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On approximating fixed points of weak enriched contraction mappings via Kirk's iterative algorithm in Banach spaces

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ABSTRACT. Recently Berinde and Păcurar [Approximating fixed points of enriched contractions in Banach spaces. *J. Fixed Point Theory Appl.* **22** (2020), no. 2., 1–10], first introduced the idea of enriched contraction mappings and proved the existence of a fixed point of an enriched contraction mapping using the well-known fact that any fixed point of the averaged mapping T_λ, where $\lambda \in (0, 1]$, is also a fixed point of the initial mapping T. In this work, we introduce the idea of weak enriched contraction mappings, and a new generalization of an averaged mapping called double averaged mapping. The first attempt is to prove the existence and uniqueness of the fixed point of a double averaged mapping associated with a weak enriched contraction mapping. Based on this result on Banach spaces, we give some sufficient conditions for the equality of all fixed points of a double averaged mapping and the set of all fixed points of a weak enriched contraction mapping. Moreover, our results show that an appropriate Kirk's iterative algorithm can be used to approximate a fixed point of a weak enriched contraction mapping. An illustrative example for showing the efficiency of our results is given.

1. INTRODUCTION AND PRELIMINARIES

First, we will introduce basic notations and needed definitions. For each a self-mapping T on a nonempty set X, $\zeta \in X$ is called a fixed point of T if $T\zeta = \zeta$, and we denote the set of all fixed points of T by Fix$\{T\}$. For each $\lambda \in [0, 1]$, the averaged mapping associated with T is defined by $T_\lambda := (1-\lambda)I + \lambda T$, where I is an identity mapping. It is well-known that $\text{Fix}(T_\lambda) = \text{Fix}(T)$ for all $\lambda \in (0, 1]$.

The useful result in the theory of metric spaces to guarantee the existence and uniqueness of the fixed point of a self-mapping on a metric space was introduced by Banach in his Ph.D. thesis [3]. The statement of this theorem is as follows:

Theorem 1.1 (Banach fixed point theorem [3]). *Let (X, d) be a complete metric space and T be a self-mapping on X. If T satisfies a Banach contractive condition, i.e., there is $k \in [0, 1)$ such that for every $x, y \in X$, we have $d(Tx, Ty) \leq kd(x, y)$. Then T has a unique fixed point. Moreover, for each $x_0 \in X$, the fixed point of T can be approximated by the Picard iteration $\{x_n\}$, which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$.*

Many researchers use this theorem to prove the existence and uniqueness of solutions in various nonlinear problems, i.e., differential equations, integral equations, optimization problems, etc. (see [1, 11, 17] and references therein).

Next, we will recall some important fixed point iterations needed in this paper. For a mapping T from a convex subset D of a normed space $(X, \|\cdot\|)$ into itself and a given $x_0 \in D$, the Picard iteration $\{x_n\} \subseteq D$ is defined by

$$x_n := Tx_{n-1}$$

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for all $n \in \mathbb{N}$. From Theorem 1.1, we know that the Picard iteration converges to a fixed point of T if D is complete and T satisfies the Banach contractive condition. However, if the Banach contractive condition is slightly weaker, the Picard iteration need not converge to a fixed point of the mapping T. So, we need to consider other iteration procedures, such as the Krasnoselskij iteration $\{x_n\} \subseteq D$, which is defined by

$$x_n := (1-\lambda)x_{n-1} + \lambda T x_{n-1}$$

for all $n \in \mathbb{N}$, where $x_0 \in D$ and $\lambda \in [0,1]$ (see more details in [10]). It is easy to see that the Krasnoselskij iteration is the generalization of the Picard iteration. The other important iteration procedure is the Kirk's iteration order $k \in \mathbb{N}$, which is a sequence $\{x_n\} \subseteq D$ defined by

$$x_n := \alpha_0 x_n + \alpha_1 T x_n + \alpha_2 T^2 x_n + \ldots + \alpha_k T^k x_n,$$

where $x_0 \in D$, $\alpha_1 > 0$ and $\alpha_i \geq 0$ for $i = 0, 2, 3, \ldots, k$ such that $\sum_{i=0}^{k} \alpha_i = 1$ (see more details in [13]).

Recently, Berinde and Păcurar [6] extended Theorem 1.1 (in the case of Banach spaces) by proving the fixed point result of enriched contraction mappings via the setting of the Picard iteration of the averaged mapping, which is the Krasnoselskij iteration. The statement of this theorem is as follows:

Theorem 1.2 ([6]). *Let T be a self-mapping on a Banach space X. If T is an enriched contraction mapping, i.e. there exist $b \in [0,\infty)$ and $\theta \in [0, b+1)$ such that for each $x, y \in X$, we have*

$$\|b(x-y) + Tx - Ty\| \leq \theta \|x-y\|. \tag{1.1}$$

Then $|Fix(T)| = 1$ and there is $\lambda \in (0,1]$ such that for each $x_0 \in X$, the Krasnoselskij iteration $\{x_n\} \subseteq D$ defined by

$$x_n := (1-\lambda)x_{n-1} + \lambda T x_{n-1}$$

for all $n \in \mathbb{N}$ converges to a unique fixed point of T.

The proof of Theorem 1.2 has three essential steps as follows:

- to prove the existence of the fixed point of the averaged mapping $T_\lambda := (1-\lambda)I + \lambda T$;
- to show the uniqueness of the fixed point of the averaged mapping;
- to use the well-known that Fix(T) = Fix(T_λ).

Now a day enriched mappings are interesting to many researchers to extend the results of a fixed point theorem (see [7], [12])

This work aims to introduce the new contractive condition covering the enriched contractive condition, which is called a weak enriched contraction mapping, and to present a new mapping called a double averaged mapping, which is a generalization of the idea of an averaged mapping.

For the first result related to such two mappings, we prove the existence and uniqueness of a fixed point of a double averaged mapping constructed from a weak enriched contraction mapping. This result also shows that an appropriate Kirk's iterative scheme can approximate a fixed point of this double averaged mapping. Moreover, some sufficient conditions for the equality of the set of all fixed points of a double averaged mapping, and the set of all fixed points of a weak contraction mapping are presented in this paper. Based on this result, an appropriate Kirk's iterative algorithm can be used to approximate a fixed point of a weak enriched contraction mapping. We also give an example to support our main result. Meanwhile, many results in the literature can not be applied in our illustrative example.

2. Weak enriched contraction mappings and double averaged mappings

Inspired by the benefit of a class of enriched contraction mappings due to Berinda and Păcurar [6], we attempt to give evolution to this class by inventing the class of mappings covering all enriched contraction mappings. Each element in this class is called a weak enriched contraction mapping defined by the following definition.

Definition 2.1. Let C be a convex subset of a normed space $(X, \|\cdot\|)$. A self-mapping T on C is called a weak enriched mapping if there exist nonnegative real numbers a, b and $w \in [0, a+b+1)$ such that

$$\|a(x-y)+Tx-Ty+b(T^2x-T^2y)\| \leq w\|x-y\| \tag{2.2}$$

for every $x, y \in C$.

If we set $b = 0$ in (2.2), it reduces to an enriched contraction mapping, which is a generalization of a Banach contraction mapping.

Since (1.1) implies that

$$\begin{aligned} \|Tx-Ty\| &\leq \|b(x-y)\| + \|b(x-y)+Tx-Ty\| \\ &\leq (b+\theta)\|x-y\| \end{aligned}$$

for all $x, y \in X$, if T is an enriched contraction mapping, then it is Lipschitz continuous with a Lipschitz constant $b+\theta$, and then it is continuous. This shows that a discontinuous mapping is not an enriched contraction mapping. However, this situation does not hold for weak enriched contraction mappings because (2.2) does not imply the continuity of T. Moreover, there is a discontinuous mapping satisfying (2.2) (see later in Example 2.1). Therefore, a presented contraction mapping is a real proper generalization of two famous contraction mappings consisting of a Banach contraction mapping and an enriched contraction mapping.

In Theorem 1.2, the proof uses the well-known fact that Fix(T) = Fix(T_λ) for all $\lambda \in (0,1)$, but this fact is not sufficient to guarantee the existence of a fixed point for weak enriched contraction mappings. It brings to the motivation for inventing a new mapping with a similar property to the abovementioned and can help prove a fixed point result for weak enriched contraction mappings. This new mapping, named a double averaged mapping, is defined by

$$T_{\alpha_1,\alpha_2} := (1-\alpha_1-\alpha_2)I + \alpha_1 T + \alpha_2 T^2, \tag{2.3}$$

where $\alpha_1 > 0, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 \in (0,1]$. It is easy to see that $T_\lambda = T_{\alpha_1,0}$, where $\alpha_0 = \lambda$, and then T_{α_1,α_2} is a generalization of T_λ. Now, we will show the existence and uniqueness of a fixed point of a double averaged mapping related to a weak enriched contraction mapping as follows:

Theorem 2.3. *Let C be a closed convex subset of a Banach space $(X, \|\cdot\|)$ and $T : C \to C$ be a weak enriched contraction mapping. Then there are $\alpha_1 > 0$ and $\alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 \in (0,1]$ such that the following assertions hold:*

(F1) $|Fix(T_{\alpha_1,\alpha_2})| = 1$;

(F2) *for any given $x_0 \in C$, the iteration $\{x_n\} \subseteq C$ given by*

$$x_n = (1-\alpha_1-\alpha_2)x_{n-1} + \alpha_1 T x_{n-1} + \alpha_2 T^2 x_{n-1} \tag{2.4}$$

for all $n \in \mathbb{N}$ converges to the unique fixed point of T_{α_1,α_2}.

Proof. Since T is a weak enriched contraction, there are a, b satisfying the condition (2.2). If $b = 0$, T is an enriched contraction and so the result follows from [6]. We need to prove for $a \geq 0$ and $b > 0$. Define $\alpha_1 := \frac{1}{a+b+1} > 0$ and $\alpha_2 := \frac{b}{a+b+1} \geq 0$. Then the equation (2.2) becomes

$$\left\| \left(\frac{1-\alpha_2}{\alpha_1} - 1 \right)(x-y) + Tx - Ty + \frac{\alpha_2}{\alpha_1}(T^2x - T^2y) \right\| \leq w\|x-y\|$$

for all $x, y \in C$. Since $\alpha_1 > 0$, the above inequality becomes

$$\|(1-\alpha_1-\alpha_2)(x-y) + \alpha_1(Tx-Ty) + \alpha_2(T^2x - T^2y)\| \leq \zeta\|x-y\|$$

for all $x, y \in C$, where $\zeta = w\alpha_1 \in [0,1)$. In view of (2.3), the above inequality implies that

$$\|T_{\alpha_1,\alpha_2}x - T_{\alpha_1,\alpha_2}y\| \leq \zeta\|x-y\| \tag{2.5}$$

for all $x, y \in C$. Now, for a given $x_0 \in C$, define a sequence $\{x_n\} \subseteq C$ by $x_n = T_{\alpha_1,\alpha_2}x_{n-1}$ for $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|T_{\alpha_1,\alpha_2}x_n - T_{\alpha_1,\alpha_2}x_{n-1}\| \\ &\leq \zeta\|x_n - x_{n-1}\|. \end{aligned}$$

By repeating the same process, we obtain

$$\|x_{n+1} - x_n\| \leq \zeta^n\|x_1 - x_0\| \tag{2.6}$$

for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is a Cauchy sequence in C. Using the completeness of C, there is a point $x^* \in C$ such that $x_n \to x^*$ as $n \to \infty$. From (2.5), we get

$$\begin{aligned} \|x_{n+1} - T_{\alpha_1,\alpha_2}x^*\| &= \|T_{\alpha_1,\alpha_2}x_n - T_{\alpha_1,\alpha_2}x^*\| \\ &\leq \zeta\|x_n - x^*\|. \end{aligned} \tag{2.7}$$

Taking $n \to \infty$ in (2.7), we get $\|x^* - T_{\alpha_1,\alpha_2}x^*\| = 0$, that is, $T_{\alpha_1,\alpha_2}x^* = x^*$. This implies that $x^* \in \text{Fix}(T_{\alpha_1,\alpha_2})$.

Finally, we assume that T_{α_1,α_2} have more than one fixed points, denoted x^* and z^* such that $x^* \neq z^*$. By (2.5), we have

$$\|x^* - z^*\| = \|T_{\alpha_1,\alpha_2}x^* - T_{\alpha_1,\alpha_2}z^*\| \leq \zeta\|x^* - z^*\| < \|x^* - z^*\|, \tag{2.8}$$

which is a contradiction and so $|\text{Fix}(T_{\alpha_1,\alpha_2})| = 1$. □

Next, we give an illustrative example supporting Theorem 2.3.

Example 2.1. Let $X = \mathbb{R}$ be a usual normed space and T be a self-mapping on $[-1,1] \subseteq X$ defined by

$$Tx = \begin{cases} x^2; & x \in [-1,0) \\ 1-x; & x \in [0,1]. \end{cases} \tag{2.9}$$

By setting $a = b = 1$, we will show that T is a weak enriched contraction mapping with such a, b and any $w \in [1, a+b+1)$. Without loss of generality, we may assume that $x, y \in [-1,1]$ with $x \leq y$. We consider three cases for x, y. First, for each $x, y \in [-1,0)$, we have

$$\begin{aligned} \|a(x-y) + Tx - Ty + b(T^2x - T^2y)\| &= \|x - y + x^2 - y^2 + (1-x^2) - (1-y^2)\| \\ &= \|x-y\| \\ &\leq w\|x-y\|. \end{aligned}$$

Next, for each $x, y \in [0,1]$, we have

$$\begin{aligned} \|a(x-y) + Tx - Ty + b(T^2x - T^2y)\| &= \|x - y + (1-x) - (1-y) + x - y\| \\ &= \|x-y\| \\ &\leq w\|x-y\|. \end{aligned}$$

Finally, for each $x \in [-1, 0)$ and $y \in [0, 1]$, we have

$$\begin{aligned}\|a(x-y)+Tx-Ty+b(T^2x-T^2y)\| &= \|x-y+x^2-(1-y)+(1-x^2)-y\| \\ &= \|x-y\| \\ &\leq w\|x-y\|.\end{aligned}$$

Hence, T is a weak enriched contraction mapping with $a = b = 1$ and any $w \in [1, a+b+1)$. Hence, Theorem 2.3 can be used in this example. From the proof of Theorem 2.3, we can define $\alpha_1 := \frac{1}{a+b+1} = \frac{1}{3}$ and $\alpha_2 := \frac{b}{a+b+1} = \frac{1}{3}$. Then $|\text{Fix}(T_{\alpha_1,\alpha_2})| = 1$ and the following iteration

$$x_n = (1-\alpha_1-\alpha_2)x_{n-1} + \alpha_1 T x_{n-1} + \alpha_2 T^2 x_{n-1} \tag{2.10}$$

for all $n \in \mathbb{N}$, where $x_0 \in [-1, 1]$, converges to the unique fixed point of $T_{\frac{1}{3},\frac{1}{3}}$.

In the above example, it can be seen that Theorem 2.3 can help to be concluded only the existence and uniqueness of a fixed point of the mapping $T_{\frac{1}{3},\frac{1}{3}}$. Therefore, in the next section, we will give some sufficient conditions to yield that the fixed point of $T_{\frac{1}{3},\frac{1}{3}}$ is also a fixed point of T.

3. Sufficient conditions for the equality of Fix(T) and Fix(T_{α_1,α_2})

The previous section shows the existence and uniqueness of a fixed point of a double averaged mapping associated with a weak enriched mapping, which is insufficient to yield the fixed point of a weak enriched mapping. This section aims to give sufficient conditions for the equality of sets of all fixed points of a double averaged mapping associated with a weak enriched mapping and of the same weak enriched mapping. We begin with the following remark.

Remark 3.1. For each self-mapping T on a closed convex subset C of a normed space X and a given $\alpha_1 > 0$ and $\alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 \in (0, 1]$, the double averaged mapping $T_{\alpha_1,\alpha_2} : C \to C$ given by

$$T_{\alpha_1,\alpha_2}x = (1-\alpha_1-\alpha_2)x + \alpha_1 Tx + \alpha_2 T^2 x \tag{3.11}$$

has the property that $\text{Fix}(T) \subseteq \text{Fix}(T_{\alpha_1,\alpha_2})$.

The inclusion in the above remark may be strict, as the next example shows.

Example 3.2. Let $T : \mathbb{C} \to \mathbb{C}$ be defined by $Tx = x^2$ for all $x \in \mathbb{C}$. It is easy to see that the fixed point equation $Tx = x$ has two solutions, that is, $\text{Fix}(T) = \{0, 1\}$. If we set $\alpha_1 = \alpha_2 = \frac{1}{3}$, then $T_{\frac{1}{3},\frac{1}{3}}x = \frac{1}{3}x + \frac{1}{3}x^2 + \frac{1}{3}x^4$, and so the fixed point equation $T_{\frac{1}{3},\frac{1}{3}}x = x$, that is,

$$\frac{1}{3}x + \frac{1}{3}x^2 + \frac{1}{3}x^4 = x$$

having four fixed points, that is, $\text{Fix}(T_{\frac{1}{3},\frac{1}{3}}) = \left\{0, 1, \frac{1}{2}(-1 \pm 7i)\right\}$. Then $\text{Fix}(T) \subsetneq \text{Fix}(T_{\alpha_1,\alpha_2})$.

Now, we give the sufficient condition to yiled that $\text{Fix}(T) = \text{Fix}(T_{\alpha_1,\alpha_2})$ as follows.

Lemma 3.1. *Let T be a self-mapping on a closed convex subset C of a normed space $(X, \|\cdot\|)$. Suppose that there are $\alpha_1 > 0$ and $\alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 \in (0, 1]$ such that the following assertion holds:*

(W1) *for each $c \in [0, 1)$ and $z \in$ Fix(T_{α_1,α_2}), we have*

$$\|z - Tz\| \leq \|z - (1-c)Tz - cT^2 z\|. \tag{3.12}$$

Then Fix$(T) =$ Fix(T_{α_1,α_2}).

Proof. From Remark 3.1, we have $\mathrm{Fix}(T) \subseteq \mathrm{Fix}(T_{\alpha_1,\alpha_2})$. If $\mathrm{Fix}(T_{\alpha_1,\alpha_2})=\emptyset$, then $\mathrm{Fix}(T) = \emptyset$ and so $\mathrm{Fix}(T_{\alpha_1,\alpha_2}) = \mathrm{Fix}(T)$. In the remaining proof, we will assume that $\mathrm{Fix}(T_{\alpha_1,\alpha_2}) \neq \emptyset$. Let $z \in \mathrm{Fix}(T_{\alpha_1,\alpha_2})$. Putting $c := \frac{\alpha_2}{\alpha_1+\alpha_2} \in [0,1)$ in (3.12), we have

$$\begin{aligned}
\|z - Tz\| &\leq \left\| z - \frac{\alpha_1}{\alpha_1+\alpha_2}Tz - \frac{\alpha_2}{\alpha_1+\alpha_2}T^2 z \right\| \\
&= \frac{1}{\alpha_1+\alpha_2}\|z - (1-\alpha_1-\alpha_2)z - \alpha_1 Tz - \alpha_2 T^2 z\| \\
&= \|z - T_{\alpha_1,\alpha_2} z\| \\
&= 0,
\end{aligned}$$

which implies that $z \in \mathrm{Fix}(T)$. Hence, $\mathrm{Fix}(T_{\alpha_1,\alpha_2}) \subseteq \mathrm{Fix}(T)$. Therefore, $\mathrm{Fix}(T)=\mathrm{Fix}(T_{\alpha_1,\alpha_2})$. □

Remark 3.2. For each self-mapping T on a closed convex subset C of a normed space $(X, \|\cdot\|)$, if there are $\alpha_1 > 0$ and $\alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 \in (0,1]$ such that $Tz \in \mathrm{Fix}(T)$ for all $z \in \mathrm{Fix}(T_{\alpha_1,\alpha_2})$, that is, $Tz = T^2 z$ for all $z \in \mathrm{Fix}(T_{\alpha_1,\alpha_2})$, then T satisfies the condition (W1). For instance, a constant mapping and an identity mapping satisfy the condition (W1).

Next, we give another sufficient condition to guarantee the equality of a set of all fixed points of a weak enriched mapping and a set of all fixed points of a double averaged mapping associated with this weak enriched mapping, where this property keeps the generality of an averaged mapping.

Lemma 3.2. *Let T be a self-mapping on a closed convex subset C of a normed space $(X, \|\cdot\|)$. Suppose that there are $\alpha_1 > 0$ and $\alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 \in (0,1]$ such that the following assertion holds:*

(W2) *there exists a nonnegative real number $k < 1$ such that*

$$\|T_{\alpha_1,\alpha_2} x - Tx\| \leq k\|x - Tx\| \tag{3.13}$$

for all $x \in C$.

Then Fix(T) = Fix(T_{α_1,α_2}).

Proof. From Remark 3.1, we have $\mathrm{Fix}(T) \subseteq \mathrm{Fix}(T_{\alpha_1,\alpha_2})$. If $\mathrm{Fix}(T_{\alpha_1,\alpha_2}) = \emptyset$, then $\mathrm{Fix}(T) = \emptyset$ and so $\mathrm{Fix}(T_{\alpha_1,\alpha_2}) = \mathrm{Fix}(T)$. In the remaining proof, we will suppose that $\mathrm{Fix}(T_{\alpha_1,\alpha_2}) \neq \emptyset$. Now, for each $z \in \mathrm{Fix}(T_{\alpha_1,\alpha_2})$, we have

$$\|z - Tz\| = \|T_{\alpha_1,\alpha_2} z - Tz\| \leq k\|z - Tz\| \tag{3.14}$$

and so $\|z - Tz\| = 0$, that is, $z = Tz$. Then $z \in \mathrm{Fix}(T)$ and so $\mathrm{Fix}(T_{\alpha_1,\alpha_2}) \subseteq \mathrm{Fix}(T)$. Therefore, $\mathrm{Fix}(T) = \mathrm{Fix}(T_{\alpha_1,\alpha_2})$. □

It is easy to see that the averaged mapping satisfies (3.13) with a constant $k = 1 - \lambda$, where $\lambda \in (0,1)$ is a constant coresponding with the averaged mapping.

Example 3.3. Let $X = \mathbb{R}$ be a usual normed space and T be a self-mapping on X defined by $Tx = 1 - x$ for all $x \in X$. There are $\alpha_1 = \frac{1}{3}$ and $\alpha_2 = \frac{1}{3}$ such that

$$\begin{aligned}
\|T_{\alpha_1,\alpha_2} x - Tx\| &= \left\| \frac{1}{3}x + \frac{1}{3}(1-x) + \frac{1}{3}x - (1-x) \right\| \\
&= \left\| \frac{2}{3}x - \frac{2}{3}(1-x) \right\| \\
&= \frac{2}{3}\|x - Tx\|.
\end{aligned}$$

Therefore, (W2) holds with $k=\frac{2}{3}$. Moreover, it is easy to see that $\text{Fix}(T)=\text{Fix}(T_{\alpha_1,\alpha_2})=\{\frac{1}{2}\}$.

Lemma 3.3. *Let T be a self-mapping on a closed convex subset C of a normed space $(X,\|\cdot\|)$. Suppose that there are $\alpha_1>0$ and $\alpha_2\geq 0$ with $\alpha_1+\alpha_2\in(0,1]$ such that Fix$(T_{\alpha_1,\alpha_2})\neq\emptyset$ and the following assertion holds:*

(W3) *for each $x\in$ Fix(T_{α_1,α_2}), there exists a closed convex subset $B\subseteq C$ that contains x such that $T(B)\subseteq B$ and T satisfies* (3.13) *only on set B.*

Then Fix$(T\mid_B)$ = Fix$(T_{\alpha_1,\alpha_2}\mid_B)$.

Proof. It straints forward from Lemma 3.2 with the restriction of T on B. □

Next, we establish the fixed point theorem for weak enriched contraction mappings via the help of all the above lemmas.

Theorem 3.4. *Let T be a self-mapping on a closed convex subset C of a Banach space $(X,\|\cdot\|)$ and $T:C\to C$ be a weak enriched contraction mapping. Then there are $\alpha_1>0$ and $\alpha_2\geq 0$ with $\alpha_1+\alpha_2\in(0,1]$ such that (F1) and (F2) hold. Moreover, if T,α_1 and α_2 satisfy (W1) or (W2) or (W3), then*

(T1) $|$*Fix*$(T)|=1$;

(T2) *for any given $x_0\in C$, the iteration $\{x_n\}\subseteq C$ given by*

$$x_n=(1-\alpha_1-\alpha_2)x_{n-1}+\alpha_1 Tx_{n-1}+\alpha_2 T^2x_{n-1} \tag{3.15}$$

for all $n\in\mathbb{N}$ converges to the unique fixed point of T.

Proof. From Theorem 2.3, there are $\alpha_1>0$ and $\alpha_2\geq 0$ with $\alpha_1+\alpha_2\in(0,1]$ such that (F1) and (F2) hold, that is, $|\text{Fix}(T_{\alpha_1,\alpha_2})|=1$ and the iteration in (3.15) converges to the fixed point of T_{α_1,α_2}. Since α_1 and α_2 satisfy (W1) or (W2) or (W3), the results follows from Lemma 3.1 or Lemma 3.2 or Lemma 3.3, respectively. □

Finally, we give an example to illustrate the condition (W3) and Theorem 3.4 as follows:

Example 3.4. Let T be a self-mapping defined in Equation (2.9). Example 2.1 concluded the existence and uniqueness of the fixed point of T_{α_1,α_2}. First, we observes that if $x\in[-1,0)$, then $T_{\frac{1}{3},\frac{1}{3}}x\in[0,1]$ and if $x\in[0,1]$, then $T_{\frac{1}{3},\frac{1}{3}}x\in[0,1]$. This means that the set of fixed points of $T_{\frac{1}{3},\frac{1}{3}}$ contained in $[0,1]$. Since $T([0,1])\subseteq[0,1]$ and there is $k=\frac{2}{3}$ such that for each $x\in[0,1]$, we have

$$\|T_{\frac{1}{3},\frac{1}{3}}x-Tx\|=\left\|\frac{2}{3}x-\frac{2}{3}(1-x)\right\|\leq\frac{2}{3}\|x-Tx\|.$$

Hence, (W3) hold. By Theorem 3.4, we get $\text{Fix}(T)=\text{Fix}(T_{\frac{1}{3},\frac{1}{3}})$. Moreover, for each $x_0\in[-1,1]$, the iteration (3.15) having the following form

$$\begin{aligned}x_1 &= \frac{1}{3}x_0+\frac{1}{3}Tx_0+\frac{1}{3}T^2x_0\\ &= \begin{cases}\frac{1}{3}x_0+\frac{1}{3}x_0^2+\frac{1}{3}(1-x_0^2) & \text{if } x_0\in[-1,0)\\ \frac{1}{3}x_0+\frac{1}{3}(1-x_0)+\frac{1}{3}x_0 & \text{if } x_0\in[0,1]\end{cases}\\ &= \frac{1}{3}(x_0+1)\in[0,1],\end{aligned}$$

$$\begin{aligned} x_2 &= \frac{1}{3}x_1 + \frac{1}{3}Tx_1 + \frac{1}{3}T^2x_1 \\ &= \frac{1}{3}x_1 + \frac{1}{3}(1-x_1) + \frac{1}{3}x_1 \\ &= \frac{1}{3}(x_1+1) \\ &= \frac{1}{3}\left(\frac{1}{3}(x_0+1)+1\right) \\ &= \frac{1}{3^2}(x_0+1) + \frac{1}{3} \in [0,1], \\ &\vdots \end{aligned}$$

$$x_n = \frac{1}{3^n}(x_0+1) + \sum_{i=1}^{n} \frac{1}{3^i} \quad \text{for all } n \in \mathbb{N} \tag{3.16}$$

converges to the unique fixed point $\frac{2}{3}$ of T. Figure 1 shows the behavior of the convergence for the above iterations with several initial points. Meanwhile, Figure 2 shows the behavior of the divergence for the Picard iterations with several initial points.

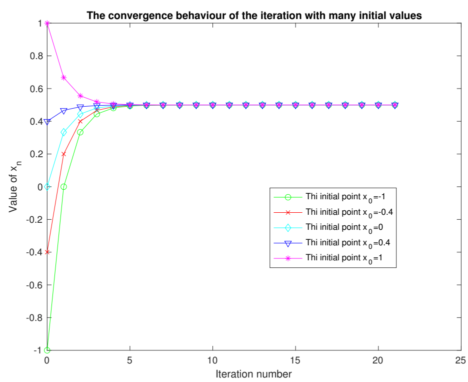

FIGURE 1. Convergence behavior of the iteration (3.16) with $x_0 = -1, -0.4, 0, 0.4, 1$.

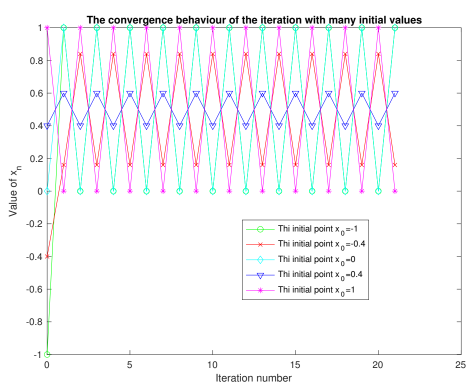

FIGURE 2. Convergence behavior of the Picard iteration with $x_0 = -1, -0.4, 0, 0.4, 1$.

4. Conclusion and open question

In this paper, we introduced two new mappings: a weak enriched contraction mapping and a double averaged mapping. Moreover, we proved that for each self-mapping T on a closed convex subset of a Banach space satisfying the weak enriched contractive condition, there are $\alpha_1 > 0$ and $\alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 \in (0, 1]$ such that T_{α_1,α_2} has the unique fixed point, and an appropriate Kirk iteration procedure can approximate it. Last but not least, we give three sufficient conditions for the equality of $\text{Fix}(T)$ and $\text{Fix}(T_{\alpha_1,\alpha_2})$. In addition, we gave an illustrative example of a mapping satisfying the weak enriched contractive condition, but it does not satisfy the enriched contractive condition. This example shows the efficiency of our main theorem.

This paper points out that we used the Krik iteration order 2 to construct a weak enriched contractive condition. However, it is easy to see that all results in this paper can be extend by using the idea of the Krik iteration order k for any $k \in \mathbb{N}$.

Finally, we give open questions to the reader for further study as follows:

- Can we remove or generalize each condition of Lemmas 3.1, 3.2, or 3.3?
- If we extend all results in this paper by using the idea of the Krik iteration order k for any $k \in \mathbb{N}$, under which conditions can we conclude the existence and uniqueness of a fixed point of weak enriched mapping?
- Can we use the idea in this paper to investigate convex contraction mappings? The reader can see more details on convex contraction mappings in [4, 8, 9, 15] and references therein.
- Can we use the idea in this paper to extend a definition of enrich nonexpansive mapping and invent fixed point results for such new mappings? The reader can see more details on enrich nonexpansive mappings in [5].
- Can we mix the idea in this paper and fixed point results for decreasing convex orbital operators in Hilbert spaces in [16] to investigate some new development?
- Can we use the sufficient condition of the equality of the fixed point of a double averaged mapping and its initial mapping to extend the results of other types of enriched mappings (see in [7, 12])?

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Data Availability Statementst. The authors declare that all data supporting the findings of this study are available within the article.

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