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A self-adaptive forward-backward-forward algorithm for solving split variational inequalities

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ABSTRACT. In this paper, we consider an iterative approximation problem of split variational inequalities in Hilbert spaces. In order to solve this split problem, we construct an iterative algorithm which combines a forward-backward-forward method and a self-adaptive rule to update the step-sizes. We prove that the constructed algorithm converges strongly to a solution of the split variational inequalities under some mild assumptions.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let $\psi : C \to H$ be a nonlinear operator. In this paper, we focus on the following variational inequality of finding a point $u^\dagger \in C$ satisfying

$$\langle \psi(u^\dagger), x - u^\dagger\rangle \geq 0,\ \forall x \in C. \tag{1.1}$$

Here, its solution is denoted by $\mathrm{Sol}(C, \psi)$.

Let $\mathrm{Sol}_d(C, \psi)$ be the solution set of the dual variational inequality of (1.1), that is,

$$\mathrm{Sol}_d(C, \psi) := \{u^\dagger \in C | \langle \psi(x), x - u^\dagger\rangle \geq 0, \forall x \in C\}. \tag{1.2}$$

It is obviously that $\mathrm{Sol}(C, \psi)$ is closed convex. If C is convex and ψ is continuous, then $\mathrm{Sol}_d(C, \psi) \subset \mathrm{Sol}(C, \psi)$.

Variational inequalities were introduced by Stampacchia ([26]) in the context of calculus of variations and optimal control theory for the study of partial differential equations with applications principally drawn from mechanics. Variational inequalities in the finite dimensional case have taken their own tangent to become an interesting research field since the late 1970s ([15]). This fact unveiled this methodology for the study of many problems such as partial differential equations, optimization problems ([11, 48]), fixed point problems ([28, 30, 32, 38, 39]), optimal control problems ([3, 12]), mathematical programming problems ([9]), management problems, equilibrium problems ([51]), network problems, and so on. A tremendous amount of work has gone into variational inequality in different directions, namely, existence theories, iterative methods and applications, see, e.g., [1, 4, 29, 36, 37, 47].

An operator $\psi : C \to H$ is called

(i) monotone if

$$\langle \psi(x) - \psi(\hat{x}), x - \hat{x}\rangle \geq 0,\ \forall x, \hat{x} \in C.$$

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(ii) τ-strongly monotone if there exists a constant $\tau > 0$ satisfying

$$\langle \psi(x) - \psi(\hat{x}), x - \hat{x} \rangle \geq \tau \|x - \hat{x}\|^2, \ \forall x, \hat{x} \in C.$$

(iii) τ-inverse-strongly monotone if there exists a constant $\tau > 0$ satisfying

$$\langle \psi(x) - \psi(\hat{x}), x - \hat{x} \rangle \geq \tau \|\psi(x) - \psi(\hat{x})\|^2, \ \forall x, \hat{x} \in C.$$

(iv) pseudomonotone if

$$\langle \psi(\hat{x}), x - \hat{x} \rangle \geq 0 \Rightarrow \langle \psi(x), x - \hat{x} \rangle \geq 0, \ \forall x, \hat{x} \in C.$$

(v) L-Lipschitz continuous if there exists a constant $L > 0$ satisfying

$$\|\psi(x) - \psi(\hat{x})\| \leq L\|x - \hat{x}\|, \ \forall x, \hat{x} \in C.$$

If $L < 1$, then ψ is called L-contraction. If $L = 1$, then ψ is called nonexpansive.

A basic algorithm for solving (1.1) is the projection algorithm ([2, 25]) defined by the following manner

$$x_{n+1} = \text{proj}_C[x_n - \lambda\psi(x_n)], \ n \geq 0,$$

where $\text{proj}_C : H \to C$ is the metric projection, the operator ψ is strongly monotone (or inverse-strongly monotone) ([12]) and L-Lipschitz continuous and $\lambda \in (0, \frac{1}{L})$ is the step-size.

In order to relax the operator ψ to more general monotone operator, Korpelevich [18] introduced an extragradient algorithm defined by

$$\begin{cases} y_n = \text{proj}_C[x_n - \lambda\psi(x_n)], \\ x_{n+1} = \text{proj}_C[x_n - \lambda\psi(y_n)], \ n \geq 0. \end{cases} \tag{1.3}$$

Extragradient algorithm has attracted much attention by many scholars who modified and extended (1.3) in several forms, see [7, 31, 33, 34, 30, 44, 50]. Especially, Tseng [32] suggested the following so-called Tseng's algorithm

$$\begin{cases} y_n = \text{proj}_C[x_n - \lambda\psi(x_n)], \\ x_{n+1} = y_n + \lambda[\psi(x_n) - \psi(y_n)], \ n \geq 0. \end{cases} \tag{1.4}$$

Very recently, Bot, Csetnek and Vuong [3] approach the solution of $\text{Sol}(C, \psi)$ from a continuous perspective by means of trajectories generated by the dynamical system of forward-backward-forward type ([3]) and propose the following algorithm

$$\begin{cases} y_n = \text{proj}_C[x_n - \lambda\psi(x_n)], \\ x_{n+1} = (1 - \sigma_n)x_n + \sigma_n[y_n + \lambda(\psi(x_n) - \psi(y_n))], \ n \geq 0, \end{cases} \tag{1.5}$$

where ψ is a pseudomonotone operator. Several related results, please refer to [5, 10, 46].

Recall that an operator $\psi : C \to H$ is said to be quasimonotone if

$$\langle \psi(y), x - y \rangle > 0 \text{ implies } \langle \psi(x), x - y \rangle \geq 0, \ \forall x, y \in C.$$

It is easy to see that if ψ is pseudomonotone, then ψ must be quasimonotone. But the reverse assertion is not true in general, see the following example.

Example 1.1. (see [23]) The function $\psi : \mathbb{R} \to \mathbb{R}$ defined by $\psi(x) = x^2$ is quasimonotone on $\mathbb{R}$, but not pseudomonotone on $\mathbb{R}$.

Recently, Yin and Hussain [45] approach the solution of quasimonotone variational inequalities by using forward-backward-forward method (1.5). On the other hand, to ensure the convergence of the sequence $\{x_n\}$, a common condition $\mathrm{Sol}(C,\psi) \subset \mathrm{Sol}_d(C,\psi)$ is needed, that is,

$$\langle \psi(x), x-u\rangle \geq 0,\ \forall u \in \mathrm{Sol}(C,\psi) \text{ and } x \in C,$$

which is a direct consequence of the pseudomonotonicity of ψ. But this conclusion (that is, $\mathrm{Sol}(C,\psi) \subset \mathrm{Sol}_d(C,\psi)$) is false, if ψ is quasimonotone.

Let $\varphi : C \to H$ and $\phi : C \to C$ be two nonlinear operators. Recall that the generalized variational inequality ([22, 42, 43]) is to find a point $x^\dagger \in C$ such that

$$\langle \varphi(x^\dagger), \phi(x) - \phi(x^\dagger)\rangle \geq 0,\ \forall x \in C. \tag{1.6}$$

The solution set of (1.6) is denoted by $\mathrm{Sol}(C,\varphi,\phi)$.

If $\phi \equiv I_C$, then the generalized variational inequality (1.6) reduces to the variational inequality (1.1).

The main purpose of this paper is to consider the following split variational inequality problem ([16, 17]) of finding a point $x^\dagger$ such that

$$x^\dagger \in \mathrm{Sol}(C,\varphi,\phi) \text{ and } \phi(x^\dagger) \in \mathrm{Sol}_d(C,\psi). \tag{1.7}$$

The prototype of split variational inequality considered in [8, 20, 24, 41] is to seek a point $x^\dagger$ such that $x^\dagger \in \mathrm{Sol}(C,\varphi)$ and $A(x^\dagger) \in \mathrm{Sol}(Q,\psi)$, where $A : C \to Q$ is a bounded linear operator. The reason why we are interested in the split variational inequality is that it is an extension of the split feasibility problem ([6]) arising from image denoising, signal processing and image reconstruction, see, [13, 14, 19, 27, 28, 38, 39, 40] for more details.

Motivated by the work in this field, in this paper, we investigate the split variational inequality (1.7) in which the involved operators φ is inverse strongly ϕ-monotone and ψ is quasimonotone. For solving this split variational inequality, we propose an iterative algorithm which combines the forward-backward-forward algorithm (1.5) and a self-adaptive rule to update the step-sizes. We show that the constructed algorithm converges strongly to a solution of the split variational inequalities under some mild conditions.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Let u be any fixed point in H. Then, there exists a unique point in C, denoted by $\mathrm{proj}_C[u]$ such that

$$\|u - \mathrm{proj}_C[u]\| \leq \|x-u\|,\ \forall x \in C,$$

where proj_C is orthogonal projection from H onto C. It is well known that proj_C satisfies the following two inequalities

$$\|\mathrm{proj}_C[x] - \mathrm{proj}_C[y]\|^2 \leq \langle \mathrm{proj}_C[x] - \mathrm{proj}_C[y], x-y\rangle,\ \forall x,y \in H. \tag{2.8}$$

and

$$\langle x - \mathrm{proj}_C[x], y - \mathrm{proj}_C[x]\rangle \leq 0,\ \forall x \in H, y \in C. \tag{2.9}$$

Let $\phi : C \to C$ and $\varphi : C \to H$ be two operators. Recall that $\varphi : C \to H$ is said to be ϑ-inverse strongly ϕ-monotone if there exists a constant $\vartheta > 0$ such that

$$\langle \varphi(x) - \varphi(y), \phi(x) - \phi(y)\rangle \geq \vartheta\|\varphi(x) - \varphi(y)\|^2,\ \forall x,y \in C.$$

Then, for all $x, y \in C$ and $\alpha > 0$, we have

$$\begin{aligned}\|(\phi(x) - \alpha\varphi(x)) - (\phi(y) - \alpha\varphi(y))\|^2 \leq{}& \alpha(\alpha - 2\vartheta)\|\varphi(x) - \varphi(y)\|^2 \\ &+ \|\phi(x) - \phi(y)\|^2.\end{aligned} \tag{2.10}$$

An operator $B : H \to 2^H$ is said to be monotone if and only if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in \text{dom}(B)$, $u \in B(x)$, and $v \in B(y)$. A monotone operator B on H is said to be maximal if and only if its graph is not strictly contained in the graph of any other monotone operator on H.

In what follows, we use " $\rightharpoonup$ " and " $\to$ " to denote weak convergence and strong convergence, respectively. Recall that an operator $\psi : C \to H$ is said to be sequentially weakly continuous if for given sequence $\{x_n\}$: $x_n \rightharpoonup u$ implies that $\psi(x_n) \rightharpoonup \psi(u)$.

Lemma 2.1. *Let H be a real Hilbert space. Then,*

$$\|rx + (1-r)x^\dagger\|^2 = r\|x\|^2 + (1-r)\|x^\dagger\|^2 - r(1-r)\|x - x^\dagger\|^2, \tag{2.11}$$

for any $x, x^\dagger \in H$ and $r \in \mathbb{R}$.

Lemma 2.2 ([35]). *Let $\{a_n\} \subset [0, +\infty)$, $\{b_n\} \subset (0,1)$ and $\{c_n\} \subset \mathbb{R}$ be three real numbers sequences. Suppose that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ satisfy the following assumptions*

(i) $a_{n+1} \leq (1-b_n)a_n + c_n, \ \forall n \geq 1$;
(ii) $\sum_{n=1}^\infty b_n = \infty$;
(iii) $\limsup\limits_{n\to\infty} c_n/b_n \leq 0$ *or* $\sum_{n=1}^\infty |c_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 ([21]). *Let $\{\Upsilon_n\}$ be a sequence of real numbers. Assume $\{\Upsilon_n\}$ does not decrease at infinity, that is, there exists at least a subsequence $\{\Upsilon_{n_k}\}$ of $\{\Upsilon_n\}$ such that $\Upsilon_{n_k} \leq \Upsilon_{n_k+1}$ for all $k \geq 0$. For every $n \geq N_0$, define an integer sequence $\{I(n)\}$ as*

$$I(n) = \max\{i \leq n : \Upsilon_{n_i} < \Upsilon_{n_i+1}\}.$$

Then $I(n) \to \infty$ as $n \to \infty$ and for all $n \geq N_0$,

$$\max\{\Upsilon_{I(n)}, \Upsilon_n\} \leq \Upsilon_{I(n)+1}.$$

3. Main results

In this section, we first propose an iterative algorithm for solving problem (1.7). Consequently, we demonstrate its convergence analysis. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\mathcal{C} : C \to H$ be a θ-contractive operator. Let $\phi : C \to C$ be a weakly continuous and τ-strongly monotone operator with $\text{Rang}(\phi) = C$. Let $\varphi : C \to H$ be a ϑ-inverse strongly ϕ-monotone operator. Let the operator ψ be quasi-monotone on H, L-Lipschitz continuous on C.

In the sequel, we suppose that ψ satisfies the following condition (p): Let $\{z_n\}$ be a known sequence in H. If $z_n \rightharpoonup q^\dagger \in H$ and $\liminf_{k\to\infty} \|\psi(z_n)\| = 0$, then we have $\psi(q^\dagger) = 0$.

Remark 3.1. It is clearly that if ψ is sequentially weakly continuous, then ψ satisfies the hypothesis (p).

Let $\{\vartheta_n\}$, $\{\sigma_n\}$ and $\{\rho_n\}$ be three real numbers sequences in $[0,1]$ and $\{\alpha_n\}$ be a real numbers sequence in $(0,\infty)$. Let $\xi \in (0,\infty)$ and $\eta \in (0,1)$ be two constants. In what follows, we suppose that $\Xi := \{x | x \in \text{Sol}(C, \varphi, \phi) \text{ and } \phi(x) \in \text{Sol}_d(C, \psi)\} \neq \emptyset$.

Next, we present an iterative algorithm for solving problem (1.7).

Algorithm 3.1. Let x_0 be an initial point in C and ς_0 be a positive constant. Set $n = 0$.

Step 1. Let x_n be constructed. Then, compute

$$\hat{y}_n = \text{proj}_C[\vartheta_n \xi \mathcal{C}(x_n) + (1-\vartheta_n)(\phi(x_n) - \alpha_n \varphi(x_n))]. \tag{3.12}$$

Step 2. Let ς_n be given. Compute

$$y_n = \text{proj}_C[\hat{y}_n - \varsigma_n \psi(\hat{y}_n)], \tag{3.13}$$

and

$$\hat{x}_n = (1-\sigma_n)\hat{y}_n + \sigma_n y_n + \sigma_n \varsigma_n[\psi(\hat{y}_n) - \psi(y_n)]. \tag{3.14}$$

Step 3. Compute

$$\phi(x_{n+1}) = (1-\rho_n)\phi(x_n) + \rho_n \hat{x}_n. \tag{3.15}$$

Step 4. Update

$$\varsigma_{n+1} = \begin{cases} \min\left\{\varsigma_n, \frac{\eta\|\hat{y}_n - y_n\|}{\|\psi(\hat{y}_n)-\psi(y_n)\|}\right\}, & \text{if } \psi(\hat{y}_n) \neq \psi(y_n), \\ \varsigma_n, & else. \end{cases} \tag{3.16}$$

Write $n := n+1$ and return to step 1.

Remark 3.2. (i) By the definition (3.16) of $\{\varsigma_n\}$, $\varsigma_{n+1} \leq \varsigma_n$. Thanks to the L-Lipschitz continuity of ψ, we have $\frac{\eta\|y_n - \hat{y}_n\|}{\|\psi(y_n)-\psi(\hat{y}_n)\|} \geq \frac{\eta}{L}$ which implies that $\varsigma_n \geq \min\{\varsigma_0, \frac{\eta}{L}\}$. Thus, $\lim_{n\to\infty} \varsigma_n$ exists. (ii) If $y_n = \hat{y}_n$, then $y_n \in \mathrm{Sol}(C,\psi)$. (iii) The following variational inequality has a unique solution

$$x \in \Xi,\ \langle \xi\mathcal{C}(x) - \phi(x), \phi(y) - \phi(x)\rangle \leq 0,\ \forall y \in \mathrm{Sol}(C,\varphi,\phi) \cap \phi^{-1}(\mathrm{Sol}(C,\psi)). \tag{3.17}$$

Theorem 3.1. *Suppose that the following conditions are satisfied:*

(C1): $\lim_{n\to\infty} \vartheta_n = 0$ *and* $\sum_{n=1}^{\infty} \vartheta_n = \infty$;
(C2): $0 < \liminf_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \rho_n < 1$;
(C3): $0 < \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 1$;
(C4): $0 < \theta\xi < \tau < 2\vartheta$ *and* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 2\vartheta$.

Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to a point in Ξ which solves (3.17).

Proof. Let $\hat{z} \in \Xi$. Then, $\hat{z} \in \mathrm{Sol}(C,\varphi,\phi)$ and $\phi(\hat{z}) \in \mathrm{Sol}_d(C,\psi) \subset \mathrm{Sol}(C,\psi)$. It follows that $\phi(\hat{z}) = \mathrm{proj}_C[\phi(\hat{z}) - \alpha\varphi(\hat{z})], \forall \alpha > 0$. By (C4), we deduce that $\phi(\hat{z}) = \mathrm{proj}_C[\phi(\hat{z}) - \alpha_n\varphi(\hat{z})], \forall n \geq 0$. Utilizing (2.10), we have

$$\begin{aligned} &\|\phi(x_n) - \alpha_n\varphi(x_n) - (\phi(\hat{z}) - \alpha_n\varphi(\hat{z}))\|^2 \\ &\quad \leq \|\phi(x_n) - \phi(\hat{z})\|^2 + \alpha_n(\alpha_n - 2\vartheta)\|\varphi(x_n) - \varphi(\hat{z})\|^2 \\ &\quad \leq \|\phi(x_n) - \phi(\hat{z})\|^2, \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} &\|\phi(x_{n+1}) - \alpha_{n+1}\varphi(x_{n+1}) - (\phi(x_n) - \alpha_{n+1}\varphi(x_n))\|^2 \\ &\quad \leq \|\phi(x_{n+1}) - \phi(x_n)\|^2 + \alpha_{n+1}(\alpha_{n+1} - 2\vartheta)\|\varphi(x_{n+1}) - \varphi(x_n)\|^2. \end{aligned} \tag{3.19}$$

By (3.12) and (3.18), we obtain

$$\begin{aligned}\|\hat{y}_n-\phi(\hat{z})\| &= \|\mathrm{proj}_C[\vartheta_n\xi\mathcal{C}(x_n)+(1-\vartheta_n)(\phi(x_n)-\alpha_n\varphi(x_n))]\\ &\quad -\mathrm{proj}_C[\phi(\hat{z})-\alpha_n\varphi(\hat{z})]\|\\ &\leq \|\vartheta_n(\xi\mathcal{C}(x_n)-\phi(\hat{z})+\alpha_n\varphi(\hat{z}))+(1-\vartheta_n)\\ &\quad \times[\phi(x_n)-\alpha_n\varphi(x_n)-(\phi(\hat{z})-\alpha_n\varphi(\hat{z}))]\|\\ &\leq \vartheta_n\xi\|\mathcal{C}(x_n)-\mathcal{C}(\hat{z})\|+\vartheta_n\|\xi\mathcal{C}(\hat{z})-\phi(\hat{z})+\alpha_n\varphi(\hat{z})\|\\ &\quad +(1-\vartheta_n)\|\phi(x_n)-\alpha_n\varphi(x_n)-(\phi(\hat{z})-\alpha_n\varphi(\hat{z}))\|\\ &\leq \vartheta_n\theta\xi\|x_n-\hat{z}\|+\vartheta_n\|\xi\mathcal{C}(\hat{z})-\phi(\hat{z})+\alpha_n\varphi(\hat{z})\|\\ &\quad +(1-\vartheta_n)\|\phi(x_n)-\phi(\hat{z})\|\\ &\leq \vartheta_n\theta\xi/\tau\|\phi(x_n)-\phi(\hat{z})\|+\vartheta_n\|\xi\mathcal{C}(\hat{z})-\phi(\hat{z})+\alpha_n\varphi(\hat{z})\|\\ &\quad +(1-\vartheta_n)\|\phi(x_n)-\phi(\hat{z})\|\\ &\leq [1-(1-\theta\xi/\tau)\vartheta_n]\|\phi(x_n)-\phi(\hat{z})\|\\ &\quad +\vartheta_n(\|\xi\mathcal{C}(\hat{z})-\phi(\hat{z})\|+2\vartheta\|\varphi(\hat{z})\|).\end{aligned}\tag{3.20}$$

Combining (3.13) and (3.20), we receive

$$\begin{aligned}\|\hat{y}_n-\phi(\hat{z})\|^2 &\leq \|\vartheta_n(\xi\mathcal{C}(x_n)-\phi(\hat{z})+\alpha_n\varphi(\hat{z}))+(1-\vartheta_n)\\ &\quad \times[\phi(x_n)-\alpha_n\varphi(x_n)-(\phi(\hat{z})-\alpha_n\varphi(\hat{z}))]\|^2\\ &\leq \vartheta_n\|\xi\mathcal{C}(x_n)-\phi(\hat{z})+\alpha_n\varphi(\hat{z})\|^2+(1-\vartheta_n)\\ &\quad \times\|\phi(x_n)-\alpha_n\varphi(x_n)-(\phi(\hat{z})-\alpha_n\varphi(\hat{z}))\|^2\\ &\leq \vartheta_n\|\xi\mathcal{C}(x_n)-\phi(\hat{z})+\alpha_n\varphi(\hat{z})\|^2+(1-\vartheta_n)[\|\phi(x_n)-\phi(\hat{z})\|^2\\ &\quad +\alpha_n(\alpha_n-2\vartheta)\|\varphi(x_n)-\varphi(\hat{z})\|^2].\end{aligned}\tag{3.21}$$

Applying (2.9) to (3.13), we deduce

$$\langle y_n-\hat{y}_n+\varsigma_n\psi(\hat{y}_n),y_n-\phi(\hat{z})\rangle\leq 0.\tag{3.22}$$

Since $\phi(\hat{z})\in\mathrm{Sol_d}(C,\psi)$ and $y_n\in C$, we have

$$\langle\psi(y_n),y_n-\phi(\hat{z})\rangle\geq 0.\tag{3.23}$$

From (3.22) and (3.23), we get

$$\langle y_n-\hat{y}_n,y_n-\phi(\hat{z})\rangle+\varsigma_n\langle\psi(\hat{y}_n)-\psi(y_n),y_n-\phi(\hat{z})\rangle\leq 0.\tag{3.24}$$

It follows that

$$\frac{1}{2}(\|y_n-\hat{y}_n\|^2+\|y_n-\phi(\hat{z})\|^2-\|\hat{y}_n-\phi(\hat{z})\|^2)+\varsigma_n\langle\psi(\hat{y}_n)-\psi(y_n),y_n-\phi(\hat{z})\rangle\leq 0,$$

which yields that

$$\|y_n-\phi(\hat{z})\|^2\leq\|\hat{y}_n-\phi(\hat{z})\|^2-2\varsigma_n\langle\psi(\hat{y}_n)-\psi(y_n),y_n-\phi(\hat{z})\rangle-\|y_n-\hat{y}_n\|^2.\tag{3.25}$$

By (3.14), we have

$$\begin{aligned}\|\hat{x}_n-\phi(\hat{z})\|^2 &= \|(1-\sigma_n)(\hat{y}_n-\phi(\hat{z}))+\sigma_n(y_n-\phi(\hat{z}))+\sigma_n\varsigma_n[\psi(\hat{y}_n)-\psi(y_n)]\|^2\\ &= \|(1-\sigma_n)(\hat{y}_n-\phi(\hat{z}))+\sigma_n(y_n-\phi(\hat{z}))\|^2+\sigma_n^2\varsigma_n^2\|\psi(\hat{y}_n)-\psi(y_n)\|^2\\ &\quad +2\sigma_n(1-\sigma_n)\varsigma_n\langle\hat{y}_n-\phi(\hat{z}),\psi(\hat{y}_n)-\psi(y_n)\rangle\\ &\quad +2\sigma_n^2\varsigma_n\langle y_n-\phi(\hat{z}),\psi(\hat{y}_n)-\psi(y_n)\rangle.\end{aligned}\tag{3.26}$$

Applying equality (2.11), we obtain

$$\begin{aligned}\|(1-\sigma_n)(\hat{y}_n-\phi(\hat{z}))+\sigma_n(y_n-\phi(\hat{z}))\|^2 &= (1-\sigma_n)\|\hat{y}_n-\phi(\hat{z})\|^2 \\ &\quad+\sigma_n\|y_n-\phi(\hat{z})\|^2-\sigma_n(1-\sigma_n)\|\hat{y}_n-y_n\|^2.\end{aligned}\tag{3.27}$$

Substituting (3.25) and (3.27) into (3.26) to deduce

$$\begin{aligned}\|\hat{x}_n-\phi(\hat{z})\|^2 &\le \|\hat{y}_n-\phi(\hat{z})\|^2-\sigma_n(2-\sigma_n)\|\hat{y}_n-y_n\|^2+\sigma_n^2\varsigma_n^2\|\psi(\hat{y}_n)-\psi(y_n)\|^2 \\ &\quad+2\sigma_n(1-\sigma_n)\varsigma_n\langle\hat{y}_n-y_n,\psi(\hat{y}_n)-\psi(y_n)\rangle \\ &\le \|\hat{y}_n-\phi(\hat{z})\|^2-\sigma_n(2-\sigma_n)\|\hat{y}_n-y_n\|^2+\sigma_n^2\varsigma_n^2\|\psi(\hat{y}_n)-\psi(y_n)\|^2 \\ &\quad+2\sigma_n(1-\sigma_n)\varsigma_n\|\hat{y}_n-y_n\|\|\psi(\hat{y}_n)-\psi(y_n)\|.\end{aligned}\tag{3.28}$$

By (3.16), $\|\psi(y_n)-\psi(\hat{y}_n)\| \le \frac{\eta\|y_n-\hat{y}_n\|}{\varsigma_{n+1}}$. This together with (3.28) implies that

$$\begin{aligned}\|\hat{x}_n-\phi(\hat{z})\|^2 &\le \|\hat{y}_n-\phi(\hat{z})\|^2-\sigma_n(2-\sigma_n)\|\hat{y}_n-y_n\|^2+\sigma_n^2\eta^2\frac{\varsigma_n^2}{\varsigma_{n+1}^2}\|y_n-\hat{y}_n\|^2 \\ &\quad+2\sigma_n(1-\sigma_n)\eta\frac{\varsigma_n}{\varsigma_{n+1}}\|\hat{y}_n-y_n\|^2 \\ &=\|\hat{y}_n-\phi(\hat{z})\|^2-\sigma_n\Big[2-\sigma_n-\sigma_n\eta^2\frac{\varsigma_n^2}{\varsigma_{n+1}^2}-2(1-\sigma_n)\eta\frac{\varsigma_n}{\varsigma_{n+1}}\Big]\|\hat{y}_n-y_n\|^2.\end{aligned}\tag{3.29}$$

By Remark 3.2, $\lim_{n\to\infty}\varsigma_n$ exists and thus $\lim_{n\to\infty}\frac{\varsigma_n}{\varsigma_{n+1}}=1$. This together with condition (C3) implies that $\lim_{n\to\infty}[2-\sigma_n-\sigma_n\eta^2\frac{\varsigma_n^2}{\varsigma_{n+1}^2}-2(1-\sigma_n)\eta\frac{\varsigma_n}{\varsigma_{n+1}}]>0$. Then, there exist a positive constant ϱ and a positive integer m such that $2-\sigma_n-\sigma_n\eta^2\frac{\varsigma_n^2}{\varsigma_{n+1}^2}-2(1-\sigma_n)\eta\frac{\varsigma_n}{\varsigma_{n+1}}\ge\varrho>0$ when $n\ge m$. Based on (3.29), we get

$$\|\hat{x}_n-\phi(\hat{z})\|^2 \le \|\hat{y}_n-\phi(\hat{z})\|^2-\varrho\|\hat{y}_n-y_n\|^2.\tag{3.30}$$

From (3.15), (3.20) and (3.30), we have

$$\begin{aligned}\|\phi(x_{n+1})-\phi(\hat{z})\| &\le (1-\rho_n)\|\phi(x_n)-\phi(\hat{z})\|+\rho_n\|\hat{x}_n-\phi(\hat{z})\| \\ &\le (1-\rho_n)\|\phi(x_n)-\phi(\hat{z})\|+\rho_n\|\hat{y}_n-\phi(\hat{z})\| \\ &\le (1-\rho_n)\|\phi(x_n)-\phi(\hat{z})\|+\rho_n[1-(1-\theta\xi/\tau)\vartheta_n] \\ &\quad\times\|\phi(x_n)-\phi(\hat{z})\|+\rho_n\vartheta_n(\xi\|\mathcal{C}(\hat{z})-\phi(\hat{z})\|+2\vartheta\|\varphi(\hat{z})\|) \\ &= [1-(1-\theta\xi/\tau)\rho_n\vartheta_n]\|\phi(x_n)-\phi(\hat{z})\|+(1-\theta\xi/\tau)\rho_n\vartheta_n \\ &\quad\times\frac{\|\xi\mathcal{C}(\hat{z})-\phi(\hat{z})\|+2\vartheta\|\varphi(\hat{z})\|}{1-\theta\xi/\tau}.\end{aligned}\tag{3.31}$$

It results in that

$$\begin{aligned}\|\phi(x_n)-\phi(\hat{z})\| \le \max\Big\{&\|\phi(x_0)-\phi(\hat{z})\|,\cdots,\|\phi(x_m)-\phi(\hat{z})\|, \\ &\frac{\|\xi\mathcal{C}(\hat{z})-\phi(\hat{z})\|+2\vartheta\|\varphi(\hat{z})\|}{1-\theta\xi/\tau}\Big\},\end{aligned}$$

which implies that the sequence $\{\phi(x_n)\}$ is bounded. Note that $\|x_n-\hat{z}\|\le\frac{1}{\tau}\|\phi(x_n)-\phi(\hat{z})\|$ and $\|y_n-\phi(\hat{z})\|\le\|\hat{y}_n-\phi(\hat{z})\|+\varsigma_n\|\psi(\hat{y}_n)]\|$. Consequently, the sequences $\{x_n\}$, $\{y_n\}$, $\{\hat{y}_n\}$ and $\{\hat{x}_n\}$ are bounded.

According to (2.11) and (3.15), we have

$$\begin{aligned}\|\phi(x_{n+1})-\phi(\hat z)\|^2 &= \|(1-\rho_n)(\phi(x_n)-\phi(\hat z))+\rho_n(\hat x_n-\phi(\hat z))\|^2\\ &= \rho_n\|\hat x_n-\phi(\hat z)\|^2+(1-\rho_n)\|\phi(x_n)-\phi(\hat z)\|^2\\ &\quad -\rho_n(1-\rho_n)\|\hat x_n-\phi(x_n)\|^2.\end{aligned} \tag{3.32}$$

In the light of (3.30) and (3.32), we derive

$$\begin{aligned}\|\phi(x_{n+1})-\phi(\hat z)\|^2 &\le \rho_n\|\hat y_n-\phi(\hat z)\|^2+(1-\rho_n)\|\phi(x_n)-\phi(\hat z)\|^2\\ &\quad -\rho_n(1-\rho_n)\|\hat x_n-\phi(x_n)\|^2.\end{aligned} \tag{3.33}$$

Using (3.20), we receive

$$\begin{aligned}\|\hat y_n-\phi(\hat z)\|^2 &\le [1-(1-\theta\xi/\tau)\vartheta_n]\|\phi(x_n)-\phi(\hat z)\|^2\\ &\quad +(1-\theta\xi/\tau)\vartheta_n\left(\frac{\|\xi\mathcal{C}(\hat z)-\phi(\hat z)\|+2\vartheta\|\varphi(\hat z)\|}{1-\theta\xi/\tau}\right)^2.\end{aligned} \tag{3.34}$$

Now, we divide the following proof into two cases. Case (i): There exists some large enough $N_1 \ge m$ such that the sequence $\{\|\phi(x_n)-\phi(\hat z)\|\}$ is decreasing when $n \ge N_1$. Case (ii): For any $N_2 \ge m$, there exists an integer $j \ge N_2$ such that $\|\phi(x_j)-\phi(\hat z)\| \le \|\phi(x_{j+1})-\phi(\hat z)\|$.

For Case (i), we can deduce that $\lim_{n\to\infty}\|\phi(x_n)-\phi(\hat z)\|$ exists. On account of (3.33) and (3.34), we have

$$\begin{aligned}\rho_n(1-\rho_n)\|\hat x_n-\phi(x_n)\|^2 &\le \rho_n\|\hat y_n-\phi(\hat z)\|^2+(1-\rho_n)\|\phi(x_n)-\phi(\hat z)\|^2\\ &\quad -\|\phi(x_{n+1})-\phi(\hat z)\|^2\\ &\le \|\phi(x_n)-\phi(\hat z)\|^2-\|\phi(x_{n+1})-\phi(\hat z)\|^2\\ &\quad +(1-\theta\xi/\tau)\vartheta_n\left(\frac{\|\xi\mathcal{C}(\hat z)-\phi(\hat z)\|+2\vartheta\|\varphi(\hat z)\|}{1-\theta\xi/\tau}\right)^2\\ &\to 0.\end{aligned}$$

It leads to

$$\lim_{n\to\infty}\|\hat x_n-\phi(x_n)\|=0. \tag{3.35}$$

By (3.15), we have $\phi(x_{n+1})-\phi(x_n)=\rho_n(\hat x_n-\phi(x_n))$. This together with (3.35) implies that

$$\lim_{n\to\infty}\|\phi(x_{n+1})-\phi(x_n)\|=0. \tag{3.36}$$

By virtue of (3.15), (3.21) and (3.30), we get

$$\begin{aligned}\|\phi(x_{n+1})-\phi(\hat z)\|^2 &= \|(1-\rho_n)(\phi(x_n)-\phi(\hat z))+\rho_n(\hat x_n-\phi(\hat z))\|^2\\ &\le (1-\rho_n)\|\phi(x_n)-\phi(\hat z)\|^2+\rho_n\|\hat x_n-\phi(\hat z)\|^2\\ &\le (1-\rho_n)\|\phi(x_n)-\phi(\hat z)\|^2+\rho_n\|\hat y_n-\phi(\hat z)\|^2\\ &\le (1-\rho_n)\|\phi(x_n)-\phi(\hat z)\|^2+\rho_n(1-\vartheta_n)\|\phi(x_n)-\phi(\hat z)\|^2\\ &\quad +\rho_n(1-\vartheta_n)\alpha_n(\alpha_n-2\vartheta)\|\varphi(x_n)-\varphi(\hat z)\|^2\\ &\quad +\rho_n\vartheta_n\|\xi\mathcal{C}(x_n)-\phi(\hat z)+\alpha_n\varphi(\hat z)\|^2\\ &\le \|\phi(x_n)-\phi(\hat z)\|^2+\rho_n\vartheta_n\|\xi\mathcal{C}(x_n)-\phi(\hat z)+\alpha_n\varphi(\hat z)\|^2\\ &\quad +\rho_n(1-\vartheta_n)\alpha_n(\alpha_n-2\vartheta)\|\varphi(x_n)-\varphi(\hat z)\|^2.\end{aligned} \tag{3.37}$$

It yields that

$$\begin{aligned}(3.38)\qquad &\rho_n(1-\vartheta_n)\alpha_n(2\vartheta-\alpha_n)\|\varphi(x_n)-\varphi(\hat{z})\|^2\\ &\quad\le \|\phi(x_n)-\phi(\hat{z})\|^2-\|\phi(x_{n+1})-\phi(\hat{z})\|^2\\ &\qquad+\rho_n\vartheta_n\|\xi\mathcal{C}(x_n)-\phi(\hat{z})+\alpha_n\varphi(\hat{z})\|^2\\ &\quad\to 0.\end{aligned}$$

Noting that $\liminf_{n\to\infty}\rho_n(1-\vartheta_n)\alpha_n(2\vartheta-\alpha_n)>0$, it follows from (3.38) that

$$(3.39)\qquad \lim_{n\to\infty}\|\varphi(x_n)-\varphi(\hat{z})\|=0.$$

As a result of (2.8), (3.12) and (3.18), we have

$$\begin{aligned}\|\hat{y}_n-\phi(\hat{z})\|^2 &= \|\mathrm{proj}_C[\vartheta_n\xi\mathcal{C}(x_n)+(1-\vartheta_n)(\phi(x_n)-\alpha_n\varphi(x_n))]\\ &\quad-\mathrm{proj}_C[\phi(\hat{z})-\alpha_n\varphi(\hat{z})]\|^2\\ &\le\langle\vartheta_n\xi\mathcal{C}(x_n)+(1-\vartheta_n)(\phi(x_n)-\alpha_n\varphi(x_n))-(\phi(\hat{z})-\alpha_n\varphi(\hat{z})),\hat{y}_n-\phi(\hat{z})\rangle\\ &=\vartheta_n\langle\xi\mathcal{C}(x_n)-\phi(\hat{z})+\alpha_n\varphi(\hat{z}),\hat{y}_n-\phi(\hat{z})\rangle\\ &\quad+(1-\vartheta_n)\langle\phi(x_n)-\alpha_n\varphi(x_n)-\phi(\hat{z})+\alpha_n\varphi(\hat{z}),\hat{y}_n-\phi(\hat{z})\rangle\\ &\le\vartheta_n\langle\xi\mathcal{C}(x_n)-\phi(\hat{z})+\alpha_n\varphi(\hat{z}),\hat{y}_n-\phi(\hat{z})\rangle+\frac{1}{2}\Big\{\|\hat{y}_n-\phi(\hat{z})\|^2\\ &\quad+\|\phi(x_n)-\alpha_n\varphi(x_n)-\phi(\hat{z})+\alpha_n\varphi(\hat{z})\|^2\\ &\quad-\|\phi(x_n)-\hat{y}_n-\alpha_n(\varphi(x_n)-\varphi(\hat{z}))\|^2\Big\}\\ &\le\vartheta_n\|\xi\mathcal{C}(x_n)-\phi(\hat{z})+\alpha_n\varphi(\hat{z})\|\|\hat{y}_n-\phi(\hat{z})\|\\ &\quad+\frac{1}{2}\Big\{\|\phi(x_n)-\phi(\hat{z})\|^2+\|\hat{y}_n-\phi(\hat{z})\|^2-\alpha_n^2\|\varphi(x_n)-\varphi(\hat{z})\|\\ &\quad-\|\phi(x_n)-\hat{y}_n\|^2+2\alpha_n\langle\phi(x_n)-\hat{y}_n,\varphi(x_n)-\varphi(\hat{z})\rangle\Big\}.\end{aligned}$$

It follows that

$$\begin{aligned}(3.40)\qquad \|\hat{y}_n-\phi(\hat{z})\|^2 &\le\|\phi(x_n)-\phi(\hat{z})\|^2-\|\phi(x_n)-\hat{y}_n\|^2\\ &\quad+2\alpha_n\|\phi(x_n)-\hat{y}_n\|\|\varphi(x_n)-\varphi(\hat{z})\|\\ &\quad+2\vartheta_n\|\xi\mathcal{C}(x_n)-\phi(\hat{z})+\alpha_n\varphi(\hat{z})\|\|\hat{y}_n-\phi(\hat{z})\|.\end{aligned}$$

Based on (3.33) and (3.40), we achieve

$$\begin{aligned}\|\phi(x_{n+1})-\phi(\hat{z})\|^2 &\le\|\phi(x_n)-\phi(\hat{z})\|^2+2\alpha_n\|\phi(x_n)-\hat{y}_n\|\|\varphi(x_n)-\varphi(\hat{z})\|\\ &\quad+2\vartheta_n\|\xi\mathcal{C}(x_n)-\phi(\hat{z})+\alpha_n\varphi(\hat{z})\|\|\hat{y}_n-\phi(\hat{z})\|\\ &\quad-\rho_n\|\phi(x_n)-\hat{y}_n\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}(3.41)\qquad \rho_n\|\phi(x_n)-\hat{y}_n\|^2 &\le\|\phi(x_n)-\phi(\hat{z})\|^2-\|\phi(x_{n+1})-\phi(\hat{z})\|^2\\ &\quad+2\alpha_n\|\phi(x_n)-\hat{y}_n\|\|\varphi(x_n)-\varphi(\hat{z})\|\\ &\quad+2\vartheta_n\|\mathcal{C}(x_n)-\phi(\hat{z})+\alpha_n\varphi(\hat{z})\|\|\hat{y}_n-\phi(\hat{z})\|.\end{aligned}$$

By conditions (C1), (C2), (3.39) and (3.41), we obtain

$$(3.42)\qquad \lim_{n\to\infty}\|\phi(x_n)-\hat{y}_n\|=0.$$

On the other hand, from (3.30) and (3.37), we get

$$\begin{aligned}\|\phi(x_{n+1})-\phi(\hat{z})\|^2 &\le (1-\rho_n)\|\phi(x_n)-\phi(\hat{z})\|^2+\rho_n\|\hat{x}_n-\phi(\hat{z})\|^2\\ &\le (1-\rho_n)\|\phi(x_n)-\phi(\hat{z})\|^2+\rho_n\|\hat{y}_n-\phi(\hat{z})\|^2-\rho_n\varrho\|\hat{y}_n-y_n\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}\rho_n\varrho\|\hat{y}_n-y_n\|^2 \le{}& (1-\rho_n)(\|\phi(x_n)-\phi(\hat{z})\|^2-\|\phi(x_{n+1})-\phi(\hat{z})\|^2)\\ &+\rho_n(\|\hat{y}_n-\phi(\hat{z})\|+\|\phi(x_{n+1})-\phi(\hat{z})\|)\|\phi(x_{n+1})-\hat{y}_n\|.\end{aligned} \tag{3.43}$$

In the light of (3.36) and (3.42), we deduce

$$\lim_{n\to\infty}\|\phi(x_{n+1})-\hat{y}_n\|=0.$$

This together with (3.43) implies that

$$\lim_{n\to\infty}\|y_n-\hat{y}_n\|=0. \tag{3.44}$$

By (3.43) and the Lipschitz continuity of ψ, we have

$$\lim_{n\to\infty}\|\psi(\hat{y}_n)-\psi(y_n)\|=0. \tag{3.45}$$

Since the sequences $\{x_n\}$ and $\{\hat{y}_n\}$ are bounded, we can choose a common subsequence $\{n_i\}$ of $\{n\}$ such that $x_{n_i}\rightharpoonup u^*$ and

$$\limsup_{n\to\infty}\langle\xi\mathcal{C}(\hat{z})-\phi(\hat{z}),\hat{y}_n-\phi(\hat{z})\rangle=\lim_{i\to\infty}\langle\xi\mathcal{C}(\hat{z})-\phi(\hat{z}),\hat{y}_{n_i}-\phi(\hat{z})\rangle. \tag{3.46}$$

Using the weak continuity of ϕ, we deduce that $\phi(x_{n_i})\rightharpoonup\phi(u^*)$. By (3.42), we obtain $\hat{y}_{n_i}\rightharpoonup\phi(u^*)$ and $y_{n_i}\rightharpoonup\phi(u^*)$ due to (3.44).

In the sequel, we prove $u^*\in\mathrm{Sol}(C,\varphi,\phi)$. Let $\hat{\phi}$ be an operator defined by

$$\hat{\phi}(\tilde{u})=\begin{cases}\varphi(\tilde{u})+N_C(\tilde{u}), & \tilde{u}\in C,\\ \emptyset, & \tilde{u}\notin C.\end{cases}$$

Then, $\hat{\phi}$ is maximal ϕ-monotone. Let $(\tilde{u},u)\in G(\hat{\phi})$. Hence, $u-\varphi(\tilde{u})\in N_C(\tilde{u})$ and $\langle\phi(\tilde{u})-\phi(x_{n_i}),u-\varphi(\tilde{u})\rangle\ge 0$. From (3.12), we have

$$\langle\phi(\tilde{u})-\hat{y}_{n_i},\hat{y}_{n_i}-[\vartheta_{n_i}\xi\mathcal{C}(x_{n_i})+(1-\vartheta_{n_i})(\phi(x_{n_i})-\alpha_{n_i}\varphi(x_{n_i}))]\rangle\ge 0.$$

It follows that

$$\langle\phi(\tilde{u})-\hat{y}_{n_i},\frac{\hat{y}_{n_i}-\phi(x_{n_i})}{\alpha_{n_i}}+\varphi(x_{n_i})\rangle+\frac{\vartheta_{n_i}}{\alpha_{n_i}}\langle\phi(\tilde{u})-\hat{y}_{n_i},\phi(x_{n_i})-\alpha_{n_i}\varphi(x_{n_i})-\xi\mathcal{C}(x_{n_i})\rangle\ge 0.$$

Then,

$$
\begin{aligned}
\langle\phi(\tilde{u})-\phi(x_{n_i}),u\rangle &\geq \langle\phi(\tilde{u})-\phi(x_{n_i}),\varphi(\tilde{u})\rangle\\
&\geq \langle\phi(\tilde{u})-\phi(x_{n_i}),\varphi(\tilde{u})\rangle-\langle\phi(\tilde{u})-\hat{y}_{n_i},\frac{\hat{y}_{n_i}-\phi(x_{n_i})}{\alpha_{n_i}}+\varphi(x_{n_i})\rangle\\
&\quad-\frac{\vartheta_{n_i}}{\alpha_{n_i}}\langle\phi(\tilde{u})-\hat{y}_{n_i},\phi(x_{n_i})-\alpha_{n_i}\varphi(x_{n_i})-\xi\mathcal{C}(x_{n_i})\rangle\\
&=\langle\phi(\tilde{u})-\phi(x_{n_i}),\varphi(\tilde{u})-\varphi(x_{n_i})\rangle+\langle\hat{y}_{n_i}-\phi(x_{n_i}),\varphi(x_{n_i})\rangle\\
&\quad-\frac{\vartheta_{n_i}}{\alpha_{n_i}}\langle\phi(\tilde{u})-\hat{y}_{n_i},\phi(x_{n_i})-\alpha_{n_i}\varphi(x_{n_i})-\xi\mathcal{C}(x_{n_i})\rangle\\
&\quad-\langle\phi(\tilde{u})-\hat{y}_{n_i},\frac{\hat{y}_{n_i}-\phi(x_{n_i})}{\alpha_{n_i}}\rangle\\
&\geq\langle\hat{y}_{n_i}-\phi(x_{n_i}),\varphi(x_{n_i})\rangle-\langle\phi(\tilde{u})-\hat{y}_{n_i},\frac{\hat{y}_{n_i}-\phi(x_{n_i})}{\alpha_{n_i}}\rangle\\
&\quad-\frac{\vartheta_{n_i}}{\alpha_{n_i}}\langle\phi(\tilde{u})-\hat{y}_{n_i},\phi(x_{n_i})-\alpha_{n_i}\varphi(x_{n_i})-\xi\mathcal{C}(x_{n_i})\rangle.
\end{aligned}
\tag{3.47}
$$

Since $\|\hat{y}_{n_i}-\phi(x_{n_i})\|\to 0$, $\vartheta_{n_i}\to 0$, $\liminf_{i\to\infty}\alpha_{n_i}>0$ and $\phi(x_{n_i})\rightharpoonup\phi(u^*)$, in (3.47), letting $i\to\infty$, we deduce that $\langle\phi(\tilde{u})-\phi(u^*),u\rangle\geq 0$. Thus, $u^*\in\phi^{-1}(0)$. Hence, $u^*\in\mathrm{Sol}(C,\varphi,\phi)$.

Next, we show $\phi(u^*)\in\mathrm{Sol}(C,\psi)$. By (3.13), $y_{n_i}=\mathrm{proj}_C[\hat{y}_{n_i}-\varsigma_{n_i}\psi(\hat{y}_{n_i})]$. Applying (2.9), we have

$$\langle y_{n_i}-\hat{y}_{n_i}+\varsigma_{n_i}\psi(\hat{y}_{n_i}),y_{n_i}-u\rangle\leq 0,\ \forall u\in C.$$

Hence,

$$\frac{1}{\varsigma_{n_i}}\langle\hat{y}_{n_i}-y_{n_i},u-y_{n_i}\rangle+\langle\psi(\hat{y}_{n_i}),y_{n_i}-\hat{y}_{n_i}\rangle\leq\langle\psi(\hat{y}_{n_i}),u-\hat{y}_{n_i}\rangle,\ \forall u\in C. \tag{3.48}$$

Based on (3.44) and (3.48), we deduce

$$\liminf_{i\to\infty}\langle\psi(\hat{y}_{n_i}),u-\hat{y}_{n_i}\rangle\geq 0,\ \forall u\in C. \tag{3.49}$$

There are two cases: (i) $\liminf_{i\to\infty}\|\psi(\hat{y}_{n_i})\|=0$ and (ii) $\liminf_{i\to\infty}\|\psi(\hat{y}_{n_i})\|>0$. For the case (i), since $\hat{y}_{n_i}\rightharpoonup\phi(u^*)$ and ψ satisfies condition (p), we conclude that $\psi(\phi(u^*))=0$. So, $\phi(u^*)\in\mathrm{Sol}(C,\psi)$.

Assume the case (ii) holds, that is, $\liminf_{i\to\infty}\|\psi(\hat{y}_{n_i})\|>0$. According to (3.49), we receive

$$\liminf_{i\to\infty}\langle\psi(\hat{y}_{n_i})/\|\psi(\hat{y}_{n_i})\|,u-\hat{y}_{n_i}\rangle\geq 0,\ \forall u\in C. \tag{3.50}$$

Let $\{\varpi_i\}$ be a positive real number sequence such that $\varpi_i\to 0$ as $i\to\infty$. Thanks to (3.50), for each ϖ_i, there exists an integer $N_i>0$ such that

$$\langle\psi(\hat{y}_{n_i})/\|\psi(\hat{y}_{n_i})\|,u-\hat{y}_{n_i}\rangle+\varpi_i>0,\ \forall u\in C,\ \forall i\geq N_i.$$

Thus,

$$\langle\psi(\hat{y}_{n_i}),u-\hat{y}_{n_i}\rangle+\varpi_i\|\psi(\hat{y}_{n_i})\|>0,\ \forall u\in C,\ \forall i\geq N_i. \tag{3.51}$$

Set $\hat{z}_{n_i}=\frac{\psi(\hat{y}_{n_i})}{\|\psi(\hat{y}_{n_i})\|^2}$ for all $i\geq 0$. Then, $\langle\psi(\hat{y}_{n_i}),\hat{z}_{n_i}\rangle=1$ for each i. Using (3.51), we obtain

$$\langle\psi(\hat{y}_{n_i}),u-\hat{y}_{n_i}+\varpi_i\|\psi(\hat{y}_{n_i})\|\hat{z}_{n_i}\rangle>0,\ \forall u\in C,\ \forall i\geq N_i. \tag{3.52}$$

Since ψ is quasimonotone, from (3.52), we have

$$\langle\psi(u+\varpi_i\|\psi(\hat{y}_{n_i})\|\hat{z}_{n_i}),u+\varpi_i\|\psi(\hat{y}_{n_i})\|\hat{z}_{n_i}-\hat{y}_{n_i}\rangle\geq 0,\ \forall u\in C,\ \forall i\geq N_i. \tag{3.53}$$

Owing to $\lim_{i\to\infty}\varpi_i\|\psi(\hat{y}_{n_i})\|\|\hat{z}_{n_i}\| = \lim_{i\to\infty}\varpi_i = 0$, letting $i\to\infty$ in (3.53), we get

$$\langle\psi(u), u-\phi(u^*)\rangle \geq 0,\ \forall u\in C, \tag{3.54}$$

which implies that $\phi(u^*)\in \mathrm{Sol}_d(C,\psi)\subset \mathrm{Sol}(C,\psi)$. Therefore, $u^*\in \mathrm{Sol}(C,\varphi,\phi)\cap\phi^{-1}(\mathrm{Sol}(C,\psi))$. According to (3.17) and (3.46), we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle\xi\mathcal{C}(\hat{z})-\phi(\hat{z}), \hat{y}_n-\phi(\hat{z})\rangle &= \lim_{i\to\infty}\langle\xi\mathcal{C}(\hat{z})-\phi(\hat{z}), \hat{y}_{n_i}-\phi(\hat{z})\rangle\\ &= \langle\xi\mathcal{C}(\hat{z})-\phi(\hat{z}), \phi(u^*)-\phi(\hat{z})\rangle \leq 0.\end{aligned} \tag{3.55}$$

Let $\tilde{z}$ be the unique solution of (3.17). Using (2.8) and (3.12), we get

$$\begin{aligned}\|\hat{y}_n-\phi(\tilde{z})\|^2 &= \|\mathrm{proj}_C[\vartheta_n\xi\mathcal{C}(x_n)+(1-\vartheta_n)(\phi(x_n)-\alpha_n\varphi(x_n))]\\ &\quad -\mathrm{proj}_C[\phi(\tilde{z})-(1-\vartheta_n)\alpha_n\varphi(\tilde{z})]\|^2\\ &\leq \vartheta_n\langle\xi\mathcal{C}(x_n)-\xi\mathcal{C}(\tilde{z}), \hat{y}_n-\phi(\tilde{z})\rangle+\vartheta_n\langle\xi\mathcal{C}(\tilde{z})-\phi(\tilde{z}), \hat{y}_n-\phi(\tilde{z})\rangle\\ &\quad +(1-\vartheta_n)\langle\phi(x_n)-\alpha_n\varphi(x_n)-(\phi(\tilde{z})-\alpha_n\varphi(\tilde{z})), \hat{y}_n-\phi(\tilde{z})\rangle\\ &\leq [1-(1-\theta\xi/\tau)\vartheta_n]\|\phi(x_n)-\phi(\tilde{z})\|\|\hat{y}_n-\phi(\tilde{z})\|\\ &\quad +\vartheta_n\langle\xi\mathcal{C}(\tilde{z})-\phi(\tilde{z}), \hat{y}_n-\phi(\tilde{z})\rangle\\ &\leq \frac{1-(1-\theta\xi/\tau)\vartheta_n}{2}\|\phi(x_n)-\phi(\tilde{z})\|^2+\frac{1}{2}\|\hat{y}_n-\phi(\tilde{z})\|^2\\ &\quad +\vartheta_n\langle\xi\mathcal{C}(\tilde{z})-\phi(\tilde{z}), \hat{y}_n-\phi(\tilde{z})\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\|\hat{y}_n-\phi(\tilde{z})\|^2 \leq{}& [1-(1-\theta\xi/\tau)\vartheta_n]\|\phi(x_n)-\phi(\tilde{z})\|^2\\ &+2\vartheta_n\langle\xi\mathcal{C}(\tilde{z})-\phi(\tilde{z}), \hat{y}_n-\phi(\tilde{z})\rangle.\end{aligned} \tag{3.56}$$

Combining (3.31) and (3.56), we have

$$\begin{aligned}\|\phi(x_{n+1})-\phi(\tilde{z})\|^2 &\leq (1-\rho_n)\|\phi(x_n)-\phi(\tilde{z})\|^2+\rho_n\|\hat{y}_n-\phi(\tilde{z})\|^2\\ &\leq [1-(1-\theta\xi/\tau)\rho_n\vartheta_n]\|\phi(x_n)-\phi(\tilde{z})\|^2\\ &\quad +2\rho_n\vartheta_n\langle\xi\mathcal{C}(\tilde{z})-\phi(\tilde{z}), \hat{y}_n-\phi(\tilde{z})\rangle.\end{aligned} \tag{3.57}$$

By virtue of Lemma 2.2 and (3.57), we conclude that $\phi(x_n)\to\phi(\tilde{z})$ and $x_n\to\tilde{z}$.

Case (ii): For any $N_2\geq m$, there exists an integer $j\geq N_2$ such that $\|\phi(x_j)-\phi(\tilde{z})\|\leq\|\phi(x_{j+1})-\phi(\tilde{z})\|$. Set $\Upsilon_n=\{\|\phi(x_n)-\phi(\tilde{z})\|^2\}$. Then, $\Upsilon_j\leq\Upsilon_{j+1}$. For all $n\geq j$, define an integer sequence $\{I(n)\}$ as follows $I(n)=\max\{i\in\mathbb{N}|j\leq i\leq n, \Upsilon_i\leq\Upsilon_{i+1}\}$. It is obvious that $I(n)$ satisfies the following properties: (i) $I(n)\leq I(n+1)$, (ii) $\lim_{n\to\infty}I(n)=\infty$ and (iii) $\Upsilon_{I(n)}\leq\Upsilon_{I(n)+1}, \forall n\geq j$.

By the similar argument as that of Case (i), we can prove

$$\limsup_{n\to\infty}\langle\xi\mathcal{C}(\tilde{z})-\phi(\tilde{z}), \hat{y}_{I(n)}-\phi(\tilde{z})\rangle\leq 0 \tag{3.58}$$

and

$$\begin{aligned}\Upsilon_{I(n)+1}\leq{}& [1-(1-\theta\xi/\tau)\vartheta_{I(n)}\rho_{I(n)}]\Upsilon_{I(n)}\\ &+2\vartheta_{I(n)}\rho_{I(n)}\langle\xi\mathcal{C}(\tilde{z})-\phi(\tilde{z}), \hat{y}_{I(n)}-\phi(\tilde{z})\rangle.\end{aligned} \tag{3.59}$$

Note that $\Upsilon_{I(n)}\leq\Upsilon_{I(n)+1}$. This together with (3.59) implies that

$$\Upsilon_{I(n)}\leq\frac{2}{1-\theta\xi/\tau}\langle\xi\mathcal{C}(\tilde{z})-\phi(\tilde{z}), \hat{y}_{I(n)}-\phi(\tilde{z})\rangle. \tag{3.60}$$

Combining (3.58) and (3.60), we obtain $\limsup_{n\to\infty}\Upsilon_{I(n)}\leq 0$ and so

$$\lim_{n\to\infty}\Upsilon_{I(n)}=0. \tag{3.61}$$

Taking into account (3.58) and (3.59), we deduce $\limsup_{n\to\infty} \Upsilon_{I(n)+1} \leq \limsup_{n\to\infty} \Upsilon_{I(n)}$. This together with (3.61) implies that $\lim_{n\to\infty} \Upsilon_{I(n)+1} = 0$. By Lemma 2.3, we obtain $0 \leq \Upsilon_n \leq \max\{\Upsilon_{I(n)}, \Upsilon_{I(n)+1}\}$. Therefore, $\Upsilon_n \to 0$. That is, $\phi(x_n) \to \phi(\tilde{z})$ and thus $x_n \to \tilde{z}$. This completes the proof. □

In Algorithm 3.1, let ϕ be the identity operator with $\tau = 1$ and $\varphi : C \to H$ be ϑ-inverse strongly monotone operator. Then, we have the following algorithm and corollary.

Algorithm 3.2. Let x_0 be an initial point in C and ς_0 be a positive constant. Set $n = 0$.
Step 1. Let x_n be constructed. Then, compute

$$\hat{y}_n = \mathrm{proj}_C[\vartheta_n \xi \mathcal{C}(x_n) + (1-\vartheta_n)(x_n - \alpha_n \varphi(x_n))].$$

Step 2. Let ς_n be given. Compute

$$y_n = \mathrm{proj}_C[\hat{y}_n - \varsigma_n \psi(\hat{y}_n)],$$

and

$$\hat{x}_n = (1-\sigma_n)\hat{y}_n + \sigma_n y_n + \sigma_n \varsigma_n[\psi(\hat{y}_n) - \psi(y_n)].$$

Step 3. Compute

$$x_{n+1} = (1-\rho_n)x_n + \rho_n \hat{x}_n.$$

Step 4. Update

$$\varsigma_{n+1} = \begin{cases} \min\left\{\varsigma_n, \frac{\eta\|\hat{y}_n - y_n\|}{\|\psi(\hat{y}_n) - \psi(y_n)\|}\right\}, & \text{if } \psi(\hat{y}_n) \neq \psi(y_n), \\ \varsigma_n, & else. \end{cases}$$

Write $n := n + 1$ and return to step 1.

Corollary 3.1. *Suppose that $\Xi_1 := \mathrm{Sol}(C, \varphi) \cap \mathrm{Sol}_d(C, \psi) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges strongly to $\hat{z} = \mathrm{proj}_{\Xi_1}\mathcal{C}(\hat{z})$.*

4. Concluding remarks

This paper is devoted to the investigation and analysis of the split variational inequality problem (1.7), which was originally introduced in [8] with the transformation ϕ being a bounded linear operator. However, in our result, we consider a more general case that the transformation ϕ is nonlinear. Namely, if we choose ϕ to be a bounded linear operator, then the investigated problem (1.7) reduces to the well-known problem proposed in [8]. In this respect, our results extend and improve some existing results in the literature ([8, 16, 17, 20]).

At the same time, in order to prove the convergence of the sequence $\{x_n\}$, a common condition $\mathrm{Sol}(C, \psi) \subset \mathrm{Sol}_d(C, \psi)$ is used. In this paper, we remove this assumption. As a matter of fact, since ψ is quasimonotone, the assumption $\mathrm{Sol}(C, \psi) \subset \mathrm{Sol}_d(C, \psi)$) may be not hold.

Generally speaking, in order to prove $\omega(x_n)$ belongs to the solution set, ψ should be sequentially weakly continuous (see [3, 33]). In this paper, we replace this conditions by a weaker condition (p). It is easy to verify that if ψ satisfies sequential weak continuity, then ψ possesses the property (p). Our method can be used to relax the sequential weak continuity imposed on ψ.

With these advantages in hand, we show that the introduced algorithm [Algorithm 3.1] converges strongly to a solution of the split variational inequality problem (1.7) which also solves VI (3.17) under some mild conditions.

An interesting problem arises: could we further weaken or remove the condition (p)?

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