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On the solution of the generalized functional equation arising in mathematical psychology and theory of learning approached by the Banach fixed point theorem

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ABSTRACT. In mathematical psychology, the model of decision practice represents the development of moral judgment that deals with the time to decide the meaning of the various choices and selecting one of them for use. Most animal behavior research classifies such situations as two distinct phenomena. On the other hand, reward plays a big part in this kind of study since, based on the selected side and food location, such circumstances may be classified into four categories. This paper intends to investigate such types of behavior and establish a general functional equation for it. The proposed functional equation can be used to describe several psychological and learning theory models in the existing literature. By using the fixed point theory tools, we obtain the results related to the existence, uniqueness, and stability of a solution to the proposed functional equation. Finally, we give two examples to support our main results.

1. INTRODUCTION AND PRELIMINARIES

Mathematical psychology is an approach to psychological study focused on mathematical modeling of perceptual, thinking, cognitive, and motor processes. On the other hand, the learning process may also be interpreted in animals or humans as a set of choices between many alternative responses. In recent mathematical learning experiments, investigators have concluded that such simple learning experiments follow the stochastic process (for the detail, see [6]).

Our emphasis is on a primary type of learning experiment. In such experiments, each series of trials collects the topic of alternate answers under the experimenters' instructions. The alternative options may be to click one of the button sets, turn right in a maze, hop over a barrier until a shock is released, or struggle to recall a phrase.

In 2019, Turab and Sintunavarat [24] discussed the experimental work of Bush and Wilson [7] and analyzed the movement of a paradise fish in a two-choice situation. If a fish is awarded by selecting the correct side in such experiments, its probability will increase in the subsequent trials. Bush and Wilson [7] concluded that the current possibility of selecting the right side takes the form $\alpha_1 x + 1 - \alpha_1$, where $\alpha_1 \in (0,1)$ is the learning parameter appropriate to this particular outcome and x is the probability of selecting the right-hand side of the tank. At the same time, if the fish chooses the other side, its probability will decrease to $\alpha_2 x$, where $\alpha_2 \in (0,1)$. Various conclusions can be obtained regarding the animals' final behavior in such processes. For example, in the reinforcement-extinction model, if more animals select the reward side, the non-reward side's risk will decline. As a result, the animals will start to divert to the reward side. In comparison, the habit-forming paradigm suggests that animals may be stable on all sides. Such type of relationship can be described in Table 1.

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Operators for reinforcement-extinction model			
Fish's Response	**Event**	**Outcome (Left side)**	**Outcomes (Right side)**
Reinforcement	E_1	$\alpha_1 x$	$\alpha_1 x + 1 - \alpha_1$
Non-reinforcement	E_2	$\alpha_2 x + 1 - \alpha_2$	$\alpha_2 x$
Operators for habit formation model			
Reinforcement	E_1	$\alpha_1 x$	$\alpha_1 x + 1 - \alpha_1$
Non-reinforcement	E_2	$\alpha_2 x$	$\alpha_2 x + +1 - \alpha_2$

TABLE 1. Operators describing the fish behavior under some models

Turab and Sintunavarat [24, 25] described such relationship by the following functional equation

$$U(x) = xU(\alpha_1 x + 1 - \alpha_1) + (1 - x)U(\alpha_2 x) \tag{1.1}$$

for all $x \in [0, 1]$, where $0 < \alpha_1 \leq \alpha_2 < 1$ and $U : [0, 1] \to \mathbb{R}$ is an unknown function such that

$$\begin{cases} U(0) = 0, \\ U(1) = 1. \end{cases} \tag{1.2}$$

They used the fixed point results to obtain the existence and uniqueness of a solution to the proposed equation (1.1) with the condition (1.2).

In 2020, Turab and Sintunavarat [26] used the above idea to observe the learning process of dogs enclosed in a tiny box with a steel grid floor and proposed the following functional equation

$$U(x) = xU(\alpha_1 x + (1 - \alpha_1)\beta_1) + (1 - x)U(\alpha_2 x + (1 - \alpha_2)\beta_2) \tag{1.3}$$

for all $x \in [0, 1]$, where $\beta_1, \beta_2 \in [0, 1]$, $0 < \alpha_1 \leq \alpha_2 < 1$ and $U : [0, 1] \to \mathbb{R}$ is an unknown function.

Recently, in [27], the authors extended the idea of (1.1) by proposing the following generalized functional equation

$$U(x) = xU(f(x)) + (1 - x)U(g(x)) \tag{1.4}$$

for all $x \in [0, 1]$, where $U : [0, 1] \to \mathbb{R}$ is an unknown function and $f, g : [0, 1] \to [0, 1]$ are Banach contraction mappings satisfying

$$g(0) = 0 \tag{1.5}$$

under several conditions. The above functional equation (1.4) is used to describe the relationship between predator animals and their two prey choices. Numerous research on human and animals behavior in such situations have generated notable findings (for the detail, see [5, 9, 14, 15, 28, 29, 30, 21]).

On the other hand, the theory of fixed points is concerned with the conditions that ensure the existence of points x in a set Y that satisfies an operator equation $x = Gx$, where G is a transformation defined on a set Y. It consists of techniques that can be used to solve problems in diverse areas of mathematics. For the recent research in this area, we refer [2, 8, 10, 18, 19] and the references therein.

In the progression, the following noted result will be needed.

Theorem 1.1 (Banach Fixed Point Theorem in [4])**.** *Let (X, d) be a complete metric space and $T : X \to X$ be a Banach contraction mapping, that is,*

$$d(Tx, Ty) \leq kd(x, y) \tag{1.6}$$

for all $x, y \in X$, where $k \in [0,1)$. Then T has precisely one fixed point. Moreover, the Picard iteration $\{x_n\}$ in X, which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, where $x_0 \in X$, converges to the unique fixed point of T.

2. Introduction to the imitation model

The word "imitation" is thought to be a fundamental phenomenon in social behavior. The experiment demonstrated that imitation is a learned behavior that can be managed via rewards and penalties. Miller and Dollard [16] classified imitation into three types:

(1) identical behavior,
(2) matched-dependent interaction, and
(3) copying behavior.

Schien [21] attempted to test Miller and Dollard's declarations [16] with the army selectees. Later on, to observe the imitation in children, Shwartz [22] organized a guessing game in which two children were brought together in a room. There were fifty trials, and each participant was instructed to predict whether the experimenter would say 'a' or 'b' to each child at any given trial. In order to prepare for the trials, a schedule was created by arranging the slots within the ten tests. At the end of the experiment, Shwartz discovered that 9- and 10-year-olds imitated more as compared to the 15- and 16-year-olds, implying that kindergarten children will imitate more.

By depending on child 2's imitation and non-imitation behavior and the confirmation and denial of the experimenter, we can divide such responses into four events (see Table 2).

Child 2's Responses	**Outcomes**	**Events**	**Probabilities**
R_1 (imitation)	O_1 (confirmation)	E_1	px
R_1 (imitation)	O_2 (denial)	E_2	$(1-p)x$
R_2 (non-imitation)	O_1 (confirmation)	E_3	$p(1-x)$
R_2 (non-imitation)	O_2 (denial)	E_4	$(1-p)(1-x)$

TABLE 2. Possible responses in the Shwartz experiment [22]

Here, by following the work discussed in [24, 26], we propose the following general functional equation to discuss the experimental work of Shwartz [22]

$$\begin{aligned} U(x) &= pxU(h_1(x)) + (1-p)xU(h_2(x)) \\ &\quad + p(1-x)U(h_3(x)) + (1-p)(1-x)U(h_4(x)) \end{aligned} \tag{2.7}$$

for all $x \in [0,1]$, where $p \in [0,1]$, $U : [0,1] \to \mathbb{R}$ is an unknown function such that $U(0) = 0$ and $h_1, h_2, h_3, h_4 : [0,1] \to [0,1]$ are given mappings such that

$$h_3(0) = h_4(0) = 0. \tag{2.8}$$

Our aim is to find the necessary conditions for the existence and uniqueness of a solution to the proposed functional equation (2.7) with (2.8) by utilizing the Banach fixed point theorem. After that, we present two examples to show the significance of our result in this area of research. Finally, we discuss the stability of a solution to the proposed model.

3. Existence and uniqueness results

Let $Y = [0,1]$. We indicate in this paper, the class of all continuous real-valued functions $U : Y \to \mathbb{R}$ such that $U(0) = 0$ and

$$\sup_{x_1 \neq x_2} \frac{|U(x_1) - U(x_2)|}{|x_1 - x_2|} < \infty \tag{3.9}$$

by B. It is straightforward that $(B, \|\cdot\|)$ is a Banach space, where $\|\cdot\|$ is defined by

$$\|U\| = \sup_{x_1 \neq x_2} \frac{|U(x_1) - U(x_2)|}{|x_1 - x_2|} \tag{3.10}$$

for all $U \in B$ (for the detail, see [25]).

Theorem 3.2. *Consider the functional equation (2.7) with (2.8). Suppose that $h_1, h_2, h_3, h_4 : Y \to Y$ are Banach contraction mappings with contractive coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, respectively, satisfying*

$$2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) < 1$$

and

$$h_1(0) = h_2(0) = 0.$$

Then (2.7) has a unique solution. Furthermore, the sequence $\{U_n\}$ in B defined for each $x \in Y$ by

$$\begin{aligned} U_n(x) &= pxU_{n-1}(h_1(x)) + (1-p)xU_{n-1}(h_2(x)) \\ &\quad + p(1-x)U_{n-1}(h_3(x)) + (1-p)(1-x)U_{n-1}(h_4(x)) \end{aligned} \tag{3.11}$$

for all $n \in \mathbb{N}$, where U_0 is given in B, converges to a unique solution of (2.7).

Proof. Let $d : B \times B \to \mathbb{R}$ be a metric induced by $\|\cdot\|$ on B. Thus (B, d) is a complete metric space. We deal with the operator G from B which is defined for each $U \in B$ by

$$\begin{aligned} (GU)(x) &= pxU(h_1(x)) + (1-p)xU(h_2(x)) \\ &\quad + p(1-x)U(h_3(x)) + (1-p)(1-x)U(h_4(x)) \end{aligned}$$

for all $x \in Y$. For each $U \in B$, we obtain

$$(GU)(0) = pU(h_3(0)) + (1-p)U(h_4(0)) = 0.$$

Also, G is continuous and $\|GU\| < \infty$ for all $U \in B$. Therefore, G is a self operator on B. Furthermore, it is clear that the solution of (2.7) is equivalent to the fixed point of G. Since G is a linear mapping, for $U_1, U_2 \in B$, we obtain

$$\|GU_1 - GU_2\| = \|G(U_1 - U_2)\|.$$

For each $x_1, x_2 \in Y$ with $x_1 \neq x_2$, we get

$$\begin{aligned} &\frac{G(U_1 - U_2)(x_1) - G(U_1 - U_2)(x_2)}{x_1 - x_2} \\ &= \frac{1}{x_1 - x_2}\left[px_1(U_1 - U_2)(h_1(x_1)) + (1-p)x_1(U_1 - U_2)(h_2(x_1))\right. \\ &\quad + p(1-x_1)(U_1 - U_2)(h_3(x_1)) + (1-p)(1-x_1)(U_1 - U_2)(h_4(x_1)) \\ &\quad - px_2(U_1 - U_2)(h_1(x_2)) - (1-p)x_2(U_1 - U_2)(h_2(x_2)) \\ &\quad \left. - p(1-x_2)(U_1 - U_2)(h_3(x_2)) - (1-p)(1-x_2)(U_1 - U_2)(h_4(x_2))\right] \\ &= \frac{1}{x_1 - x_2}\left[px_1(U_1 - U_2)(h_1(x_1)) - px_1(U_1 - U_2)(h_1(x_2))\right. \\ &\quad + (1-p)x_1(U_1 - U_2)(h_2(x_1)) - (1-p)x_1(U_1 - U_2)(h_2(x_2)) \\ &\quad + p(1-x_1)(U_1 - U_2)(h_3(x_1)) - p(1-x_1)(U_1 - U_2)(h_3(x_2)) \end{aligned}$$

$$+(1-p)(1-x_1)(U_1-U_2)(h_4(x_1)) - (1-p)(1-x_1)(U_1-U_2)(h_4(x_2))$$
$$+px_1(U_1-U_2)(h_1(x_2)) - px_2(U_1-U_2)(h_1(x_2))$$
$$+(1-p)x_1(U_1-U_2)(h_2(x_2)) - (1-p)x_2(U_1-U_2)(h_2(x_2))$$
$$+p(1-x_1)(U_1-U_2)(h_3(x_2)) - p(1-x_2)(U_1-U_2)(h_3(x_2))$$
$$+(1-p)(1-x_1)(U_1-U_2)(h_4(x_2)) - (1-p)(1-x_2)(U_1-U_2)(h_4(x_2))\big].$$

Then we have

$$\begin{aligned}
&\left|\frac{G(U_1-U_2)(x_1)-G(U_1-U_2)(x_2)}{x_1-x_2}\right| \\
&= \Big|\frac{1}{x_1-x_2}\left[px_1(U_1-U_2)(h_1(x_1)) - px_1(U_1-U_2)(h_1(x_2))\right] \\
&\quad +\frac{1}{x_1-x_2}\left[(1-p)x_1(U_1-U_2)(h_2(x_1)) - (1-p)x_1(U_1-U_2)(h_2(x_2))\right] \\
&\quad +\frac{1}{x_1-x_2}\left[p(1-x_1)(U_1-U_2)(h_3(x_1)) - p(1-x_1)(U_1-U_2)(h_3(x_2))\right] \\
&\quad +\frac{1}{x_1-x_2}\big[(1-p)(1-x_1)(U_1-U_2)(h_4(x_1)) \\
&\quad -(1-p)(1-x_1)(U_1-U_2)(h_4(x_2))\big] \\
&\quad +\frac{1}{x_1-x_2}\left[px_1(U_1-U_2)(h_1(x_2)) - px_2(U_1-U_2)(h_1(x_2))\right] \\
&\quad +\frac{1}{x_1-x_2}\left[(1-p)x_1(U_1-U_2)(h_2(x_2)) - (1-p)x_2(U_1-U_2)(h_2(x_2))\right] \\
&\quad +\frac{1}{x_1-x_2}\left[p(1-x_1)(U_1-U_2)(h_3(x_2)) - p(1-x_2)(U_1-U_2)(h_3(x_2))\right] \\
&\quad +\frac{1}{x_1-x_2}\big[(1-p)(1-x_1)(U_1-U_2)(h_4(x_2)) \\
&\quad -(1-p)(1-x_2)(U_1-U_2)(h_4(x_2))\big]\Big| \\
&\le \alpha_1 px_1\|U_1-U_2\| + \alpha_2(1-p)x_1\|U_1-U_2\| + \alpha_3 p(1-x_1)\|U_1-U_2\| \\
&\quad +\alpha_4(1-p)(1-x_1)\|U_1-U_2\| \\
&\quad +|p(U_1-U_2)(h_1(x_2)) - p(U_1-U_2)(h_1(0))| \\
&\quad +|(1-p)(U_1-U_2)(h_2(x_2)) - (1-p)(U_1-U_2)(h_2(0))| \\
&\quad +|p(U_1-U_2)(h_3(x_2)) - p(U_1-U_2)(h_3(0))| \\
&\quad +|(1-p)(U_1-U_2)(h_4(x_2)) - (1-p)(U_1-U_2)(h_4(0))| \\
&\le \alpha_1 px_1\|U_1-U_2\| + \alpha_2(1-p)x_1\|U_1-U_2\| + \alpha_3 p(1-x_1)\|U_1-U_2\| \\
&\quad +\alpha_4(1-p)(1-x_1)\|U_1-U_2\| + \alpha_1 px_2\|U_1-U_2\| + \alpha_2(1-p)x_2\|U_1-U_2\| \\
&\quad +\alpha_3 px_2\|U_1-U_2\| + \alpha_4(1-p)x_2\|U_1-U_2\| \\
&\le 2(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\|U_1-U_2\|.
\end{aligned}$$

This gives that

$$\begin{aligned}
d(GU_1, GU_2) &= \|GU_1 - GU_2\| \\
&\le 2(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\|U_1-U_2\| \\
&= 2(\alpha_1+\alpha_2+\alpha_3+\alpha_4)d(U_1,U_2).
\end{aligned}$$

As $0 \le 2(\alpha_1+\alpha_2+\alpha_3+\alpha_4) < 1$, by Theorem 1.1, we get the conclusion of this theorem. □

From Theorem 3.2, we have the following result.

Corollary 3.1. *Consider the functional equation (2.7) with (2.8). Suppose that* $h_1, h_2, h_3, h_4 : Y \to Y$ *are Banach contraction mappings with contractive coefficients* $\alpha_1, \alpha_2, \alpha_3, \alpha_4$*, respectively, satisfying* $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4$ *with* $8\alpha_4 < 1$*. Also, assume that*

$$h_1(0) = h_2(0) = 0.$$

Then (2.7) has a unique solution. Furthermore, the sequence $\{U_n\}$ *in* B *defined for each* $x \in Y$ *by*

$$\begin{aligned} U_n(x) &= pxU_{n-1}(h_1(x)) + (1-p)xU_{n-1}(h_2(x)) \\ &\quad +p(1-x)U_{n-1}(h_3(x)) + (1-p)(1-x)U_{n-1}(h_4(x)) \end{aligned} \tag{3.12}$$

for all $n \in \mathbb{N}$*, where* U_0 *is given in* B*, converges to a unique solution of (2.7).*

Theorem 3.3. *Consider the functional equation (2.7) with (2.8). Suppose that* $h_1, h_2, h_3, h_4 : Y \to Y$ *are Banach contraction mappings with contractive coefficients* $\alpha_1, \alpha_2, \alpha_3, \alpha_4$*, respectively, satisfying* $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4$*. Also, there exist* $\alpha_5, \alpha_6 \geq 0$ *such that*

$$\begin{cases} h_1(x) \leq \alpha_5, \\ h_2(x) \leq \alpha_6 \end{cases} \tag{3.13}$$

for all $x \in Y$ *and assume that* $6\alpha_4 + \alpha_5 + \alpha_6 < 1$*. Then (2.7) has a unique solution. Furthermore, the sequence* $\{U_n\}$ *in* B *defined for each* $x \in Y$ *by*

$$\begin{aligned} U_n(x) &= pxU_{n-1}(h_1(x)) + (1-p)xU_{n-1}(h_2(x)) \\ &\quad +p(1-x)U_{n-1}(h_3(x)) + (1-p)(1-x)U_{n-1}(h_4(x)), \end{aligned} \tag{3.14}$$

for all $n \in \mathbb{N}$*, where* U_0 *is given in* B*, converges to a unique solution of (2.7).*

Proof. The line of the proof of this theorem is the same as Theorem 3.2. Here, we highlight those parts which are different from the previous theorem. For each $x_1, x_2 \in Y$ with $x_1 \neq x_2$, we get

$$\begin{aligned} &\left| \frac{G(U_1 - U_2)(x_1) - G(U_1 - U_2)(x_2)}{x_1 - x_2} \right| \\ &\leq \alpha_1 p x_1 \|U_1 - U_2\| + \alpha_2(1-p)x_1 \|U_1 - U_2\| \\ &\quad +\alpha_3 p(1-x_1) \|U_1 - U_2\| + \alpha_4(1-p)(1-x_1) \|U_1 - U_2\| \\ &\quad + |p(U_1 - U_2)(h_1(x_2)) - p(U_1 - U_2)(0)| \\ &\quad + |(1-p)(U_1 - U_2)(h_2(x_2)) - (1-p)(U_1 - U_2)(0)| \\ &\quad + |p(U_1 - U_2)(h_3(x_2)) - p(U_1 - U_2)(h_3(0))| \\ &\quad + |(1-p)(U_1 - U_2)(h_4(x_2)) - (1-p)(U_1 - U_2)(h_4(0))| \\ &\leq \alpha_1 p x_1 \|U_1 - U_2\| + \alpha_2(1-p)x_1 \|U_1 - U_2\| + \alpha_3 p(1-x_1) \|U_1 - U_2\| \\ &\quad +\alpha_4(1-p)(1-x_1) \|U_1 - U_2\| + \alpha_5 p \|U_1 - U_2\| \\ &\quad +\alpha_6(1-p) \|U_1 - U_2\| + \alpha_3 p x_2 \|U_1 - U_2\| + \alpha_4(1-p)x_2 \|U_1 - U_2\| \\ &\leq (6\alpha_4 + \alpha_5 + \alpha_6) \|U_1 - U_2\| . \end{aligned}$$

This gives that

$$d(GU_1, GU_2) \leq (6\alpha_4 + \alpha_5 + \alpha_6) d(U_1, U_2).$$

As $0 \leq (6\alpha_4 + \alpha_5 + \alpha_6) < 1$, by Theorem 1.1, we get the conclusion of this theorem. □

From Theorem 3.3, we get the following corollary.

Corollary 3.2. *Consider the functional equation (2.7). Suppose that* $h_1, h_2, h_3, h_4 : Y \to Y$ *are contraction mappings with contractive coefficients* $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ *respectively, satisfying* $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4$*. Also, there exists* $\alpha \geq 0$ *such that*

$$\begin{cases} h_1(x) \leq \alpha, \\ h_2(x) \leq \alpha \end{cases} \tag{3.15}$$

for all $x \in Y$ *and assume that* $6\alpha_4 + 2\alpha < 1$*. Then (2.7) has a unique solution. Furthermore, the sequence* $\{U_n\}$ *in* B *defined for each* $x \in Y$ *by*

$$\begin{aligned} U_n(x) &= pxU_{n-1}(h_1(x)) + (1-p)xU_{n-1}(h_2(x)) \\ &\quad + p(1-x)U_{n-1}(h_3(x)) + (1-p)(1-x)U_{n-1}(h_4(x)), \end{aligned} \tag{3.16}$$

for all $n \in \mathbb{N}$*, where* U_0 *is given in* B*, converges to a unique solution of (2.7).*

Remark 3.1. Our proposed model (2.7) with (2.8) is a generalization of many mathematical models existing in the particular research.

(1) If we put $p = 0$ (or $p = 1$) and define $h_2, h_4 : Y \to Y$ by

$$h_2(x) = \alpha_1 x + 1 - \alpha_1 \text{ and } h_4(x) = \alpha_2 x,$$

for all $x \in Y$, where $0 < \alpha_1 \leq \alpha_2 < 1$ (or define $h_1, h_3 : Y \to Y$ as the same rule, respectively), then our functional equation (2.7) reduces to the functional equation (1.1).

(2) If we put $p = 0$ (or $p = 1$) and define $h_2, h_4 : Y \to Y$ (or $h_1, h_3 : Y \to Y$) as Banach contraction mappings with $h_4(0) = 0$ (or $h_3(0) = 0$), then our functional equation (2.7) reduces to (1.4), which is the generalization functional equation in [27, 5, 23].

To support our argument, we now present the following examples.

Example 3.1. Consider the following functional equation

$$U(x) = pxU\left(\frac{x}{8}\right) + (1-p)xU\left(\frac{x}{9}\right) + p(1-x)U\left(\frac{x}{7}\right) + (1-p)(1-x)U\left(\frac{x}{11}\right) \tag{3.17}$$

for all $x \in Y$, where $U : Y \to \mathbb{R}$ is an unknown function. If we set the mappings $h_1, h_2, h_3, h_4 : Y \to Y$ by

$$h_1(x) = \frac{x}{8},\ h_2(x) = \frac{x}{9},\ h_3(x) = \frac{x}{7} \text{ and } h_4(x) = \frac{x}{11}$$

for all $x \in Y$, then the functional equation (2.7) reduces to the functional equation (3.17).

Here, h_1, h_2, h_3, h_4 are Banach contraction mappings with contractive coefficients $\alpha_1 = \frac{1}{8}$, $\alpha_2 = \frac{1}{9}$, $\alpha_3 = \frac{1}{7}$ and $\alpha_4 = \frac{1}{11}$, respectively, and thus

$$2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = \frac{2605}{2772} < 1.$$

Also,

$$h_1(0) = h_2(0) = h_3(0) = h_4(0) = 0.$$

Now, all assumptions of Theorem 3.2 hold. Thus, the functional equation (3.17) has a unique solution.

Moreover, if we choose an initial approximation $U_0(x) = x$ for all $x \in Y$, then the following iteration converges to a unique solution of (3.17):

$$\begin{aligned} U_1(x) &= \frac{1}{5544}[-211px^2 + 112x^2 + 288px + 504x], \\ U_2(x) &= \frac{1}{170400029184}\left[\begin{array}{c} 57488005p^2x^3 - 56982800px^3 + 14049280x^3 \\ -415653120p^2x^2 - 422537472px^2 + 341397504x^2 \\ +459841536p^2x + 1609445376px + 1408264704x \end{array}\right], \\ &\vdots \\ U_n(x) &= pxU_{n-1}\left(\frac{x}{8}\right) + (1-p)xU_{n-1}\left(\frac{x}{9}\right) + p(1-x)U_{n-1}\left(\frac{x}{7}\right) \\ &\quad +(1-p)(1-x)U_{n-1}\left(\frac{x}{11}\right) \end{aligned}$$

for all $n \in \mathbb{N}$.

Example 3.2. Consider the following functional equation

$$U(x) = pxU\left(\frac{x+1}{23}\right) + (1-p)xU\left(\frac{x+2}{21}\right) + p(1-x)U\left(\frac{x}{19}\right) + (1-p)(1-x)U\left(\frac{x}{17}\right) \tag{3.18}$$

for all $x \in Y$, where $U : Y \to \mathbb{R}$ is an unknown function. If we set the mappings $h_1, h_2, h_3, h_4 : Y \to Y$ by

$$h_1(x) = \frac{x+1}{23},\ h_2(x) = \frac{x+2}{21},\ h_3(x) = \frac{x}{19} \text{ and } h_4(x) = \frac{x}{17}$$

for all $x \in Y$, then the functional equation (2.7) reduces to the functional equation (3.18).

Here, h_1, h_2, h_3, h_4 are Banach contraction mappings with contractive coefficients $\alpha_1 = \frac{1}{23}$, $\alpha_2 = \frac{1}{21}$, $\alpha_3 = \frac{1}{19}$ and $\alpha_4 = \frac{1}{17}$, respectively. Also,

$$|h_1(x)| \leq \frac{2}{23} =: \alpha_1 \text{ and } |h_2(x)| \leq \frac{1}{7} =: \alpha_1 \text{ for all } x \in Y$$

and

$$h_3(0) = h_4(0) = 0.$$

Thus,

$$6\alpha_4 + \alpha_5 + \alpha_6 = \frac{6}{17} + \frac{2}{23} + \frac{1}{7} = \frac{1595}{2737} < 1.$$

Now, all hypotheses of Theorem 3.3 hold. Thus, the functional equation (3.18) has a unique solution.

Moreover, if we choose an initial approximation $U_0(x) = x$ for all $x \in Y$, then the following iteration converges to a unique solution of (3.18):

$$\begin{aligned}
U_1(x) &= \frac{1}{156009}[320px^2 - 1748x^2 - 9041px + 24035x], \\
U_2(x) &= \frac{1}{36395183601}\left[\begin{array}{c} -28160p^2x^3 + 323104px^3 - 924692x^3 \\ +8338726p^2x^2 - 120820127px^2 + 263306047x^2 \\ +108634075p^2x - 487490543px + 530310862x \end{array}\right] \\
&\quad + \frac{1}{16276262961}\left[\begin{array}{c} 23040p^2x^3 - 241376px^3 + 631028x^3 \\ -5863526p^2x^2 + 71252603px^2 - 148133823x^2 \\ +5840486p^2x - 71011227px + 147502795x \end{array}\right], \\
&\vdots \\
U_n(x) &= pxU_{n-1}\left(\frac{x+1}{23}\right) + (1-p)xU_{n-1}\left(\frac{x+2}{21}\right) + p(1-x)U_{n-1}\left(\frac{x}{19}\right) \\
&\quad +(1-p)(1-x)U_{n-1}\left(\frac{x}{17}\right)
\end{aligned}$$

for all $n \in \mathbb{N}$.

4. Stability analysis

The consistency of solutions is of considerable significance in the theory of mathematical modeling. Therefore, it is essential to discuss the stability of the proposed mathematical model (2.7) here. For instance of the Hyers-Ulam and Hyers-Ulam-Rassias stability for various types of equations, we refer [23, 1, 3, 11, 12, 13, 17, 23, 20, 31]. Now, we state the following result related to the Hyers-Ulam-Rassias type stability of a solution to the proposed model (2.7).

Theorem 4.4. *Under the assumption of Theorem 3.2, the fixed point equation of G, where $G : B \to B$ is defined for each $U \in B$ by*

$$\begin{aligned}
(GU)(x) &= pxU(h_1(x)) + (1-p)xU(h_2(x)) \\
&\quad +p(1-x)U(h_3(x)) + (1-p)(1-x)U(h_4(x))
\end{aligned} \tag{4.19}$$

for all $x \in Y$, has the Hyers-Ulam-Rassias stability, that is, for a fixed function $\varphi : B \to [0, \infty)$, we have that for every $U \in B$ with $d(GU, U) \le \varphi(U)$, there exists a unique $U' \in B$ such that $GU' = U'$ and $d(U, U') \le C\varphi(U)$ for some $C > 0$.

Proof. Let $U \in B$ such that $d(GU, U) \le \varphi(U)$. From Theorem 3.2, there exists a unique $U' \in B$ such that $GU' = U'$. Then we have

$$\begin{aligned}
d(U, U') &\le d(U, GU) + d(GU, U') \\
&\le \varphi(U) + d(GU, GU') \\
&\le \varphi(U) + 2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)d(U, U')
\end{aligned}$$

and so

$$d(U, U_0) \le C\varphi(U),$$

where $C := \dfrac{1}{1 - 2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)}$. □

From the above analysis, we obtain the following result related to the Hyers-Ulam stability.

Corollary 4.3. *Under the assumption of Theorem 3.2, the fixed point equation of G, where $G : B \to B$ is defined for each $U \in B$ by*

$$\begin{aligned} (GU)(x) &= pxU(h_1(x)) + (1-p)xU(h_2(x)) \\ &\quad + p(1-x)U(h_3(x)) + (1-p)(1-x)U(h_4(x)), \end{aligned} \tag{4.20}$$

for all $x \in Y$, has Hyers-Ulam stability, that is, for $\tau > 0$ (a fixed number), we have that for every $U \in B$ with $d(GU, U) \leq \tau$, there exists a unique $U' \in B$ such that $GU' = U'$ and $d(U, U') \leq C\tau$ for some $C > 0$.

Using the same line in the proof of Theorem 4.4, we can use Theorem 3.3 to obtain the Hyers-Ulam-Rassias stability of the fixed point equation of G, where G is defined as (4.19). Based on the large of our proposed model (2.7), the results in this section cover several stability results of the existing model in the literature including results in [23].

5. Conclusion and open problems

Bush and Wilson's model [7] and Shwartz's imitation model [22] play a vital role in mathematical psychology and learning theory. The authors of [7] developed a concept of "reward" based on animals choosing the right side in a two-choice situation and classified such circumstances into four categories: left-reward, right-reward, right non-reward, and left non-reward. On the other hand, Shwartz used such aspects to observe children's imitative and non-imitative behavior in specific circumstances (see [22]). In this paper, by using the idea presented in [7, 22], we proposed a general functional equation that can be used to discuss the psychological behavior of animals and humans in a two-choice situation. The Banach fixed point theorem has been used to examine the existence and uniqueness of a generalized functional equation's solution. Furthermore, we discussed the stability of a solution to the proposed model. In the end, we present the following questions as open problems for those interested in this type of research.

Problem 1: Is the condition $h_3(0) = 0 = h_4(0)$ necessary for the functional equation (2.7)?

Problem 2: Are the conditions mentioned in Theorem 3.3 sufficient to prove the existence and uniqueness of a solution to the functional equation (2.7)?

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