CARPATHIAN J. MATH.
Volume **39** (2023), No. 2,
Pages 529 - 539

Online version at https://www.carpathian.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2023.02.13

A note on the generators of the polynomial algebra of six variables and application

NGUYEN KHAC TIN

ABSTRACT. Let $\mathcal{P}_n := H^*((\mathbb{R}P^\infty)^n) \cong \mathbb{Z}_2[x_1, x_2, \ldots, x_n]$ be the graded polynomial algebra over $\mathcal{K}$, where $\mathcal{K}$ denotes the prime field of two elements. We investigate the Peterson hit problem for the polynomial algebra $\mathcal{P}_n$, viewed as a graded left module over the mod-2 Steenrod algebra, $\mathcal{A}$. For $n > 4$, this problem is still unsolved, even in the case of $n = 5$ with the help of computers.

In this paper, we study the hit problem for the case $n = 6$ in degree $d_k = 6(2^k - 1) + 9.2^k$, with k an arbitrary non-negative integer. By considering $\mathcal{K}$ as a trivial $\mathcal{A}$-module, then the hit problem is equivalent to the problem of finding a basis of $\mathcal{K}$-graded vector space $\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_n$. The main goal of the current paper is to explicitly determine an admissible monomial basis of the $\mathcal{K}$-graded vector space $\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6$ in some degrees. At the same time, the behavior of the sixth Singer algebraic transfer in degree $d_k = 6(2^k - 1) + 9.2^k$ is also discussed at the end of this article. Here, the Singer algebraic transfer is a homomorphism from the homology of the mod-2 Steenrod algebra, $\mathrm{Tor}^{\mathcal{A}}_{n,n+d}(\mathcal{K}, \mathcal{K})$, to the subspace of $\mathcal{K} \otimes_{\mathcal{A}} \mathcal{P}_n$ consisting of all the GL_n-invariant classes of degree d.

1. INTRODUCTION

Throughout the paper, the coefficient ring for homology and cohomology is always $\mathcal{K}$ the field of two elements. Let $\mathbb{R}\mathcal{P}^\infty$ be the infinite dimensional real projective space. Then, $H^*(\mathbb{R}\mathcal{P}^\infty) \cong \mathcal{K}[x_1]$, and therefore, the mod-2 cohomology algebra of the direct product of n copies of $\mathbb{R}\mathcal{P}^\infty$ is isomorphic to the graded polynomial algebra $\mathcal{K}[x_1, x_2, \ldots, x_n]$, reviewed as an unstable $\mathcal{A}$-module on n generators $x_1, x_2, \ldots, x_n$, each of degree one.

The $\mathcal{A}$-module structure of $\mathcal{P}_n$ is determined by the properties of the Steenrod operation and the Cartan formula (see Steenrod and Epstein [12]).

A homogeneous polynomial g of degree d in $\mathcal{P}_n$ is called *hit* if there is an equation in the form of a finite sum $g = \sum_{i\geqslant 0} Sq^{2^i}(g_i)$, where the degree of the polynomials g_i is less than d. This means, g belongs to $\mathcal{A}^+\mathcal{P}_n$. Here, $\mathcal{A}^+$ is an ideal of $\mathcal{A}$ generated by all Steenrod squares Sq^k, with $k > 0$.

The *Peterson hit problem* in Algebraic Topology is to find a minimal generating set for $\mathcal{P}_n$, reviewed as a module over the mod-2 Steenrod algebra $\mathcal{A}$. If we consider $\mathcal{K}$ as a trivial $\mathcal{A}$-module, then the hit problem is equivalent to the problem of finding a basis of $\mathcal{K}$-graded vector space:

$$\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_n = \bigoplus_{d\geqslant 0}(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_n)_d \cong \mathcal{P}_n/\mathcal{A}^+\mathcal{P}_n$$

in each degree $d \in \mathbb{N}$. Here, $(\mathcal{P}_n)_d$ is the subspace of $\mathcal{P}_n$ consisting of all the homogeneous polynomials of degree d in $\mathcal{P}_n$ and $(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_n)_d$ is the subspace of $\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_n$ consisting of all the classes represented by the elements in $(\mathcal{P}_n)_d$.

In [6], Peterson conjectured that as a module over the Steenrod algebra $\mathcal{A}$, the polynomial algebra $\mathcal{P}_n$ is generated by monomials in degree d that satisfy $\alpha(d+n) \leqslant n$, where

Received: 28.09.2021. In revised form: 31.07.2022. Accepted: 06.08.2022
2010 *Mathematics Subject Classification.* 55S10, 55S05, 55T15.
Key words and phrases. *Polynomial algebra, hit problem, graded rings.*

$\alpha(d)$ denotes the number of ones in dyadic expansion of d, and proved it for $n \leqslant 2$. The conjecture was established in general by Wood [22]. This is a useful tool for determining $\mathcal{A}$-generators for $\mathcal{P}_n$. And then, the hit problem was investigated by many authors (see Repka-Selick [9], Silverman [11], Mothebe-Kaelo-Ramatebele [5], Sum [14], Sum-Tin [16], Tin [19] and others).

Let r, s, t be non-negative intergers. Based on the results of Wood [22], Kameko [3], and Sum [14], the hit problem is reduced to the case of degree d of the form $d = r(2^t - 1) + 2^t s$ such that $0 \leqslant \mu(s) < r \leqslant n$, where

$$\mu(d) = \min\{a \in \mathbb{Z} \ : \ \alpha(d+a) \leqslant a\}.$$

Now, the hit problem was completely determined for $n \leqslant 4$, (see F.P.Peterson [6] for $n = 1, 2$, see M.Kameko for $n = 3$ in his thesis [3], see N.Sum [14] for $n = 4$). For $n > 4$, it is still unsolved, even in the case of $n = 5$ with the help of computers.

In the presnt paper, we study the hit problem for the case $n = 6$ in degree $d_k = 6(2^k - 1) + 9.2^k$, with k an arbitrary non-negative integer. The main goal of the current paper is to explicitly determine an admissible monomial basis of the $\mathcal{K}$-graded vector space $\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6$ in some degrees. The behavior of the sixth Singer algebraic transfer in degree $d_k = 6(2^k-1)+9.2^k$ is also discussed at the end of this article. Here, the Singer algebraic transfer is a homomorphism from the homology of the mod-2 Steenrod algebra, $\mathrm{Tor}^{\mathcal{A}}_{n,n+d}(\mathcal{K},\mathcal{K})$, to the subspace of $\mathcal{K} \otimes_{\mathcal{A}} \mathcal{P}_n$ consisting of all the GL_n-invariant classes of degree d.

Next, in Section 2, we recall some needed information on admissible monomials in $\mathcal{P}_n$. The proofs of the main results will be presented in Section 3.

2. Preliminaries

First, we recall some necessary results in Singer [10], Kameko [3], and Sum [14], which will be used in the next section.

Let $\alpha_i(d)$ be the i-th coefficient in dyadic expansion of d. Then, $d = \sum_{i\geqslant 0} \alpha_i(d).2^i$ where $\alpha_i(d) \in \{0,1\}$.

Let $u = x_1^{d_1}x_2^{d_2}\ldots x_n^{d_n} \in \mathcal{P}_n$. The weight vector of u is defined by

$$\omega(u) = (\omega_1(x), \omega_2(x), \ldots, \omega_k(x), \ldots),$$

where $\omega_i(x) = \sum_{1\leqslant j\leqslant n} \alpha_{i-1}(d_j),\ i \geqslant 1$.

A sequence of non-negative intergers $(\omega_1, \omega_2, \ldots, \omega_i, \ldots)$ is called the weight vector ω if $\omega_i = 0$ for $i \gg 0$. Then, we define $\deg \omega = \sum_{i\geqslant 0} \omega_i.2^{i-1}$.

Remarkably, the order on the set of sequences of nonnegative integers is given the left lexicographical order. Let $\mathcal{P}_n(\omega)$ denotes the subspace of $\mathcal{P}_n$ spanned by all monomials u such that $\deg u = \deg \omega$, $\omega(u) \leqslant \omega$, and we will denote by $\mathcal{P}_n^-(\omega)$ the subspace of $\mathcal{P}_n$ spanned by all monomials $u \in \mathcal{P}_n(\omega)$ such that $\omega(u) < \omega$.

Definition 2.1. Let $u,\ v$ be two polynomials of the same degree in $\mathcal{P}_n$, and ω a weight vector.

(i) $u \equiv v$ if and only if $u - v \in \mathcal{A}^+\mathcal{P}_n$. If $u \equiv 0$ then u is called *hit*.

(ii) $u \equiv_\omega v$ if and only if $u - v \in ((\mathcal{A}^+\mathcal{P}_n \cap \mathcal{P}_n(\omega)) + \mathcal{P}_n^-(\omega))$.

It is very easy to check that the relations $\equiv$ and $\equiv_\omega$ are equivalence ones. Denote by $Q\mathcal{P}_n(\omega)$ the quotient of $\mathcal{P}_n(\omega)$ by the equivalence relation $\equiv_\omega$. Then, one has

$$Q\mathcal{P}_n(\omega) = \mathcal{P}_n(\omega)/((\mathcal{A}^+\mathcal{P}_n \cap \mathcal{P}_n(\omega)) + \mathcal{P}_n^-(\omega)).$$

Definition 2.2. Let $u = x_1^{d_1}x_2^{d_2}\ldots x_n^{d_n}, v = x_1^{e_1}x_2^{e_2}\ldots x_n^{e_n}$ be monomials of the same degree in P_k. We say that $u < v$ if and only if one of the following holds:

(i) $\omega(u) < \omega(v)$;

(ii) $\omega(u) = \omega(v)$, and $(d_1, d_2, \ldots, d_n) < (e_1, e_2, \ldots, e_n)$.

Definition 2.3. A monomial u is said to be inadmissible if there exist monomials $v_1, v_2, \ldots, v_m$ such that $v_i < u$ for $i = 1, 2, \ldots, m$ and $u - \sum_{t=1}^{m} v_i \in \mathcal{A}^+\mathcal{P}_n$. We say u is admissible if it is not inadmissible.

Obviously, the set of all the admissible monomials of degree d in $\mathcal{P}_n$ is a minimal set of $\mathcal{A}$-generators for $\mathcal{P}_n$ in degree d.

Definition 2.4. Let $u \in \mathcal{P}_n$. We say u is strictly inadmissible if and only if there exist monomials $v_1, v_2, \ldots, v_m$ such that $v_j < u$, for $j = 1, 2, \ldots, m$ and $u = \sum_{j=1}^{m} v_j + \sum_{i=1}^{2^s-1} Sq^i(f_i)$ with $s = \max\{k : \omega_k(u) > 0\}$ and suitable polynomials $f_i \in \mathcal{P}_n$.

It is easy to check that if u is strictly inadmissible monomial, then it is inadmissible monomial.

Theorem 2.1 (Kameko [3], Sum [14]). *Let u, v, w be monomials in $\mathcal{P}_n$ such that $\omega_i(u) = 0$ for $i > r > 0$, $\omega_s(w) \neq 0$ and $\omega_i(w) = 0$ for $i > s > 0$.*
(i) *If w is inadmissible, then uw^{2^r} is also inadmissible.*
(ii) *If w is strictly inadmissible, then wv^{2^s} is also strictly inadmissible.*

Let $z = x_1^{d_1} x_2^{d_2} \ldots x_n^{d_n} \in \mathcal{P}_n$. The monomial z is called a spike if $d_j = 2^{t_j} - 1$ for t_j a non-negative integer and $j = 1, 2, \ldots, n$. Moreover, z is called a minimal spike, if it is a spike such that $t_1 > t_2 > \ldots > t_{r-1} \geqslant t_r > 0$ and $t_j = 0$ for $j > r$.

The following is a Singer's criterion on the hit monomials in $\mathcal{P}_n$.

Theorem 2.2 (Singer [10]). *Assume that $u \in \mathcal{P}_n$ is a monomial of degree d, where $\mu(d) \leqslant n$. Let z be the minimal spike of degree d. Then, u is hit if $\omega(u) < \omega(z)$.*

In what follows, let us denote by $\mathcal{D}_n(d)$ the set of all admissible monomials of degree d in $\mathcal{P}_n$. The cardinality of a set M is denoted by $|M|$.

3. The main results

In this section, we study the hit problem for the polynomial algebra of six variables in some degrees.

For $k = 0$, then $d_0 = 6(2^0 - 1) + 9.2^0$. We explicitly determine an admissible monomial basis of the $\mathcal{K}$-vector space $\big(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6\big)_{6(2^0-1)+9.2^0}$. Let us denote by $\mathcal{P}_n^0$ and $\mathcal{P}_n^+$ the $\mathcal{A}$-submodules of $\mathcal{P}_n$ spanned by all the monomials $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ such that $\prod_{i=1}^{n} a_i = 0$, and $\prod_{i=1}^{n} a_i > 0$, respectively. It is easy to see that $\mathcal{P}_n^0$ and $\mathcal{P}_n^+$ are the $\mathcal{A}$-submodules of $\mathcal{P}_n$.

Since $\mathcal{P}_n = \oplus_{d\geqslant 0}(\mathcal{P}_n)_d$ is the graded polynomial algebra, we have a direct summand decomposition of the $\mathcal{K}$-vector spaces

$$\big(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6\big)_{6(2^0-1)+9.2^0} = (\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_n^0)_{6(2^0-1)+9.2^0} \oplus (\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_n^+)_{6(2^0-1)+9.2^0}$$

Consider the homomorphism $\mathcal{L}_t : \mathcal{P}_5 \to \mathcal{P}_6$, for $1 \leqslant t \leqslant 6$ by substituting:

$$\mathcal{L}_t(x_k) = \begin{cases} x_k, & \text{if } 1 \leqslant k \leqslant t-1, \\ x_{k+1}, & \text{if } t \leqslant k \leqslant 5. \end{cases}$$

It is easy to check that $\mathcal{L}_t$ is a homomorphism of $\mathcal{A}$-modules.

Recall that $(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_5)_{6(2^0-1)+9.2^0}$ is a $\mathcal{K}$-vector space of dimension 191 with a basis consisting of all the classes represented by the monomials $a_j,\ 1 \leqslant j \leqslant 191$. Consequently, $|\mathcal{D}_5\big(6(2^0-1)+9.2^0\big)| = 191$ (see Tin [20]).

By a simple computation, we see that $|\bigcup_{k=1}^{6}\mathcal{L}_k(\mathcal{D}_5(9))| = 596$. Moreover, we get the set

$$\mathcal{B}^0 = \{b_i : b_i \in \bigcup_{k=1}^{6}\mathcal{L}_k(a_j), 1 \leqslant j \leqslant 191, 1 \leqslant i \leqslant 596\}$$

is a minimal set of generators for $\mathcal{A}$-module $\mathcal{P}_6^0$ in degree $6(2^0-1)+9.2^0$. More specifically, we obain the following proposition.

Proposition 3.1. *The set* $[\mathcal{B}^0] = \{[v] : v \in \mathcal{B}^0\}$ *is a basis of* $\mathcal{K}$*-vector space* $(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6^0)_{6(2^0-1)+9.2^0}$. *This implies* $(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6^0)_{6(2^0-1)+9.2^0}$ *has dimension* 596.

Remark 3.1. Put $Z_{(n,m)} = \{I = (i_1, i_2, \ldots, i_m) : 1 \leqslant i_1 < \ldots < i_m \leqslant n\}$, $1 \leqslant m < n$. For $I \in Z_{(n,m)}$, consider the homomorphism $f_I : \mathcal{P}_m \to \mathcal{P}_n$ of algebras by substituting $f_I(x_\ell) = x_{i_\ell}$ with $1 \leqslant \ell \leqslant m$. Then, f_I is a monomorphism of $\mathcal{A}$-modules. The following is a quote from Mothebe-Kaelo-Ramatebele [5]:

$$\mathcal{K} \otimes_{\mathcal{A}} \mathcal{P}_n^0 = \bigoplus_{1\leqslant m\leqslant n-1}\ \bigoplus_{I\in Z_{(n,m)}} (\mathcal{K} \otimes_{\mathcal{A}} f_I(\mathcal{P}_m^+)),$$

where $\dim(\mathcal{K} \otimes_{\mathcal{A}} f_I(\mathcal{P}_m^+))_d = \dim(\mathcal{K} \otimes_{\mathcal{A}} \mathcal{P}_m^+)_d$, and $|Z_{(n,m)}| = \binom{n}{m}$. Combining with the results in Wood [22], we obtain

$$\dim(\mathcal{K} \otimes_{\mathcal{A}} \mathcal{P}_n^0)_d = \sum_{\mu(d)\leqslant m\leqslant n-1} \binom{n}{m} \dim(\mathcal{K} \otimes_{\mathcal{A}} \mathcal{P}_m^+)_d.$$

Since $\mu(6(2^0-1)+9.2^0) = 3$, it follows that if $m < 3$ then $(\mathcal{K} \otimes_{\mathcal{A}} \mathcal{P}_m^+)_{6(2^0-1)+9.2^0}$ are trivial.

As is well-known, $|\mathcal{D}_5(6(2^0-1)+9.2^0)| = 191$, where $(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_5^+)_{6(2^0-1)+9.2^0}$ has dimension 31. According to Sum [14], we also see that the space $(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_4)_{6(2^0-1)+9.2^0}$ is a $\mathcal{K}$-vector space of dimension 46, where $\dim(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_4^+)_{6(2^0-1)+9.2^0} = 18$, and the space $(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_3^+)_{6(2^0-1)+9.2^0}$ has dimension 7.

Combining the aforementioned results with $\mu(6(2^0-1)+9.2^0) = 3$ yields the following result.

$$\dim(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6^0)_{6(2^0-1)+9.2^0} = \binom{6}{3}.7 + \binom{6}{4}.18 + \binom{6}{5}.31 = 596.$$

Next, we explicitly determine an admissible monomial basis of the $\mathcal{K}$-vector space $(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6^+)_{6(2^0-1)+9.2^0}$. Set $Q\mathcal{P}_n^+(\omega) := Q\mathcal{P}_n(\omega) \cap (\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_n^+)$, $\omega_{(1)} := (5,2)$, and $\omega_{(2)} := (3,3)$. Then, we have the following theorem.

Theorem 3.3. *Suppose that* $u \in \mathcal{D}_6(6(2^0-1)+9.2^0) \cap \mathcal{P}_6^+$, *then* $\omega(u) = \omega_{(j)}$ *with* $j = 1, 2$. *Moreover, we have an isomorphism of the* $\mathcal{K}$*-vector spaces:*

$$(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6^+)_{6(2^0-1)+9.2^0} \cong Q\mathcal{P}_6^+(\omega_{(1)}) \oplus Q\mathcal{P}_6^+(\omega_{(2)}).$$

Proof. Let ω be the weight vector of degree nine. We put $\mathcal{D}_6^{\otimes}(\omega) := \mathcal{D}_6(9) \cap \mathcal{P}_6(\omega)$. It is easy to see that $\mathcal{D}_6(9) = \bigcup_{\deg\omega=9} \mathcal{D}_6^{\otimes}(\omega)$.

Denote by $\mathcal{AP}_6^{\omega}$ the subspace of $\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6$ spanned by all the classes represented by the admissible monomials of weight vector ω in $\mathcal{P}_6$. It is simple to check that the map $Q\mathcal{P}_6(\omega) \longrightarrow \mathcal{AP}_6^{\omega}$, $[v]_\omega \longrightarrow [v]$ is an isomorphism of $\mathcal{K}$-vector spaces. Hence, we can identify the vector space $Q\mathcal{P}_6(\omega)$ with $\mathcal{AP}_6^{\omega} \subset \mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6$. From this, we can deduce

$$(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6)_9 = \bigoplus_{\deg \omega=9} \mathcal{A}\mathcal{P}_6^{\omega} \cong \bigoplus_{\deg \omega=9} Q\mathcal{P}_6(\omega) \qquad (3.1)$$

Hence $(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6^+)_{6(2^0-1)+9.2^0} = \bigoplus_{\deg \omega=9} Q\mathcal{P}_6^+(\omega)$.

On the other hand, it is easy to check that $z = x_1^7x_2x_3$ is the minimal spike of degree nine in $\mathcal{P}_6$ and $\omega(z) = (3,1,1)$. Suppose that x is an admissible monomial of degree nine in $\mathcal{P}_6^+$. By Theorem 2.2, it shows that $\omega_1(x) \geqslant \omega_1(z) = 3$. Since $\deg(u)$ is odd number, it implies either $\omega_1(x) = 3$ or $\omega_1(x) = 5$.

If $\omega_1(x) = 5$ then, $x = x_ix_jx_kx_\ell x_t v^2$ with $1 \leqslant i < j < k < \ell < t \leqslant 6$, where $v \in (\mathcal{P}_6)_2$. By Theorem 2.1, v is also admissible. It is easy to see that $\omega(v) = (2,0)$. And therefore, $\omega(x) = \omega_{(1)}$.

If $\omega_1(x) = 3$, then $u = x_ix_jx_ku^2$ with u a monomial of degree three in $\mathcal{P}_6$. By Theorem 2.1, u is an admissible monomial. An easy computation shows that

$$\mathcal{D}_6(3) = \{x_i^3 : 1 \leqslant i \leqslant 6\} \cup \{x_ix_j^2 : 1 \leqslant i < j \leqslant 6\} \cup \{x_ix_jx_k : 1 \leqslant i < j < k \leqslant 6\},$$

where $1 \leqslant i,j,k,\ell \leqslant 6$.

Since $u \in \mathcal{D}_6(3)$, and $x \in \mathcal{P}_6^+$, it shows that $\omega(u) = (3,0)$. So, $\omega(x) = \omega_{(2)}$.

From these above, we have $(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6^+)_{6(2^0-1)+9.2^0} \cong Q\mathcal{P}_6^+(\omega_{(1)}) \oplus Q\mathcal{P}_6^+(\omega_{(2)})$. Therefore, the theorem is proved. □

Theorem 3.4. *Let $\mathcal{D}_6^+(\omega)$ be the set of all admissible monomials in $\mathcal{P}_6^+(\omega)$. Then,*

$$|\mathcal{D}_6^+(\omega_{(j)})| = \begin{cases} 24, & \text{if } \ j = 1, \\ 10, & \text{if } \ j = 2. \end{cases}$$

This implies that $(\mathcal{K} \otimes_{\mathcal{A}} \mathcal{P}_6^+)_{6(2^0-1)+9.2^0}$ has dimension 34.

Proof. We prove the above theorem by explicitly determining all admissible monomials in $\mathcal{P}_6^+(\omega_{(j)})$ with $j \in \{1,2\}$. The proof is divided into the following cases.

Case 1. Consider the weight vector $\omega = \omega_{(1)} = (5,2)$. Suppose that X is an admissible monomial in $\mathcal{P}_6^+$ such that $\omega(X) = (5,2)$. Thus, $X = x_ix_jx_kY^2$ with $1 \leqslant i < j < k \leqslant 6$, $Y \in \mathcal{D}_6(2)$.

Consider the set $C_6^1 := \{x_ix_jx_k.Y^2 : 1 \leqslant i < j < k \leqslant 6, Y \in \mathcal{C}_6(2)\}$. Then, we have $\mathcal{P}_6^+(\omega_{(1)}) =$ Span$\{C_6^1\}$, and $|C_6^1| = 30$.

Using Theorem 2.1, it follows that if $X \in \mathcal{D}_6(9)$ such that $\omega(X) = (5,2)$, then $X \in C_6^1$.

It is easy to check that the monomials $x_1^2x_ix_jx_\ell x_kx_t^3$, $x_1^3x_2^2x_3x_4x_5x_6$ in C_6^1 are inadmissible (more precisely by Sq^1), where (i,j,ℓ,k,t) is an arbitrary permutation of $(2,3,4,5,6)$.

From the above results, it shows that $\mathcal{P}_6^+(\omega_{(1)})$ is generated by 24 elements c_i, for all $1 \leqslant i \leqslant 24$ as follows:

1. $x_1^3x_2^1x_3^1x_4^1x_5^1x_6^2$	2. $x_1^1x_2^3x_3^1x_4^1x_5^1x_6^2$	3. $x_1^1x_2^1x_3^3x_4^1x_5^1x_6^2$	4. $x_1^1x_2^1x_3^1x_4^3x_5^1x_6^2$
5. $x_1^1x_2^1x_3^1x_4^1x_5^3x_6^2$	6. $x_1^3x_2^1x_3^1x_4^1x_5^2x_6^1$	7. $x_1^1x_2^3x_3^1x_4^1x_5^2x_6^1$	8. $x_1^1x_2^1x_3^3x_4^1x_5^2x_6^1$
9. $x_1^1x_2^1x_3^1x_4^3x_5^2x_6^1$	10. $x_1^1x_2^1x_3^1x_4^1x_5^2x_6^3$	11. $x_1^3x_2^1x_3^1x_4^2x_5^1x_6^1$	12. $x_1^1x_2^3x_3^1x_4^2x_5^1x_6^1$
13. $x_1^1x_2^1x_3^3x_4^2x_5^1x_6^1$	14. $x_1^1x_2^1x_3^1x_4^2x_5^3x_6^1$	15. $x_1^1x_2^1x_3^1x_4^2x_5^1x_6^3$	16. $x_1^3x_2^1x_3^2x_4^1x_5^1x_6^1$
17. $x_1^1x_2^3x_3^2x_4^1x_5^1x_6^1$	18. $x_1^1x_2^1x_3^2x_4^3x_5^1x_6^1$	19. $x_1^1x_2^1x_3^2x_4^1x_5^3x_6^1$	20. $x_1^1x_2^1x_3^2x_4^1x_5^1x_6^3$
21. $x_1^1x_2^2x_3^3x_4^1x_5^1x_6^1$	22. $x_1^1x_2^2x_3^1x_4^3x_5^1x_6^1$	23. $x_1^1x_2^2x_3^1x_4^1x_5^3x_6^1$	24. $x_1^1x_2^2x_3^1x_4^1x_5^1x_6^3$

We next prove that the vectors $[c_i], 1 \leqslant i \leqslant 24$, are linearly independent in $\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6$. Denote

$$\mathcal{N}_n = \{(j;J) : J = (j_1, j_2, \ldots, j_t), 1 \leqslant j < j_1 < \ldots < j_t \leqslant n, 0 \leqslant t < n\}.$$

For $n = 6$, and for any $(j;J) \in \mathcal{N}_6$, we define $\varphi_{(j;J)} : \mathcal{P}_6 \to \mathcal{P}_5$ by substituting:

$$\varphi_{(j;J)}(x_i) = \begin{cases} x_i, & \text{if } 1 \leqslant i \leqslant j-1, \\ \sum_{s\in J} x_{s-1}, & \text{if } i = j, \\ x_{i-1}, & \text{if } j < i \leqslant 6. \end{cases}$$

It is easy to check that these homomorphisms are $\mathcal{A}$-modules homomorphisms. We use them to prove that a certain set of monomials is actually the set of admissible monomials in $\mathcal{P}_6$ by showing these monomials are linearly independent in $\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6$.

Suppose that there is a linear relation:

$$\mathcal{U} = \sum_{1\leqslant i\leqslant 24} \gamma_i c_i \equiv 0,$$

with $\gamma_i \in \mathcal{K},\ 1 \leqslant i \leqslant 24$.

Using the results in [20], we compute $\varphi_{(j;J)}(\mathcal{U})$ in terms of the admissible monomials in $\mathcal{P}_5(\mathrm{mod}(\mathcal{A}^+\mathcal{P}_5))$. By direct computation, from the relations $\varphi_{(j;J)}(\mathcal{U}) \equiv 0$, one gets $\gamma_i = 0$ for all $1 \leqslant i \leqslant 24$.

In summary, the set $\{[c_i] : 1 \leqslant i \leqslant 24\}$ is a basis of the $\mathcal{K}$-vector space $Q\mathcal{P}_6^+(\omega_{(1)})$. Consequently, $\dim Q\mathcal{P}_6^+(\omega_{(1)}) = 24$.

Case 2. Consider the weight vector $\omega = \omega_{(2)} = (3,3)$. Assume that Y is an admissible monomial in $\mathcal{P}_6^+$ such that $\omega(Y) = (3,3)$. Thus, one has $Y = x_ix_jx_k.w^2$ with $1 \leqslant i < j < k \leqslant 6, w \in \mathcal{D}_6(3)$.

Putting $C_6^2 := \{x_ix_jx_k.w^2 : 1 \leqslant i < j < k \leqslant 6, w \in \mathcal{D}_6(3)\}$. Then, one gets $\mathcal{P}_6^+(\omega_{(2)}) = \mathrm{Span}\{C_6^2\}$, and if $X \in \mathcal{D}_6(11)$ such that $\omega(X) = (3,3)$, then $X \in C_6^2$.

By direct calculations, we see that $\mathcal{P}_6^+(\omega_{(2)})$ is generated by 10 elements $d_i, 1 \leqslant i \leqslant 10$ as follows:

1. $x_1x_2x_3x_4^2x_5^2x_6^2$	2. $x_1x_2x_3^2x_4x_5^2x_6^2$	3. $x_1x_2x_3^2x_4^2x_5x_6^2$	4. $x_1x_2x_3^2x_4^2x_5^2x_6$
5. $x_1x_2^2x_3x_4x_5^2x_6^2$	6. $x_1x_2^2x_3x_4^2x_5x_6^2$	7. $x_1x_2^2x_3x_4^2x_5^2x_6$	8. $x_1x_2^2x_3^2x_4x_5x_6^2$
9. $x_1x_2^2x_3^2x_4x_5^2x_6$	10. $x_1x_2^2x_3^2x_4^2x_5x_6$		

We now prove that the vectors $[d_i], 1 \leqslant i \leqslant 10$ are linearly independent in $\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6$. Suppose that there is a linear relation:

$$\mathcal{S} = \sum_{1\leqslant i\leqslant 10} \gamma_i d_i \equiv 0,$$

with $\gamma_i \in \mathcal{K},\ 1 \leqslant i \leqslant 10$. Using the results in [20], we compute $\varphi_{(j;J)}(\mathcal{S})$ in terms of the admissible monomials in $\mathcal{P}_5(\mathrm{mod}(\mathcal{A}^+\mathcal{P}_5))$. From the relations $\varphi_{(j;J)}(\mathcal{S}) \equiv 0$, one gets $\gamma_i = 0$ for all $1 \leqslant i \leqslant 10$.

Hence, $Q\mathcal{P}_6^+(\omega_{(2)})$ is an $\mathcal{K}$-vector space of dimension 10 with a basis consisting of all the classes represented by the monomials $d_i,\ 1 \leqslant i \leqslant 10$. Consequently, $\dim Q\mathcal{P}_6^+(\omega_{(2)}) = 10$. And therefore, the theorem is proved. □

From the results of Proposition 3.1, Theorems 3.3 and 3.4, we obtain the following corollary.

Corollary 3.1. *The set* $\{b_i : 1 \leqslant i \leqslant 596\} \cup \{c_j : 1 \leqslant j \leqslant 24\} \cup \{d_\ell : 1 \leqslant \ell \leqslant 10\}$ *is a minimal set of* $\mathcal{A}$*-generators for* $\mathcal{P}_6$ *in degree* $6(2^0-1)+9.2^0$. *Consequently,* $\dim(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6)_{6(2^0-1)+9.2^0} = 630$.

It is worth noting that Mothebe-Kaelo-Ramatebele [5] utilized a different method to verify the dimension result of the vector space $(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6)_{6(2^0-1)+9.2^0}$.

For $k = 1$, then $d_1 = 6(2^1 - 1) + 9.2^1$. Recall the Kameko's squaring operation

$$\widetilde{Sq}^0_* := (\widetilde{Sq}^0_*)_{(n;n+2d)} : (\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_n)_{n+2d} \to (\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_n)_d,$$

which is induced by an $\mathcal{K}$-linear map $\mathcal{S}_n : \mathcal{P}_n \to \mathcal{P}_n$, given by

$$\mathcal{S}_n(x) = \begin{cases} y, & \text{if } x = \prod_{i=1}^n x_i y^2 \\ 0, & \text{otherwise} \end{cases}$$

for any monomial $x \in \mathcal{P}_n$ (see Kameko [3]).

Since Kameko's homomorphism $(\widetilde{Sq}^0_*)_{(6;24)}$ is a $\mathcal{K}$-epimorphism, and $\mathcal{P}_n$ is the graded polynomial algebra, it shows that

$$(\mathcal{K} \otimes_{\mathcal{A}} \mathcal{P}_6)_{24} \cong (\mathcal{K} \otimes_{\mathcal{A}} \mathcal{P}^0_6)_{24} \bigoplus \big(\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;24)} \cap (\mathcal{K} \otimes_{\mathcal{A}} \mathcal{P}^+_6)_{24}\big) \bigoplus \mathrm{Im}(\widetilde{Sq}^0_*)_{(6;24)}$$

First, we have the following theorem.

Theorem 3.5. *The following statements are true:*

(i) *Set* $\mathcal{D}^{\otimes 6}_{Im}(24) := \big\{[x] : x = \Gamma_6(u), \text{ for all } u \in \mathcal{D}_6(6(2^0-1)+9.2^0)\big\}$, *where* $\Gamma_6 : \mathcal{P}_6 \to \mathcal{P}_6$ *is the homomorphism determined by* $\Gamma_6(u) = \prod_{i=1}^6 x_i u^2,\ u \in \mathcal{P}_6$. *Then* $|\mathcal{D}^{\otimes 6}_{Im}(24)| = 630$, *and the space* $\mathrm{Im}(\widetilde{Sq}^0_*)_{(6;24)}$ *is isomorphic to a subspace of* $(\mathcal{K} \otimes_{\mathcal{A}} \mathcal{P}_6)_{6(2^1-1)+9.2^1}$ *generated by all the classes* $[x]$ *of* $\mathcal{D}^{\otimes 6}_{Im}(24)$.

(ii) *Let us denote by* $\mathcal{D}^{\otimes 6}_0(24) := \big\{v : v \in \bigcup_{k=1}^6 \mathcal{L}_k(\mathcal{D}_5(6(2^1-1)+9.2^1))\big\}$. *Then, we have* $|\mathcal{D}^{\otimes 6}_0(24)| = 4716$, *and the set* $\{[v] : v \in \mathcal{D}^{\otimes 6}_0(24)\}$ *is a basis of the* $\mathcal{K}$*-vector space* $(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}^0_6)_{6(2^1-1)+9.2^1}$. *This implies that* $(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}^0_6)_{6(2^1-1)+9.2^1}$ *has dimension* 4716.

Proof. We have $\mu(6(2^1-1)+9.2^1) = 4$. Using the same arguments as in Remark 3.1, we also get

$$\dim(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}^0_6)_{6(2^1-1)+9.2^1} = \sum_{4\leqslant m\leqslant 5} \binom{6}{m} \dim(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}^+_m)_{6(2^1-1)+9.2^1}.$$

Using the results in Sum [14], and Tin [18], we obtain

$$\dim(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}^+_m)_{6(2^1-1)+9.2^1} = \begin{cases} 70, & \text{if } m = 4, \\ 611, & \text{if } m = 5. \end{cases}$$

And therefore, we get

$$\dim(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}^0_6)_{6(2^1-1)+9.2^1} = \binom{6}{4}.70 + \binom{6}{5}.611 = 4716.$$

On the other hand, Tin showed in [18] that $(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_5)_{6(2^1-1)+9.2^1}$ is a $\mathcal{K}$-vector space of dimension 961 with a basis consisting of all the classes represented by the monomials $u_j,\ 1 \leqslant j \leqslant 961$. We set $\mathcal{F}^0 := \{\bigcup_{k=1}^6 \mathcal{L}_k(u_j) : 1 \leqslant j \leqslant 961\}$. An easy computation shows that $|\mathcal{F}^0| = 4716$, and the set $\{[v] : v \in \mathcal{F}^0\}$ is a basis of the $\mathcal{K}$-vector space $(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}^0_6)_{6(2^1-1)+9.2^1}$. The theorem is proved. □

Next, we explicitly determine the $\mathcal{K}$-vector space $\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;24)} \cap (\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}^+_6)_{24}$. We have the following theorem.

Theorem 3.6. *Let us denote by* $\widetilde{\omega_1} := (4,2,4)$, $\widetilde{\omega_2} := (4,2,2,1)$, $\widetilde{\omega_3} := (4,4,3)$, *and* $\widetilde{\omega_4} := (4,4,1,1)$. *Then, we have*

(i) *Assume that* x *belongs to* $(\mathcal{D}_6(24) \cap \mathcal{P}_6^+)$ *such that* $(\widetilde{Sq_*^0})_{(6;24)}([x])$ *is not an element of* $Im(\widetilde{Sq_*^0})_{(6;24)}$. *Then* $\omega(x) = \widetilde{\omega_i}$, *with* $i = 1,2,3,4$. *Moreover, we have an isomorphism of the* $\mathcal{K}$*-vector spaces:*

$$\big(\mathrm{Ker}(\widetilde{Sq_*^0})_{(6;24)} \cap (\mathcal{K} \otimes_{\mathcal{A}} \mathcal{P}_6^+)_{24}\big) \cong \bigoplus_{i=1}^{4} Q\mathcal{P}_6^+(\widetilde{\omega_i}).$$

(ii) *We have* $\dim\big(\mathrm{Ker}(\widetilde{Sq_*^0})_{(6;24)} \cap (\mathcal{K} \otimes_{\mathcal{A}} \mathcal{P}_6^+)_{24}\big) = \sum_{i=1}^{4} \dim Q\mathcal{P}_6^+(\widetilde{\omega_i}) = 2781.$

Proof. We set $Q\mathcal{P}_6^{\omega} := \mathrm{Span}\{[x] \in \mathcal{K} \otimes_{\mathcal{A}} \mathcal{P}_6 : x \text{ is admissible and } \omega(x) = \omega\}$. Using the results in Walker-Wood [21], we obtain

$$(\mathcal{K} \otimes_{\mathcal{A}} \mathcal{P}_6)_{24} = \bigoplus_{\deg \omega = 24} Q\mathcal{P}_6^{\omega} \cong \bigoplus_{\deg \omega = 24} Q\mathcal{P}_6(\omega) \tag{3.2}$$

Suppose that x is an admissible monomial of degree twenty-four in $\mathcal{P}_6^+$ such that $[x]$ belongs to $\mathrm{Ker}(\widetilde{Sq_*^0})_{(6;24)}$. Observe that $z = x_1^{15}x_2^7x_3x_4$ is the minimal spike of degree twenty-four in $\mathcal{P}_6$ and $\omega(z) = (4,2,2,1)$. Using Theorem 2.2, we get $\omega_1(x) \geqslant 4$. Since the degree of (x) is even, one gets either $\omega_1(x) = 4$, or $\omega_1(x) = 6$.

If $\omega_1(x) = 4$ then $x = x_ix_jx_kx_\ell u^2$ with u an admissible monomial of degree ten in $\mathcal{P}_6$ and $1 \leqslant i < j < k < \ell \leqslant 6$. Since x is admissible, by Theorem 2.1, u is also admissible. By an easy computation show that $\omega(u) = (2,4)$, or $\omega(u) = (2,2,1)$, or $\omega(u) = (4,1,1)$, or $\omega(u) = (4,3)$, or $\omega(u) = (6,2)$. We see that if v is a monomial in $\mathcal{P}_6$ such that $\omega(v) = (4,6,2)$, then v is strictly inadmissible (see Sum [13], Prop. 4.3). And therefore, v is inadmissible. From this, $\omega(x) = (4,2,4)$, or $\omega(x) = (4,2,2,1)$, or $\omega(x) = (4,4,3)$, or $\omega(x) = (4,4,1,1)$.

If $\omega_1(x) = 6$ then $x = \prod_{i=1}^{6} x_i w^2$, with w a monomial of degree nine in $\mathcal{P}_6$. Using Theorem 2.1, y is an admissible monomial. Hence, $(\widetilde{Sq_*^0})_{(6;24)}([x]) = [y] \neq 0$. This contradicts the fact that $[x] \in \mathrm{Ker}(\widetilde{Sq_*^0})_{(6;24)}$.

From the above results, we obtain

$$\mathrm{Ker}(\widetilde{Sq_*^0})_{(6;24)} \cap (Q\mathcal{P}_6^+)_{24} = \bigoplus_{m=1}^{4} Q\mathcal{P}_6^+(\widetilde{\omega_m}).$$

Remarkably, to list all the elements of the admissible monomial basis of the vector space $\mathrm{Ker}(\widetilde{Sq_*^0})_{(6;24)} \cap (Q\mathcal{P}_6^+)_{24}$ is far too long and computationally very technical. The following is a sketch of its proof with the aid of computers.

Let us denote by $\mathcal{M}_{\omega}^{\otimes 6}$ the set of classes represented by the admissible monomials of the vector space $\mathrm{Ker}(\widetilde{Sq_*^0})_{(6;24)} \cap (Q\mathcal{P}_6^+)_{24}$. Consider the set

$$B_6^{\otimes >}(\omega) := \{x_ix_jx_kx_\ell v^2 : 1 \leqslant i < j < k < \ell \leqslant 6, v \in \mathcal{D}_6(10)\} \ \cap\ \mathcal{P}_6^+.$$

Using Theorem 2.1, we see that if u is an admissible monomial of degree 24 in $\mathcal{P}_6^+$ such that $(\widetilde{Sq_*^0})_{(6;24)}\big([u]\big)$ does not belong to $\mathrm{Im}(\widetilde{Sq_*^0})_{(6;24)}$, then $u \in B_6^{\otimes >}(\omega)$.

We set up an algorithm implemented in Microsoft Excel software to eliminate the inadmissible monomials in $B_6^{\otimes >}(\omega)$ by observing that each monomial $x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}x_5^{a_5}x_6^{a_6}$ corresponds to a series of numbers of the type $(a_1; a_2; a_3; a_4; a_5; a_6)$.

By direct calculations, using Theorem 2.1, we filter out and remove the inadmissible monomials in $B_6^{\otimes>}(\omega)$, so we get $|\mathcal{M}_\omega^{\otimes 6}| = 2781$.

Therefore, $\dim\big(\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;24)} \cap (\mathcal{K}\otimes_{\mathcal{A}} \mathcal{P}_6^+)_{24}\big) = 2781$. The theorem is proved. □

From the results of Theorem 3.5 and Theorem 3.6, we obtain the following corollary.

Corollary 3.2. *There exist exactly* 8127 *admissible monomials of degree twenty-four in* $\mathcal{P}_6$. *Consequently,* $|\mathcal{D}_6(6(2^1-1)+9.2^1)| = 8127$.

Consider the degrees $d_k = 6(2^k-1)+9.2^k$, for any $k \geqslant 2$. Let $GL_n(\mathcal{K})$ be the general linear group over the field $\mathcal{K}$. Note that $GL_n(\mathcal{K})$ acts naturally on $\mathcal{P}_n$ by matrix substitution. Since the two actions of $GL_n(\mathcal{K})$ and $\mathcal{A}$ upon $\mathcal{P}_n$ commute with each other, hence there is an inherited action of $GL_n(\mathcal{K})$ on $\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_n$. We set

$$\zeta(n;d) = \max\{0, n-\alpha(d+n)-\zeta(d+n)\},$$

where $\zeta(n)$ is the greatest integer m such that n is divisible by 2^m. We recall the following result in Tin-Sum [17].

Theorem 3.7. *Let d be an arbitrary non-negative integer. Then*

$$(\widetilde{Sq}^0_*)^{r-s} : (\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_n)_{n(2^r-1)+2^r d} \longrightarrow (\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_n)_{n(2^s-1)+2^s d}$$

is an isomorphism of $GL_n(\mathcal{K})$-modules for every $r \geqslant s$ if and only if $s \geqslant \zeta(n;d)$.

It is easy to see that for $n=6$ and $d=54$ then $\alpha(d+n) = \alpha(60) = 4$, and $\zeta(d+n) = \zeta(2^2.15) = 2$, and therefore $\zeta(6;54) = 0$. Using the above theorem, we get an isomorphism of $\mathcal{K}$-vector spaces:

$$(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6)_{6(2^r-1)+54.2^r} \cong (\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6)_{6(2^0-1)+54.2^0} \text{ for all } r \geqslant 0.$$

And therefore, we obtain

$$\dim(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6)_{6(2^r-1)+54.2^r} = \dim(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6)_{6(2^0-1)+54.2^0} \text{ for } r \geqslant 0.$$

So, we get the set $\big\{[x] : x \in \Gamma_6^{k-2}\big(\mathcal{D}_6(6(2^2-1)+9.2^2)\big)\big\}$ is a basis of the $\mathcal{K}$-vector space $(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_6)_{6(2^k-1)+9.2^k}$, for all $k > 2$. Here, $\Gamma_n : \mathcal{P}_n \to \mathcal{P}_n$ is the homomorphism determined by $\Gamma_n(x) = \prod_{i=1}^n x_i x^2$, for all $x \in \mathcal{P}_n$.

Remark 3.2. Let $(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_n)_d^{GL_n(\mathcal{K})}$ be the subspace of $(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_n)_d$ consisting of all the $GL_n(\mathcal{K})$-invariant classes of degree d, and let us denote by $\mathcal{K}\otimes_{GL_n(\mathcal{K})} PH_d((\mathbb{R}\mathcal{P}^\infty)^n)$ the dual to $(\mathcal{K}\otimes_{\mathcal{A}}\mathcal{P}_n)_d^{GL_n(\mathcal{K})}$. One of the major applications of hit problem is in surveying a homomorphism introduced by W. M. Singer. It is a useful tool in describing the cohomology groups of the Steenrod algebra, $Ext_{\mathcal{A}}^{n,n+*}(\mathcal{K},\mathcal{K})$.

In [10], Singer defined the algebraic transfer, which is a homomorphism

$$Tr_n : \mathcal{K}\otimes_{GL_n(\mathcal{K})} PH_*((\mathbb{R}\mathcal{P}^\infty)^n) \longrightarrow Ext_{\mathcal{A}}^{n,n+*}(\mathcal{K},\mathcal{K}).$$

Singer has indicated the importance of the algebraic transfer by showing that Tr_n is a isomorphism with $n = 1, 2$ and at some other degrees with $n = 3, 4$, but he also disproved this for Tr_5 at degree 9, and then gave the following conjecture.

Conjecture 3.1. The algebraic transfer Tr_n is a monomorphism for any $n \geqslant 0$.

It could be seen from the work of Singer the meaning and necessity of the hit problem. In [1], Boardman confirmed this again by using the modular representation theory of linear groups to show that Tr_3 is also an isomorphism.

For $n \geqslant 4$, the Singer algebraic transfer was studies by many authors (See Boardman [1], Bruner-Ha-Hung [2], Minami [4], Sum-Tin [15], Phuc [7] and others). However, Singer's conjecture is still open for $n \geqslant 4$.

In the future, we will use the results of the hit problem to study and verify the Singer conjecture for the algebraic transfer in the above degrees. More specifically, by using the admissible monomial basis of degree $6(2^k - 1) + 9.2^k$ in $\mathcal{P}_6$ to explicitly compute the vector space $(\mathcal{K} \otimes_{\mathcal{A}} \mathcal{P}_6)^{GL_6(\mathcal{K})}_{6(2^k-1)+9.2^k}$ and combining the computation of the groups $Ext^{6,6(2^k-1)+9.2^k+6}_{\mathcal{A}}(\mathcal{K}, \mathcal{K})$, to obatin information about the behavior of the sixth Singer algebraic transfer in these degrees.

By Theorem 3.7, we also obtain the following theorem.

Theorem 3.8. *We have an isomorphism of $\mathcal{K}$-vector spaces:*

$$(\mathcal{K} \otimes_{\mathcal{A}} \mathcal{P}_6)^{GL_6(\mathcal{K})}_{6(2^k-1)+9.2^k} \cong (\mathcal{K} \otimes_{\mathcal{A}} \mathcal{P}_6)^{GL_6(\mathcal{K})}_{6(2^2-1)+9.2^2}, \text{ for all } k > 2.$$

By passing to the dual, we obtain the following result.

$$\mathcal{K}\otimes_{GL_6(\mathcal{K})} PH_{6(2^k-1)+9.2^k}((\mathbb{R}\mathcal{P}^\infty)^6) \cong (\mathcal{K}\otimes_{GL_6(\mathcal{K})} PH_{6(2^2-1)+9.2^2}((\mathbb{R}\mathcal{P}^\infty)^6)),$$

for $k > 2$. And therefore, we need only to compute the dimension of the vector spaces $\mathcal{K}\otimes_{GL_6(\mathcal{K})} PH_{6(2^k-1)+9.2^k}((\mathbb{R}\mathcal{P}^\infty)^6)$ for $k \leqslant 2$. In the not-too-distant future, we will investigate and validate Singer's conjecture for the sixth algebraic transfer in these circumstances.

Acknowledgments. I would like to express my warmest thanks to my advisor, Prof. Nguyen Sum for helpful conversations. This research is supported by Ho Chi Minh City University of Technology and Education (HCMUTE), Vietnam. Finally, I would like to express my warmest thanks to the referees for many helpful suggestions and detailed comments, which have led to the improvement of the paper's exposition.

References

[1] Boardman, J. M. Modular representations on the homology of power of real projective space, in: M. C. Tangora (Ed.). Algebraic Topology. Oaxtepec, 1991, in: *Contemp. Math.*, **146** (1993), 49-70.

[2] Bruner, R. R.; Ha, L. M.; Hung, N. H. V. On behavior of the algebraic transfer. *Trans. Amer. Math. Soc.* **357** (2005), 473-487.

[3] Kameko, M. *Products of projective spaces as Steenrod modules*, Ph.D. Thesis, The Johns Hopkins University, ProQuest LLC, Ann Arbor, MI, 1990. 29 pp.

[4] Minami, N. The iterated transfer analogue of the new doomsday conjecture. *Trans. Amer. Math. Soc.* **351** (1999), 2325-2351.

[5] Mothebe, M. F.; Kaelo, P.; Ramatebele, O. Dimension formulae for the polynomial algebra as a module over the Steenrod algebra in degrees less than or equal to 12. *J. Math. Reasearch* **8** (2016), 92-100.

[6] Peterson, F. P. Generators of $H^*(\mathbb{R}P^\infty \times \mathbb{R}P^\infty)$ as a module over the Steenrod algebra. *Abstracts Amer. Math. Soc.* **833** (1987), 55-89.

[7] Phuc, D. V. On Peterson's open problem and representations of the general linear groups. *Journal of the Korean Mathematical Society.* **58** (2021), no. 3, 643-702.

[8] Priddy, S. On characterizing summands in the classifying space of a group I. *American J. Math.* **112** (1990), 737-748.

[9] Repka, J.; Selick, P. On the subalgebra of $H_*((\mathbb{R}P^\infty)^n; \mathbb{F}_2)$ annihilated by Steenrod operations. *J. Pure Appl. Algebra.* **127** (1998), 273-288.

[10] Singer, W. M. The transfer in homological algebra. *Math. Zeit.* **202** (1989), 493-523.

[11] Silverman, J. H. Hit polynomials and the canonical antiautomorphism of the Steenrod algebra. *Proc. Amer. Math. Soc.* **123** (1995), 627-637.

[12] Steenrod, N. E.; Epstein, D. B. A. *Cohomology operations.* Annals of Mathematics Studies 50, Princeton University Press, Princeton N.J 1962.

[13] Sum, N. The negative answer to Kameko's conjecture on the hit problem. *Adv. Math.* **225** (2010), 2365-2390.

[14] Sum, N. On the Peterson hit problem. *Adv. Math.* **274** (2015), 432-489.

[15] Sum, N.; Tin, N. K. Some results on the fifth Singer transfer. *East-West J. Math.* **17** (2015), no. 1, 70-84.
[16] Sum, N.; Tin, N. K. The hit problem for the polynomial algebra in some weight vectors. *Topology Appl.* **290** (2021), 107579.
[17] Tin, N. K.; Sum, N. Kameko's homomorphism and the algebraic transfer. *C. R. Acad. Sci. Paris, Ser. I.* **354** (2016), 940-943.
[18] Tin, N. K. A note on the hit problem for the Steenrod algebra and its applications. Preprint 2021, available online at http:/arxiv.org/abs/2103.04393v1.
[19] Tin, N. K. A note on the Peterson hit problem for the Steenrod algebra. *Proc. Japan Acad. Ser. A, Math. Sci.* **97** (2021), no. 4, 25-28.
[20] Tin, N. K. Hit problem for the polynomial algebra as a module over Steenrod algebra in some degrees. *Asian-European J. Math.* **15** (2022), no. 1, 2250007.
[21] Walker, G.; Wood, R. M. W. *Polynomials and the mod 2 Steenrod algebra, Vol. 1. The Peterson hit problem.* London Mathematical Society Lecture Note Series, **441**. Cambridge University Press, 2018.
[22] Wood, R. M. W. Steenrod squares of polynomials and the Peterson conjecture. *Math. Proc. Cambridge Phil. Soc.* **105** (1989) 307-309.

HO CHI MINH CITY UNIVERSITY OF TECHNOLOGY AND EDUCATION
FACULTY OF APPLIED SCIENCES
01-VO VAN NGAN, THU DUC, HO CHI MINH CITY, VIETNAM
Email address: tinnk@hcmute.edu.vn