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Approximating G-variational inequality problem by G-subgradient extragradient method in Hilbert space endowed with graphs

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ABSTRACT. In this article, we introduce G-subgradient extragradient method for solving the G- variational inequality problem in Hilbert space endowed with a direct graph. Utilizing our mathematical tools, weak and strong convergence theorem are established for the proposed algorithm. In addition, we provide numerical experiments to illustrate the convergence behavior of our proposed algorithm.

1. INTRODUCTION

Let H be a real Hilbert space and D be a nonempty closed convex subset of a real Hilbert space H. The set of fixed points is denoted by $F(T) = \{x \in C : Tx = x\}$, where $T : D \to D$ is a mapping. The following symbols will be used throughout this research:

i) $G = (Eed(G), Ver(G))$ is a *directed graph* where $Ver(G)$ is *vertices set* and $Eed(G)$ is set of its edges with $\{(x,x) : x \in Ver(G)\} \subseteq Eed(G)$

ii) $Eed\big(G^{-1}\big) = \{(y,x) : (x,y) \in Eed(G)\}$.

The variational inequality problem (VIP) is to find a point $z^* \in D$ such that

$$\langle y - z^*, Bz^* \rangle \geqslant 0,$$

for all $y \in D$, where $B : D \to H$ is a mapping. The Variational inequality problems can be used to solve problems in engineering, economics, and physics; see more details in [2, 5, 9, 11].

The most famous technique for solving the problem (VIP) is the extragradient method suggested by Korpelevich [7]. This process must enumerate two projections onto the feasible set D in each iteration. If the set D is a half-space or a closed ball, effectiveness is completed in the result of the projection onto D. In the recent years, the extragradient method has approved meaningful awareness by numerous authors, who developed it in different ways, see, e.g. [2, 3, 5] and the several citations therein.

In [1], Censor et al. introduced a new extragradient method as follows:

$$\begin{cases} w_n = P_D(I - \lambda B)v_n \\ T^n = \{w \in D : \langle (I-\lambda B)v_n - w_n, w_n - w\rangle \geqslant 0\} \\ v_{n+1} = P_{T^n}\left(v_n - \lambda B w_n\right), \end{cases} \tag{1.1}$$

for all $n \in \mathbb{N}$ and $\lambda > 0$. They proved that $\{v_n\}$ generated by (1.1) converges weakly to a solution of VIP. In this technique they have renovated the second projection in Korpelevich 's extragradient method with a projection onto a half-space, which is estimated explicitly. Such method is called *subgradient extragradient*.

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Jachymski [4] was the first to analyze the fixed point problem in metric space endowed with graph and introduce the crucial conclusion in this space by integrating fixed point properties and graph theory, see more detail in [4].

Let $D = Ver(G)$ and the mapping $T : D \to D$ is called G-nonexpansive if the following conditions hold:

1) T is edge-preserving i.e., for each $x, y \in D$ such that $(x, y) \in Eed(G) \Rightarrow (Tx,Ty) \in Eed(G)$,

2) $\|Tx - Ty\| \leqslant \|x - y\|$, whenever $(x, y) \in Eed(G)$ for all $x, y \in D$.

Tiammee et al. were the first to prove the strong convergence theorem of a sequence generated by Halpern iteration for approximating fixed point problem of G-nonexpansive mapping in Hilbert space endowed with a directed graph. See more detail [10].

Using concepts related to the variational inequality problem and graph theory, Kangtunyakarn [6] introduced the G-variational inequality problem, which is to find a point $x^* \in D$ such that

$$\langle y - x^*, Bx^* \rangle \geqslant 0,$$

for all $y \in D$ with $(x^*, y) \in Eed(G)$ and $B : D \to H$ is a mapping, where $D = Ver(G)$. The set of all solution of such problem denoted by $G - Var(D, B)$. He proved strong convergence theorem to solve G-variational inequality problem.

By combining the concepts of subgradient extragadient method and graph theory in this research, we introduce G-subgradient extragadient method for approximating the solution of G-variational inequality problem. To use such a method, we introduce G-Half space by

$$T_G = \{w \in D : \langle (I - \lambda B)x - y, y - w \rangle \geqslant 0\},$$

where $\lambda > 0$, $B : D \to H$ is a mapping and $y = P_D(I - \lambda B)x$ for all $x \in H$ with $(w, x) \in Eed(G)$.

Example 1.1. Let $H = \mathbb{R}^2$ and $D = [-100, 100] \times [-100, 100]$ and metric projection $P_D : H \to D$ define by

$$P_D(z_1, z_2) = (\max\{\min\{z_1, 100\}, -100\}, \max\{\min\{z_2, 100\}, -100\}),$$

for all $z = (z_1, z_2) \in H$.

Let $B : D \to H$ define by $Bx = \left(\frac{v_1}{3}, \frac{v_2}{3}\right)$ for all $x = (v_1, v_2) \in D$ and $Ver(G) = D$, $Eed(G) = \{(u, v) : u = (u_1, u_2) \in [0, 100] \times [0, 100]$ and $v = (v_1, v_2) \in (300, \infty) \times (300, \infty)\}$.

Putting $\lambda = 2$. From definitions of P_D and B, we have $P_D(1 - \lambda B)x = P_D\left(\frac{v_1}{3}, \frac{v_2}{3}\right)$ for all $x = (v_1, v_2) \in H$.

Let $(w, x) \in Eed(G)$, where $w = (w_1, w_2)$, $x = (v_1, v_2)$. From definition of P_D, we have $P_D(I - \lambda B)x = (100, 100)$ and $T_G = [0, 100] \times [0, 100]$.

In this paper, motivated by the research [7, 1] and [6], we introduce a G-subgradient extragradient method for solving the G-variational inequality problem in Hilbert space endowed with a direct graph. Then we establish weak and strong convergence theorems under some proper conditions. Furthermore, we also give some examples to support our main result.

2. Preliminaries

This section collects well known definitions and lemmas as an essential tool for proving our main theorems.

Let D be a nonempty closed convex subset of a real Hilbert space H. We denote strong convergence and weak convergence by notations $\to$ and $\rightharpoonup$, respectively. For every $x \in H$, there exists a unique nearest point $P_D x \in D$ such that

$$||x - P_D x|| \leq ||x - y||, \quad \forall y \in D.$$

P_D is called metric projection of H onto D.
For each $x \in H$ and $y \in D$. It follows that

$$||x-y||^2 \geq ||x-P_Dx||^2 + ||y-P_Dx||^2. \tag{2.2}$$

Lemma 2.1. *Let P_D be the metric projection from H onto D. Then*
i) P_D is a nonexpansive mapping, i.e.

$$\|P_Dx - P_Dy\| \leqslant \|x-y\|, \ \forall\, x, y \in H.$$

ii) $y = P_Dx \Leftrightarrow \langle x-y, y-z\rangle \geqslant 0 \quad \forall x \in D$

Definition 2.1. [8] A subset X of $Ver(G)$ is called a *dominating set* if for every v belong to $Ver(G) - X$ there exists a point x belong to X such that (x, v) belong to $Eed(G)$ and we said that x *dominates* v or v *is dominated by* x. A subset Z of $Ver(G)$ is *dominated by* $v \in Ver(G)$ if $(v, z) \in Eed(G), \forall z \in Z$ and we said that X *dominates* v if $(x, v) \in Eed(G), \forall x \in X$.

Definition 2.2. [8] A graph G is called *transitive* if for every $x, y \in Ver(G)$ with$(x, y), (y, z) \in Eed(G)$, then $(x, z) \in Eed(G)$.

Property G [6] Vertices set $Ver(G) = D$ is said to have *Property G* if every sequence $\{a_n\}$ in D converging weakly to $x \in D$, there is a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $(a_{n_k}, x) \in Eed(G)\ \forall\, k \in \mathbb{N}$.

Definition 2.3. [6]Let $Ver(G) = D$. The mapping $B : D \to H$ is called *G-α-inverse strongly monotone*(G-α-ism) if there is $\alpha > 0$ such that

$$\langle Bx - By, x-y\rangle \geqslant \alpha \,\|Bx - By\|^2$$

$\forall x, y \in D$ with $(x, y) \in Eed(G)$.

The difference between G-α-ism and α-inverse strongly monotone is found in the reference [6].

Lemma 2.2. [6] *Let $Eed(G)$ be a convex and $Ver(G) = D$. Let $G = (Ver(G), Eed(G))$ be a direct graph and G be transitive with $Eed(G) = Eed\big(G^{-1}\big)$. Let $B : D \to H$ is G-α-ism operator with $B^{-1}(0) \neq \emptyset$. Then $G - Var(D, B) = B^{-1}(0) = F\left(P_D(I - \lambda B)\right)$, for all $\lambda > 0$.*

Lemma 2.3. [6] *Let $Eed(G)$ be a convex and $Ver(G) = D$. Let $G = (Ver(G), Eed(G))$ be a direct graph and let $B : D \to H$ is G-α-ism operator. For every $\forall\ \lambda \in (0, 2\alpha)$, if $F\left(P_D(I - \lambda B)\right) \times F\left(P_D(I - \lambda B)\right) \subseteq Eed(G)$, then $F\left(P_D(I - \lambda B)\right)$ is closed and convex.*

Lemma 2.4. [9] *Let $\{a_n\}$ and $\{b_n\}$ be subset of $[0, \infty)$ satisfying*

$$a_{n+1} \leqslant a_n + b_n,$$

for all $n \in \mathbb{N}$.

i) if $\displaystyle\sum_{n=1}^{\infty} b_n < \infty$, then $\displaystyle\lim_{n\to\infty} a_n$ exists

ii) if $\displaystyle\sum_{n=1}^{\infty} b_n < \infty$ and there exist a subsequence of $\{a_n\}$ converging to zero, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. [9] *Let $\{v_n\}$ be a sequence in H. Suppose that, for all $u \in D$,*

$$\|v_{n+1} - u\| \leqslant \|v_n - u\| + b_n,$$

for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} b_n < \infty$. Then $\{P_Dv_n\}$ converges strongly to some $z \in D$.

Lemma 2.6. [11] *Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||$$

holds for every $y \in H$ with $x \neq y$.

3. Main results

Theorem 3.1. *Let $G, Ver(G), Eed(G), B$ as in Lemma 2.2. Assume that $G - Var(D,B) \neq \emptyset$ with $G - Var(D,B) \times G - Var(D,B) \subseteq Eed(G)$. Let $\{v_n\}$ be a sequence defined by $v_0 \in D$ and*

$$\begin{cases} w_n = P_D(I-\lambda B)v_n \\ T_G^n = \{w \in D : \langle (I-\lambda B)v_n - w_n, w_n - w\rangle \geqslant 0\} \\ v_{n+1} = P_{T_G^n}\left(v_n - \lambda B w_n\right), \end{cases}$$

for all $n \in N$ where $\lambda \in (0,\alpha)$ and T_G^n is G- Half space. Then sequence $\{v_n\}$ converges weakly to an element $\bar{x} \in G - Var(D,B)$ and the sequence $\left\{P_{G-Var(D,B)}v_n\right\}$ converges strongly to $\bar{x}$, where $G - Var(D,B)$ dominates v_n, $\{v_n\}$ dominates v_0 and $\{w_n\}$ is dominated by v_0.

Proof. Let $v^* \in G - Var(D,B)$. Since $G - Var(D,B)$ dominates by v_n, we have $(v_{n,}, v^*) \in Eed(G)$ for all $n \in \mathbb{N}$. From Lemma 2.2, we have $v^* = P_D(I-\lambda B)v^*$.
Utilizing Definition 2.3, we have

$$\begin{aligned} \|w_n - v^*\|^2 &\leq \|v_n - v^*\|^2 - 2\lambda\langle Bv_n - Bv^*, v_n - \nu^*\rangle + \lambda^2\|Bv_n - Bv^*\|^2 \\ &\leq \|v_n - \nu^\star\|^2 - 2\lambda\alpha\|Bv_n - Bv^*\|^2 + \lambda^2\|Bv_n - Bv^*\|^2 \\ &= \|v_n - v^*\|^2 - \lambda(2\alpha - \lambda)\|Bv_n - Bv^*\|^2 \\ &\leq \|v_n - v^*\|^2. \end{aligned}$$

Due to $\{v_n\}$ dominates v_0 and $\{w_n\}$ is dominated by v_0, we have $(v_n, v_0), (v_0, w_n) \in Eed(G)$.
Exploiting of G is transitive, we get $(v_n, w_n) \in Eed(G)$.
From the assumption that $Eed(G) = Eed\big(G^{-1}\big)$, we deduce that $(w_n, v_n) \in Eed(G)$.
From $(w_n, v_n), (v_{n,}, v^*) \in Eed(G)$ and the assumption that G is transitive, we get $(w_n, v^*) \in Eed(G)$.
From iteration of sequence $\{v_n\}$, we have

$$\begin{aligned} \|v_{n+1} - v^*\|^2 &\leqslant \|v_n - \lambda Bw_n - v^*\|^2 - \left\|v_n - \lambda Bw_n - P_{T_G^n}\left(v_n - \lambda Bw_n\right)\right\|^2 \\ &= \|v_n - v^*\|^2 - 2\lambda\langle Bw_n, v_n - v^*\rangle + \|\lambda Bw_n\|^2 \\ &\quad - \left\|v_n - P_{T_G^n}\left(v_n - \lambda Bw_n\right)\right\|^2 + 2\lambda\left\langle Bw_n, v_n - P_{T_G^n}\left(v_n - \lambda Bw_n\right)\right\rangle \\ &\quad - \|\lambda Bw_n\|^2 \\ &= \|v_n - v^*\|^2 - 2\lambda\langle Bw_n, P_{T_G}\left(v_n - \lambda Bw_n\right) - v^*\rangle \\ &\quad - \left\|v_n - P_{T_G^n}\left(v_n - \lambda Bw_n\right)\right\|^2 \end{aligned} \tag{3.3}$$

From $(w_n, v^*) \in Eed(G)$ and monotonicity of B, we have

$$\begin{aligned} 0 &\leqslant \langle Bw_n - Bv^*, w_n - v^*\rangle \\ &= \langle Bw_n, w_n - v^*\rangle - \langle Bv^*, w_n - v^*\rangle \\ &\leqslant \langle Bw_n, w_n - v^*\rangle \\ &= \left\langle Bw_n, w_n - P_{T_G^n}\left(v_n - \lambda Bw_n\right)\right\rangle + \left\langle Bw_n, P_{T_G^n}\left(v_n - \lambda Bw_n\right) - v^*\right\rangle. \end{aligned}$$

It implies that

$$-2\lambda \left\langle Bw_n, P_{T_G^n}(v_n - \lambda Bw_n) - v^* \right\rangle \leqslant 2\lambda \left\langle Bw_n, w_n - P_{T_G^n}(v_n - \lambda Bw_n) \right\rangle \tag{3.4}$$

From (3.3) and (3.4), we have

$$\begin{aligned}
\|v_{n+1} - v^*\|^2 \leq\ & \|v_n - v^*\|^2 - 2\lambda \left\langle Bw_n, P_{T_G^n}(v_n - \lambda Bw_n) - v^* \right\rangle \\
& - \left\|v_n - P_{T_G^n}(v_n - \lambda B_{v_n})\right\|^2 \\
\leq\ & \|v_n - v^*\|^2 + 2\lambda \left\langle Bw_n, w_n - P_{T_G^n}(v_n - \lambda Bw_n) \right\rangle \\
& - \left\|v_n - P_{T_G^n}(v_n - \lambda B_{v_n})\right\|^2 \\
=\ & \|v_n - v^*\|^2 + 2\lambda \left\langle Bw_n, w_n - P_{T_G^n}(v_n - \lambda Bw_n) \right\rangle \\
& - \|v_n - w_n\|^2 - 2\left\langle v_n - w_n, w_n - P_{T_G^n}(v_n - \lambda Bw_n) \right\rangle \\
& - \left\|w_n - P_{T_G^n}(v_n - \lambda Bw_n)\right\|^2 \\
=\ & \|v_n - v^*\|^2 - \|v_n - w_n\|^2 - \left\|w_n - P_{T_G^n}(v_n - \lambda Bw_n)\right\|^2 \\
& +2\left\langle \lambda Bw_n - v_n + w_n, w_n - P_{T_G^n}(v_n - \lambda Bw_n) \right\rangle \\
=\ & \|v_n - v^*\|^2 - \|v_n - w_n\|^2 - \left\|w_n - P_{T_G^n}(v_n - \lambda Bw_n)\right\|^2 \\
& +2\left\langle (I - \lambda B)v_n - w_n, P_{T_G^n}(v_n - \lambda Bw_n) - w_n \right\rangle \\
& +2\lambda \left\langle Bv_n - Bw_n, P_{T_G^n}(v_n - \lambda Bw_n) - w_n \right\rangle \\
\leq\ & \|v_n - v^*\|^2 - \|v_n - w_n\|^2 - \left\|w_n - P_{T_G^n}(v_n - \lambda Bw_n)\right\|^2 \\
& +2\lambda \left\langle Bv_n - Bw_n, P_{T_G^n}(v_n - \lambda Bw_n) - w_n \right\rangle \\
\leq\ & \|v_n - v^*\|^2 - \|v_n - w_n\|^2 - \left\|w_n - P_{T_G^n}(v_n - \lambda Bw_n)\right\|^2 \\
& +2\frac{\lambda}{\alpha}\|Bv_n - Bw_n\| \cdot \|P_{T_G^n}(v_n - \lambda Bw_n) - w_n\| \\
\leq\ & \|v_n - v^*\|^2 - \|v_n - w_n\|^2 - \left\|w_n - P_{T_G^n}(v_n - \lambda Bw_n)\right\|^2 \\
& +\frac{\lambda}{\alpha}\left(\|v_n - w_n\|^2 + \|P_{T_G^n}(v_n - \lambda Bw_n) - w_n\|^2\right) \\
=\ & \|v_n - v^*\|^2 - \left(1 - \frac{\lambda}{\alpha}\right)\|v_n - w_n\|^2 \\
& - \left(1 - \frac{\lambda}{\alpha}\right)\left\|P_{T_G^n}(v_n - \lambda Bw_n) - w_n\right\|^2 .
\end{aligned} \tag{3.5}$$

From Lemma 2.4, we have $\lim_{n\to\infty} \|v_n - v^*\|^2$ exists for all $v^* \in G - Var(D, B)$ and $\{v_n\}$ is a bounded sequence.
From (3.5) and $\lim_{n\to\infty} \|v_n - v^*\|^2$ exists, we have

$$\lim_{n\to 2} \|P_D(I - \lambda B)v_n - v_n\| = \lim_{n\to\infty} \|v_n - w_n\| = 0.$$

Since $\{v_n\}$ is a bounded sequence, there is a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ converses weakly to $\bar{x}$.
Since D have a property G, we have $(v_{n_k}, \bar{x}) \in Eed(G)$.
Assume that $P_D(I - \lambda B)\bar{x} \neq \bar{x}$. By opial property and using the same method as (3.3), we

have

$$\begin{aligned}\limsup_{k\to\infty}\|v_{n_k}-\bar{x}\| &< \limsup_{k\to\infty}\|v_{n_k}-P_D(I-\lambda B)\bar{x}\| \\ &\leq \limsup_{k\to\infty}\left(\|v_{n_k}-P_D(I-\lambda B)v_{n_k}\|+\|P_D(I-\lambda B)v_{n_k}-P_D(I-\lambda B)\bar{x}\|\right) \\ &\leq \limsup_{k\to\infty}\|v_{n_k}-\bar{x}\|.\end{aligned}$$

Contradiction. So, we have $P_D(I-\lambda B)\bar{x}=\bar{x}$.
Let $y\in D$ with $(\bar{x},y)\in D$, then

$$\langle (I-\lambda B)\tilde{x}-\tilde{x},\bar{x}-y\rangle \geqslant 0.$$

It follows that

$$\langle y-\bar{x},B\tilde{x}\rangle \geqslant 0,$$

for all $y\in D$ with $(\bar{x},y)\in D$. Then, we have$\bar{x}\in G-Var(D,B)$.
Therefore $v_{n_k}\rightharpoonup \bar{x}\in G-Var(D,B)$ as $k\to\infty$.
Since$(v_{n_k},\bar{x})\in Eed(G)$ and using the same method as $\lim_{n\to\infty}\|v_n-v^*\|$ exists, we have $\lim_{k\to\infty}\|v_{n_k}-\bar{x}\|$ exists.
At the end of this theorem we demonstrate that $\{v_n\}$ converses weakly to $\bar{x}$. Assume that $v_{n_n}\rightharpoonup \hat{x}$ as $k\to\infty$ and $\bar{x}\neq\hat{x}$. Thank to the Opial's condition, we obtain

$$\begin{aligned}\lim_{n\to\infty}\|v_n-\bar{x}\| &= \limsup_{k\to\infty}\|v_{n_k}-\bar{x}\| \\ &< \limsup_{k\to\infty}\|v_{n_k}-\hat{x}\| \\ &< \limsup_{k\to\infty}\|v_{n_k}-\bar{x}\| \\ &= \lim_{n\to\infty}\|v_n-\bar{x}\|.\end{aligned}$$

Contradiction. So, we get $\bar{x}=\hat{x}$. We can conclude that a sequence $\{v_n\}$ converges weakly to $\bar{x}\in G-Var(D,B)$.
Due to (3.5) and exploiting of Lemma 2.5, we have $\left\{P_{G-Var(D,B)}v_n\right\}$ converges strongly to $z\in G-Var(D,B)$.
From property of $P_{G-Var(D,B)}$, we have

$$\left\langle v_n-P_{G-Var(D,B)}v_n, P_{G-Var(D,B)}v_n-\bar{x}\right\rangle \geqslant 0.$$

Take $n\to\infty$, we have $\|z-\bar{x}\|=0$. So, we have $z=\bar{x}$. Therefore we can conclude that$\left\{P_{G-Var(D,B)}v_n\right\}$ converges strongly to $\bar{x}\in G-Var(D,B)$.This is ultimately the prove. □

4. Application

To resolved a fixed point problem in Hilbert space endowed with a direct graph by using G-subgradient extragradient method, we required the following lemma;

Lemma 4.7. [6] *Let D be a nonempty closed convex subset of a real Hilbert space H and let $G=(Ver(G),Eed(G))$ be a directed graph with $D=Ver(G)$ having property G. Let $Eed(G)$ be a convex set with $Eed(G)=Eed(G^{-1})$. Let $T:D\to D$ be G-nonexpansive mapping with $F(T)\neq\emptyset$ and $F(T)\times F(T)\subseteq Eed(G)$. Then*
i) $I-T$ is $G-\frac{1}{2}$- inverse strongly monotone,
ii) $G-Var(D,I-T)=F(T)$.

The following theorem is an immediate result of Theorem 3.1 and Lemma 4.7.

Theorem 4.2. *Let D be a nonempty closed convex subset of a real Hilbert space H and let $G = (Ver(G), Eed(G))$ be a directed graph with $D = Ver(G)$ having property G. Let $Eed(G)$ be a convex set and G be transitive with $Eed(G) = Eed\left(G^{-1}\right)$ and let $T : D \to D$ be G-nonexpansive mapping with $F(T) \neq \emptyset$ and $F(T) \times F(T) \subseteq Eed(G)$. Let $\{v_n\}$ be a sequence defined by $v_0 \in D$ and*

$$\begin{cases} w_n = P_D(I - \lambda(I - T)v_n \\ T_G^n = \{w \in D : \langle (I - \lambda(I - T)v_n - w_n, w_n - v\rangle \geq 0\} \\ v_{n+1} = P_{T_G^n}\left(v_n - \lambda(I - T)w_n\right), \end{cases}$$

for all $n \in N$ where $\lambda \in (0, \alpha)$ and T_G^n is G- Half space. Then sequence $\{v_n\}$ converses weakly to an element $\bar{v} \in F(T)$ and the sequence $\left\{P_{F(T)}v_n\right\}$ converges strongly to $\bar{v}$, where $F(T)$ dominates v_n, $\{v_n\}$ dominates v_0 and $\{w_n\}$ is dominated by v_0.

Following that, we provide an example to support our main result.

Example 4.2. Let $D = [-1, 1]$ and $G = (D, Eed(G))$ be a directed graph, where $Eed(G) = \{(x, y) : x, y \in [0, 1]\}$. Let the mappings $B : D \to \mathbb{R}$ define by $Bx = x - \frac{x^3}{4} - \frac{15}{32}$, and $S : D \to \mathbb{R}$ define by $Sx = \frac{x^3}{4} + \frac{15}{32}$, for all $x \in D$.
Suppose that the sequence $\{v_n\}$ is generated by $v_0 = 1$ and

$$\begin{cases} w_n = P_D(I - \lambda B)v_n \\ T_G^n = \{w \in D : \langle (I - \lambda B)v_n - w_n, w_n - w\rangle \geq 0\} \\ v_{n+1} = P_{T_G^n}\left(v_n - \lambda B w_n\right), \end{cases} \tag{4.6}$$

for all $n \in N$ where $\lambda \in (0, \alpha)$ and T_G^n is G- Half space. Then sequence $\{v_n\}$ converses weakly to an element of $\bar{v} \in G - Var(D, B)$ and the sequence $\left\{P_{G-Var(D,B)}v_n\right\}$ converges strongly to $\bar{v}$, where $G - Var(D, B)$ dominates v_n, $\{v_n\}$ dominates v_0 and $\{w_n\}$ is dominated by v_0.
Solution. It is obvious that $\frac{1}{2} \in F(S)$, and $Eed(G) = Eed(G^{-1})$.
First, we show that S is a G-nonexpansive mapping. Let $x, y \in D$ with $(x, y) \in Eed(G)$. Then, we have $x, y \in [0, 1]$. Since x^3, y^3 , $\frac{5}{8} \in [0, 1]$ and $[0, 1]$ is a convex set, we have

$$Sx = \frac{1}{4}x^3 + \frac{3}{4}(\frac{5}{8}) \ \in \ [0, 1]$$

and

$$Sy = \frac{1}{4}y^3 + \frac{3}{4}(\frac{5}{8}) \ \in \ [0, 1].$$

From definition of S, we have

$$\begin{aligned} |Sx - Sy| &= |(\frac{x^3}{4} + \frac{15}{32}) - (\frac{y^3}{4} + \frac{15}{32})| = |\frac{x^3}{4} - \frac{y^3}{4}| \\ &= \frac{1}{4}|x^2 + xy + y^2||x - y| \leq \frac{1}{4}(3)|x - y| \\ &\leq |x - y|. \end{aligned}$$

Then $(Sx, Sy) \in Eed(G)$. Therefore S is a G-nonexpansive mapping.
Since $Bx = (I - S)x$, S is a G-nonexpansive mapping and Lemma 4.7, we have B is G-$\frac{1}{2}$-inverse strongly monotone. It is obvious that $G - Var(D, B) = \{\frac{1}{2}\}$.
Putting $\lambda = \frac{1}{4}$. From convexity of $[0, 1]$, we have

$$(I - \frac{1}{4}B)z \quad = \quad \frac{3}{4}z + \frac{1}{4}(\frac{z^3}{4} + \frac{15}{32}) \in [0, 1],$$

for all $z \in [0,1]$.
From definition of P_D, it follows that

$$P_D(I - \frac{1}{4}B)z \in [0,1], \tag{4.7}$$

for all $z \in [0,1]$.
Let $(w,z) \in Eed(G)$. From definition of T_G^n and (4.7), we have $T_G^n \subseteq [0,1]$.
Since $v_0 \in [0,1]$ and (4.7), we have

$$w_0 = P_D(I - \frac{1}{4}B)v_0 \in [0,1], \tag{4.8}$$

From $T_G^n \subseteq [0,1]$, we have

$$v_1 = P_{T_G^n}(v_0 - \frac{1}{4}Bw_0) \in [0,1], \tag{4.9}$$

Continue the method of (4.8) and (4.9), we have $w_n, v_n \in [0,1]$ for all $n \in \mathbb{N}$.
Since $v_0, \frac{1}{2}, v_n$ and $w_n \in [0,1]$, it follows that $(\frac{1}{2}, v_n)$, (v_n, v_0) and $(w_n, v_0) \in Eed(G)$.
We can conclude that $G - Var(D,B)$ dominates v_n, $\{v_n\}$ dominates v_0 and $\{w_n\}$ is dominated by v_0. All conditions of Example 4.2 satisfies Theorem 3.1, so we can conclude that sequence $\{v_n\}$ converses weakly to an element of $\frac{1}{2} \in G - Var(D,B)$ and the sequence $\{P_{G-Var(D,B)}v_n\}$ converges strongly to $\frac{1}{2}$.

n	$\mathbf{v}_n$
1	1.0000000
2	0.9349873
3	0.8699746
4	0.8108711
$\vdots$	$\vdots$
19	0.5159698
20	0.5101756

TABLE 1. Detailed analysis of computational methods (4.6) for Example 4.1 with $\mathbf{v_0} = 1$, $N = 20$.

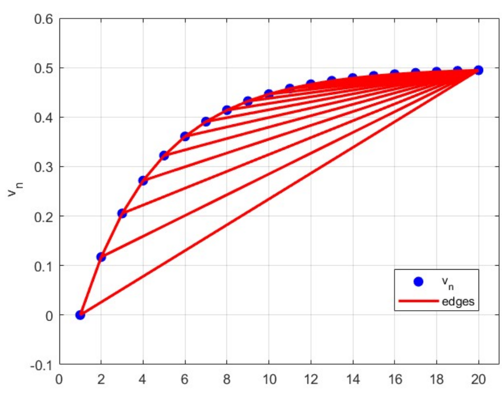

FIGURE 1. *The convergence behavious of* $\{\mathbf{v_n}\}$ with $v_0 = 1$ and $N = 20$.

Example 4.3. Let $D = [-5, 5]$ and $G = (D \times D, Eed(G))$ be a directed graph, where $Eed(G) = \{(x, y) : x = (x_1, x_2), y = (y_1, y_2) \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]\}$. Let the mappings $B : D \times D \to \mathbb{R}^2$ define by $B(x_1, x_2) = (\frac{4x_1}{5} - \frac{8}{5}, \frac{x_2}{4})$, for all $x_1, x_2 \in D$.
Let metric projection $P_D : H \times H \to D \times D$ define by

$$P_D\left(z_1, z_2\right) = \left(\max\left\{\min\left\{z_1, 5\right\}, -5\right\}, \max\left\{\min\left\{z_2, 5\right\}, -5\right\}\right),$$

for all $z = (z_1, z_2) \in H \times H$.
Suppose that the sequence $\{v^n\}$ is generated by $v^0 = (v_1^0, v_2^0) = (1, 1)$ and

$$\left\{\begin{array}{l} w^n = P_D(I - \lambda B)v^n \\ T_G^n = \{w \in D \times D \,\langle (I - \lambda B)v^n - w^n, w^n - w\rangle \geq 0\} \\ v^{n+1} = P_{T_G^n}\left(v^n - \lambda B w^n\right), \end{array}\right. \tag{4.10}$$

for all $n \in N$ where $v^n = (v_1^n, v_2^n), w^n = (w_1^n, w_2^n)$,$\lambda \in (0, \alpha)$ and T_G^n is G- Half space. Then sequence $\{v^n\}$ converses weakly to an element of $\bar{v} \in G - Var(D, B)$ and the sequence $\left\{P_{G-Var(D,B)} v^n\right\}$ converges strongly to $\bar{v}$, where $G - Var(D, B)$ dominates v^n, $\{v^n\}$ dominates v^0 and $\{w^n\}$ is dominated by v^0.

Solution. It is easy to see that $(2, 0) \in G - Var(D, B)$, and $Eed(G) = Eed(G^{-1})$. It is obvious that B is G-$\frac{1}{3}$-inverse strongly monotone.
Putting $\lambda = \frac{1}{4}$. From the definition of B, we have

$$(I - \frac{1}{4}B)z \quad = \quad (\frac{4z_1}{5} + \frac{2}{5}, \frac{15z_2}{16}) \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2], \tag{4.11}$$

for all $z = (z_1, z_2) \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$.

From definition of P_D, it follows that

$$P_D(I - \frac{1}{4}B)z \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}0, 2], \tag{4.12}$$

for all $z = (z_1, z_2) \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$.
Let $(w, z) \in Eed(G)$. From definition of T_G^n and (4.12), we have $T_G^n \subseteq [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$.
Since $v^0 \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$ and (4.12), we have

$$w^0 = P_D(I - \frac{1}{4}B)v^0 \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2], \tag{4.13}$$

where $w^0 = (w_1^0, w_2^0) \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$. From $T_G^n \subseteq [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$, we have

$$v^1 = P_{T_G^n}(v^0 - \frac{1}{4}Bw^0) \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2], \tag{4.14}$$

where $v^1 = (v_1^1, v_2^1) \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$.
Continue the method of (4.13) and (4.14), we have $w^n, v^n \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$ for all $n \in \mathbb{N}$, $v^n = (v_1^n, v_2^n), w^n = (w_1^n, w_2^n)$.
Since $v^0, (2, 0), v^n$ and $w^n \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$, it follows that $((2, 0), v^n)$, (v^n, v^0) and $(w^n, v^0) \in Eed(G)$.
We can conclude that $G - Var(D, B)$ dominates v^n, $\{v^n\}$ dominates v^0 and $\{w^n\}$ is dominated by v^0. All conditions of Example 4.3 satisfies Theorem 3.1, so we can conclude that sequence $\{v^n\}$ converses weakly to an element of $(2, 0) \in G - Var(D, B)$ and the sequence $\left\{P_{G-Var(D,B)} v^n\right\}$ converges strongly to $(2, 0)$.

n	$\mathbf{v}_1^n$	$\mathbf{v}_2^n$
1	1.0000000	1.0000000
2	1.1600000	0.9414062
3	1.2944000	0.8862457
4	1.4072960	0.8343173
$\vdots$	$\vdots$	$\vdots$
99	2.0000000	0.0028601
100	2.0000000	0.0026925

TABLE 2. Detailed analysis of computational methods (4.10) for Example 4.2 with $\mathbf{v^0} = (1, 1)$, $N = 100$.

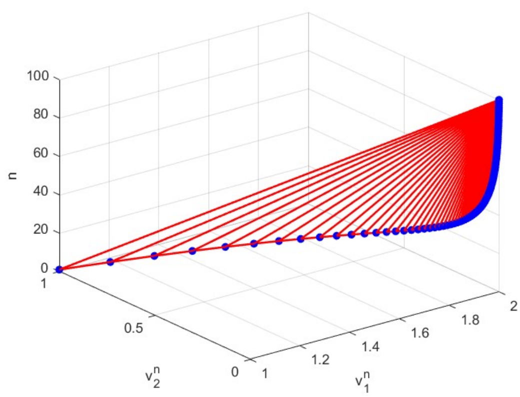

FIGURE 2. *The convergence behavious of* $\{\mathbf{v^n}\}$ with $v^0 = (1, 1)$ and $N = 100$.

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