu, R.; Mihail, A. Caristi-Kirk type and Boyd&Wong-Browder-Matkowski-Rus type fixed point results in b-metric spaces. *Filomat* **31** (2017), no. 14, 4331–4340.

[103] Miculescu, R.; Mihail, A. A generalization of Matkowski's fixed point theorem and Istrăţescu's fixed point theorem concerning convex contractions. *J. Fixed Point Theory Appl.* **19** (2017), no. 2, 1525–1533.

[104] Miculescu, R.; Mihail, A. New fixed point theorems for set-valued contractions in b-metric spaces. *J. Fixed Point Theory Appl.* **19** (2017), no. 3, 2153–2163.

[105] Mitrea, D.; Mitrea, I.; Mitrea, M.; Monniaux, S. *Groupoid metrization theory. With applications to analysis on quasi-metric spaces and functional analysis*. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, New York, 2013.

[106] Mohanta, S. K. Common fixed points in b-metric spaces endowed with a graph. *Mat. Vesnik* **68** (2016), no. 2, 140–154.

[107] Mohanta, S. K.; Patra, S. Coincidence points and common fixed points for hybrid pair of mappings in b-metric spaces endowed with a graph. *J. Linear Topol. Algebra* **6** (2017), no. 4, 301–321.

[108] Mohanta, S. K.; Patra, S. Coincidence points for graph preserving generalized almost $F - G$-contractions in b-metric spaces. *Nonlinear Stud.* **27** (2020), no. 4, 897–914.

[109] Mustafa, Z.; Roshan, J. R.; Parvaneh, V.; Kadelburg, Z. Fixed point theorems for weakly T-Chatterjea and weakly T-Kannan contractions in b-metric spaces. *J. Inequal. Appl.* **2014**, 2014:46, 14 pp.

[110] Olatinwo, M. O.; Imoru, C. O. A generalization of some results on multi-valued weakly Picard mappings in b-metric space. *Fasc. Math.* **40** (2008), 45–56.

[111] Olatinwo, M. O. Some results on multi-valued weakly Jungck mappings in b-metric space. *Cent. Eur. J. Math.* **6** (2008), no. 4, 610–621.

[112] Olatinwo, M. O. An extension of some fixed point theorems for single-valued and multi-valued Picard operators in b-metric spaces. *Nonlinear Anal. Forum* **14** (2009), 103–111.

[113] Oprea, A.; Petruşel, G. Coupled fixed point theorems for rational type contractions. *Stud. Univ. Babeş-Bolyai Math.* **61** (2016), no. 4, 463–488.

[114] Özturk, V.; Turkoglu, D. Fixed points for generalized α-ψ-contractions in b-metric spaces. *J. Nonlinear Convex Anal.* **16** (2015), no. 10, 2059–2066.

[115] Özturk, V.; Turkoglu, D.; Ansari, A. H. Fixed points for generalized $(\mathcal{F}, h, \alpha, \mu)$-$\psi$-contractions in b-metric spaces. *Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat.* **67** (2018), no. 2, 306–316.

[116] Paesano, D.; Vetro, P. Fixed point theorems for α-set-valued quasi-contractions in b-metric spaces. *J. Nonlinear Convex Anal.* **16** (2015), no. 4, 685–696.

[117] Păcurar, M. *Iterative methods for fixed point approximation*. Editura Risoprint, Cluj-Napoca, 2009.

[118] Păcurar, M. A fixed point result for ϕ-contractions on b-metric spaces without the boundedness assumption. *Fasc. Math.* **43** (2010), 127–137.

[119] Paluszyński, M.; Stempak, K. On quasi-metric and metric spaces. *Proc. Amer. Math. Soc.* **137** (2009), no. 12, 4307–4312.

[120] Panja, S.; Roy, K.; Saha, M. Fixed points for a class of extended interpolative ψF-contraction maps over a b-metric space and its application to dynamical programming. *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* **83** (2021), no. 1, 59–70.

[121] Petruşel, A.; Petruşel, G.; Samet, B.; Yao, J.-C. Coupled fixed point theorems for symmetric contractions in b-metric spaces with applications to operator equation systems. *Fixed Point Theory* **17** (2016), no. 2, 457–475.

[122] Phiangsungnoen, S.; Kumam, P. Generalized Ulam-Hyers stability and well-posedness for fixed point equation via α-admissibility. *J. Inequal. Appl.* **2014**, 2014:418, 12 pp.

[123] Pietsch, A. *History of Banach spaces and linear operators*. Birkhäuser Boston, Inc., Boston, MA, 2007.

[124] Prabakar, G. Arockia; Uthayakumar, R. Coincident and common fixed point theorems in b-metric space. *J. Adv. Res. Pure Math.* **7** (2015), no. 1, 41–51.

[125] Radenović, S.; Došenović, T.; Lampert, T. A.; Golubovć, Z. A note on some recent fixed point results for cyclic contractions in b-metric spaces and an application to integral equations. *Appl. Math. Comput.* **273** (2016), 155–164.
[126] Rahrovi, S.; Piri, H. Fixed point theorems concerning rational Geraghty contraction in a b-metric space. *Thai J. Math.* **18** (2020), no. 4, 1961–1977.
[127] Rashwan, R. A.; Moustafa, S. I. On coupled coincidence and common fixed point theorems for different types of mappings satisfying rational type contractions in b-metric spaces. *J. Adv. Stud. Topol.* **8** (2017), no. 1, 40–58.
[128] Rus, I. A.; Petruşel, A.; Petruşel, G. *Fixed point theory*. Cluj University Press, Cluj-Napoca, 2008.
[129] Salimi, P.; Vetro, P. Common fixed point results on quasi-Banach spaces and integral equations. *Georgian Math. J.* **20** (2013), no. 4, 789–804.
[130] Samreen, M.; Kamran, T.; Shahzad, N. Some fixed point theorems in b-metric space endowed with graph. *Abstr. Appl. Anal.* **2013**, Art. ID 967132, 9 pp.
[131] Samreen, M.; Kiran, Q.; Kamran, T. Fixed point theorems for φ-contractions. *J. Inequal. Appl.* **2014**, 2014:266, 16 pp.
[132] Schroeder, V. Quasi-metric and metric spaces. *Conform. Geom. Dyn.* **10** (2006), 355–360.
[133] Shahkoohi, R. J.; Razani, A. Some fixed point theorems for rational Geraghty contractive mappings in ordered b-metric spaces. *J. Inequal. Appl.* **2014**, 2014:373, 23 pp.
[134] Shatanawi, W. Fixed and common fixed point for mappings satisfying some nonlinear contractions in b-metric spaces. *J. Math. Anal.* **7** (2016), no. 4, 1–12.
[135] Shatanawi, W.; Pitea, A.; Lazović, R. Contraction conditions using comparison functions on b-metric spaces. *Fixed Point Theory Appl.* **2014**, 2014:135, 11 pp.
[136] Singh, S. L.; Bhatnagar, C.; Mishra, S. N. Stability of iterative procedures for multivalued maps in metric spaces. *Demonstratio Math.* **38** (2005), no. 4, 905–916.
[137] Singh, S. L.; Prasad, Bhagwati. Some coincidence theorems and stability of iterative procedures. *Comput. Math. Appl.* **55** (2008), no. 11, 2512–2520.
[138] Som, S.; Petruşel, A.; Dey, L. K. Some remarks on the metrizability of some metric-like structures. *Carpathian J. Math.* **37** (2021), No. 2, 265–272.
[139] Suzuki, T. Basic inequality on a b-metric space and its applications. *J. Inequal. Appl.* **2017**, Paper No. 256, 11 pp.
[140] Świątkowski, T. b-metric spaces. *Zeszyty Nauk. Politech. Lódz. Mat.* **26** (1994), 69–80.
[141] Teweldemedhin, S. G.; Tola, K. K. Fixed point results for an almost generalized α-admissible Z-contraction in the setting of partially ordered b-metric spaces. *Abstr. Appl. Anal.* **2021**, Art. ID 5988007, 11 pp.
[142] Tsegaye Leyew, B.; Abbas, M. Fixed point results of generalized Suzuki-Geraghty contractions on f-orbitally complete b-metric spaces. *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* **79** (2017), no. 2, 113–124.
[143] Tychonoff, A. Ein Fixpunktsatz. (German) *Math. Ann.* **111** (1935), 767–776.
[144] Vulpe, I. M.; Ostrajkh, D.; Khojman, F. The topological structure of a quasi-metric space. (Russian) in *Investigations in functional analysis and differential equations*, Math. Sci., Interuniv. Work Collect., Kishinev 1981, 14–19 (1981).
[145] Wilson, W. A. On Quasi-Metric Spaces. *Amer. J. Math.* **53** (1931), no. 3, 675–684.
[146] Wirtu Geleta, K.; Tola, K. K.; Teweldemedhin, S. G. Alpha-F-convex contraction mappings in b-metric space and a related fixed point theorem. *J. Funct. Spaces* **2021**, Art. ID 5720558, 7 pp.
[147] Yamaod, O.; Sintunavarat, W.; Cho, Y. J. Existence of a common solution for a system of nonlinear integral equations via fixed point methods in b-metric spaces. *Open Math.* **14** (2016), no. 1, 128–145.
[148] Yang, Z.; Sadati, H.; Sedghi, S.; Shobe, N. Common fixed point theorems for non-compatible self-maps in b-metric spaces. *J. Nonlinear Sci. Appl.* **8** (2015), no. 6, 1022–1031.

[1] TECHNICAL UNIVERSITY OF CLUJ-NAPOCA, NORTH UNIVERSITY CENTRE AT BAIA MARE, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, VICTORIEI 76, RO-430122 BAIA MARE, ROMANIA
Email address: vasile.berinde@mi.utcluj.ro

[2] ACADEMY OF ROMANIAN SCIENTISTS
Email address: vasile.berinde@gmail.com

[3] DEPARTMENT OF ECONOMICS AND BUSSINESS ADMINISTRATION IN GERMAN LANGUAGE, FACULTY OF ECONOMICS AND BUSSINESS ADMINISTRATION, BABEŞ-BOLYAI UNIVERSITY OF CLUJ-NAPOCA, T. MIHALI 58-60, 400591 CLUJ-NAPOCA, ROMANIA
Email address: madalina.pacurar@econ.ubbcluj.ro