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Unpredictable solutions of Duffing type equations with Markov coefficients

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ABSTRACT. The paper considers a stochastic differential equation of Duffing type with Markov coefficients. The existence of unpredictable solutions is considered. The unpredictability is a property of bounded functions characterized by unbounded sequences of moments of divergence and convergence in Bebutov dynamics. Markov components of the equation coefficients admit the unpredictability property. The components of the equation coefficients are derived from a Markov chain. The existence, uniqueness and exponential stability of an unpredictable solution are proved. The sequences of divergence and convergence of the coefficients and the solution are synchronized. Numerical examples that support the theoretical results are provided.

1. INTRODUCTION

The original Duffing equation was introduced in the paper [14], and has the form

$$x'' + ax' + bx + cx^3 = F_0 cos(\lambda t), \tag{1.1}$$

where a is the damping coefficient, b and c are stiffness (restoring) coefficients, F_0 is the coefficient of excitation, λ is the frequency of excitation and t is the time. The major part of papers on the equation assume that the coefficients a, b, c and F_0 are constant [20, 35]. Considering the original model one can assume mechanical reasons for variable coefficients. For example, not constant damping and driving force [15].

It is of great importance to study a Duffing equation with variable coefficients. It was emphasized in the book by Moon [24] that the case when the coefficients are irregular is of strong interest. The question is, what if the perturbations are random, and are of noise type, and the noise is articulated with asymptotic properties of divergence and convergence. Obviously, there are two problems appear. The first one is, how to insert the deviations in coefficients and to be able for proper evaluations. The second task is, how the stochastic processes relate to what we understand as deterministic chaos. That is, not to say that a random process is a deterministic phenomenon, but to demonstrate that some significant features recognized for the chaos can be seen in dynamics originated, for example, from Markov events, which happen with probabilities. This two questions are concerned in the present article. The most related results which are utilized to answer the questions, have been already obtained in our previous research. In papers [1, 2], we introduced a new type of recurrence, the unpredictable point, which is an unpredictable function in Bebutov dynamics. In the research of article [3], it was proved that any infinite time realization of a Markov process with finite state space and without memory is an unpredictable sequence.

Deterministic approaches are often used to describe nonlinear phenomena. However, they are not resistant to changes in system parameters and do not provide confidence for

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the identified model. Any real system is uncertain due to imperfections of material, noise, etc. This is why, a reliable method of identifying must be taken into account, namely uncertainties of the model parameters. In paper [32], the authors use the following stochastic model to describe the nonlinear random dynamics of a mechanical system emulated by an experimental apparatus,

$$m(\theta)\ddot{x}(\theta,t)+c(\theta)\dot{x}(\theta,t)+F(x(\theta,t))=U(t), \tag{1.2}$$

where the random processes $(\theta,t)\in\Theta\times R\mapsto x(\theta,t)$, $(\theta,t)\in\Theta\times R\mapsto\dot{x}(\theta,t)$, and $(\theta,t)\in\Theta\times R\mapsto\ddot{x}(\theta,t)$, respectively represent the displacement, velocity and acceleration in the beam free extremity. It was shown that the stochastic Duffing type model (1.2) can predict the beam's vibration responses, which ensure the robustness of the stochastic identification method.

A nonlinear model for random oscillators, which includes a stochastic version of the Duffing oscillator is studied in paper [29]. It was obtained that the effect of the type and size of nonlinearity under investigation does not affect appreciably the qualitative long-time behavior of the moments of the displacement of the oscillator, in both the hard and the soft spring models.

The results in [34], put forward that the stochastic Duffing equation can be a fitting model to certain financial time series or the time series with the same characteristics.

In our research, we follow the suggestion to consider realizations of random dynamics as functions, in general, and sequences, in particular. As it is said in the book [25] "We have described stochastic dynamics in terms of probability distributions and their various moments. A complimentary, and for many purposes especially illuminating approach, is the study of individual outcomes of the stochastic process of interest." We agree with the authors, and think that the outcomes have to be considered not only for applications, but also as perturbations for various theoretical models. In the present paper, both: inputs, which are the coefficients of the equation of the Duffing type, and correspondingly, outputs, that is, the solutions of the equation are individualized.

In the present study, stochastic processes appear in various roles. The first, it is the discrete Markov chain with finite state space and without memory (one can use processes with memory in future extension of the method). A realization of the chain is applied as an input for a dissipative stochastic inhomogeneous equations. And the random solutions of the equations in their own turn are used as coefficients and inputs for the stochastic Duffing equation. Finally, it is approved that the solution of the equation of Duffing type is a continuous unpredictable function. It is clear that the scheme of the present study is about a novel type of stochastic differential equations and can be extended for other many theoretical tasks as well as applications.

The concept of unpredictability was introduced in papers [1, 2], and has been applied for various problems of differential equations, neural networks and synchronization of gas discharge-semiconductor systems [4, 5, 6, 7, 31]. It is powerful instrument for chaos indication [8, 9, 10, 11, 23, 30].

The Markov research [21] was considered to show that random processes of dependent events can also behave as independent events. Thus, simple dynamics were invented, which have been approved as most effective for many applications. It is impossible underestimate the role of the Markov processes in development of random dynamics theory and its applications. For example, the ergodic theorem was strictly approved at the first time for the dynamics. There are several observations that the chains are strongly connected to symbolic dynamics and to Bernoulli scheme. The final step for the comprehension was done by Donald Ornstein, who verified that B-automorphisms such as subshifts of finite type and Markov shifts, Anosov flows and Sinai's billiards, ergodic automorphisms of the n-torus and the continued fraction transforms are, in fact, isomorphic [26]. Considering these results it is of great necessity to show that various random processes can be described in terms of chaos, and that they relate equally in the sense. Investigators have worked in both directions, for chaos in random dynamics, as well as for stochastic features in deterministic motions [13]. Thus, the problem of chaos in Markov chains, which is in focus of our interest, is a part of the more general and significant project.

The unpredictable orbit [12] as a single isolated motion, presenting the Poincaré chaos [1], was identified as a certain event in the Markov chains [3], and our present results are not surprising in this sense, if one issues from the research in [26] and [3].

The main subject of this article is the following stochastic differential equation

$$x''(t)+(p_0(t)+p_1(t))x'(t)+(q_0(t)+q_1(t))x(t)+(r_0(t)+r_1(t))x^3(t)=$$
$$(F_0(t)+F_1(t))cos(\lambda t), \tag{1.3}$$

where $t,x\in\mathbb{R}$; λ is a real constant; $p_0(t)$, $q_0(t)$, $r_0(t)$ and $F_0(t)$ are continuous periodic functions; coefficient components $p_1(t)$, $q_1(t)$, $r_1(t)$ and $F_1(t)$ are derived from realizations of Markov processes. This is why, we say that the coefficients are Markovian. Let us remind that the right-hand-side of the equation is also assumed as a coefficient. If the periodic components of the coefficients are inserted for the stability of the solution of equation (1.3), whereas Markov components cause irregularity of solutions. It is important to emphasize that the main goal of the research not to approve a chaos for the output. But to show existence of the stochastic output for the equation, which admits the unpredictability property, and moreover, the property of the output is synchronized with the asymptotic characteristics of stochastic perturbations in the model.

2. Preliminaries

In this section the definitions of unpredictable and Poisson stable functions as well as definition of unpredictable sequences are given. Moreover, the algorithm of construction for Markovian coefficients of equation (1.3) is provided.

2.1. **Unpredictable functions.** The following definitions are basic in the theory of unpredictable points, orbits and functions introduced [1, 2] and developed further in papers [4, 6, 7, 8, 9, 10, 11, 23, 30].

Definition 2.1. [2] A uniformly continuous and bounded function $\psi:\mathbb{R}\to\mathbb{R}$ is unpredictable if there exist positive numbers ϵ_0,σ and sequences t_n,s_n both of which diverge to infinity such that $|\psi(t+t_n)-\psi(t)|\to 0$ as $n\to\infty$ uniformly on compact subsets of $\mathbb{R}$ and $|\psi(t+t_n)-\psi(t)|\geq\epsilon_0$ for each $t\in[s_n-\sigma,s_n+\sigma]$ and $n\in\mathbb{N}$.

In what follows, we shall call t_n and s_n as convergence and divergence sequences, respectively. The presence of the convergence sequence is the argument that any unpredictable function is Poisson stable [3, 12, 28], but not vice versa.

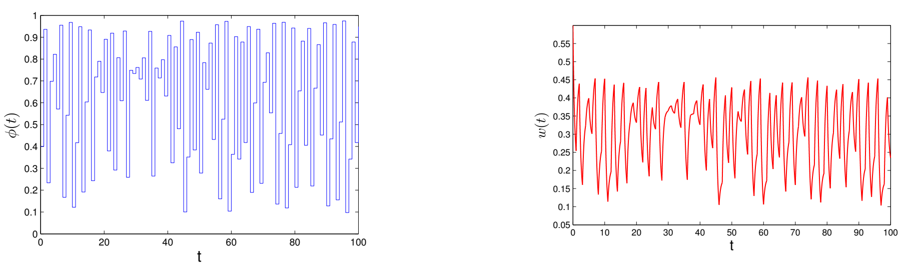
Definition 2.2. [28] A function $\phi(t)$: $\mathbb{R}\to\mathbb{R}$, bounded and continuous, is said to be Poisson stable if there is a sequence of moments t_n, $t_n\to\infty$ as $n\to\infty$, such that the sequence $\phi(t+t_n)$ uniformly converges to $\phi(t)$ on each bounded interval of the real axis.

The discrete version of the Definition 2.1 is as follows.

Definition 2.3. [2] A bounded sequence $\{k_i\}\in\mathbb{R}$, $i\in\mathbb{Z}$, is called unpredictable if there exist a positive number ϵ_0 and the sequences $\{\zeta_n\},\{\eta_n\}$, $n\in\mathbb{N}$, of positive integers both of which diverge to infinity such that $|k_{i+\eta_n}-k_i|\to 0$ as $n\to\infty$ for each i in bounded intervals of integers and $|k_{\zeta_n+\eta_n}-k_{\eta_n}|\geq\epsilon_0$ for each $n\in\mathbb{N}$.

In this paper, we shall consider unpredictable sequences with non-negative arguments and call them also unpredictable sequences [8].

Let us give examples of unpredictable functions. Using an unpredictable sequence, k_i, one can construct a piecewise constant function $\phi(t)$, such that $\phi(t) = k_i$ on intervals $t \in [hi, h(i+1))$, where h is a real number. In paper [4], the function $\phi(t)$ is determined through the solution of the logistic map and the Bernoulli process is used. Another unpredictable function, $W(t)$, is a continuous solution of differential equation $W'(t) = \alpha W(t) + \phi(t)$, where α is a negative number. In Figure 1 (a) the graph of function $\phi(t) = \lambda_i$, for $t \in [i, i+1)$, $i = 0, 1, 2, ...$, is shown, where λ_i is the unpredictable solution [1] of the logistic map, $\lambda_{i+1} = \mu\lambda_i(1-\lambda_i)$, $i \in \mathbb{Z}$, with $\lambda_0 = 0.4$, $\mu = 3.9$. Figure 1 (b) depicts the graph of the solution, $w(t)$, of the equation with $w(0) = 0.6$ and $\alpha = -2$, which exponentially approaches to the unique unpredictable solution, $W(t)$, of the non-homogeneous equation. This is why, the red line can be considered [6] for t¿40 as the graph of an unpredictable function. In the present paper, the coefficients of the equation (1.3) are determined by applying the algorithm for $W(t)$, but randomly such that a Markov chain is used instead of the logistic equation.



a) b)

FIGURE 1. The graphs of the discontinuous and continuous functions, $\phi(t)$ and $w(t)$.

2.2. **Markovian coefficients.** In this part of the paper, we demonstrate algorithms how to construct Markovian coefficients for Duffing type equation (1.3).

A Markov chain is a stochastic model, which describes a sequence of possible events such that the probability of each event depends only on the state attained in the previous one [17, 19, 22].

Since we expect for the chaotic dynamics realizations to be bounded, the special Markov chain with boundaries is constructed below. Let the real valued scalar dynamics

$$X_{n+1} = X_n + Y_n, n \geq 0, \tag{2.4}$$

be given such that Y_n is a random variable with values in $\{-2, 2\}$ with probability distribution $P(2) = P(-2) = 1/2$, if $X_n \neq -4, 6$, and certain events $Y_n = 2$, if $X_n = -4$, and $Y_n = -2$, if $X_n = 6$. To satisfy the construction of the present research, we will make the following agreements. First of all, denote $s_0 = -4$, $s_1 = -2$, $s_2 = 0$, $s_3 = 2$, $s_4 = 4$, $s_5 = 6$. Consider, the state space of the process $S = \{s_0, s_1, s_2, s_3, s_4, s_5\}$, and the value $X_n \in S$ is the state of the process at time n. The Markov chain, is a random process which satisfy property $P\{X_{n+1} = s_j | X_0, ..., X_n\} = P\{X_{n+1} = s_j | X_n\}$ for all $s_i, s_j \in S$ and $n \geq 0$, and, moreover, $P\{X_{n+1} = s_j | X_n = s_i\} = p_{ij}$, where p_{ij} is the transition probability that the chain jumps from state i to state j. It is clear that $\sum_{j=0}^{5} p_{ij} = 1$ for all $i = 0, ..., 5$. The unpredictability of infinite realizations of the dynamics is approved by Theorem 2.2 [3].

Next, we shall need the $\rho-$type piecewise constant unpredictable functions, which are defined through the Markov chain such that $\rho(t) = X_n$, if $t \in [hn, h(n+1))$. To visualize the $\rho-$type functions in Figure 2 (a) the graph of the function $\rho(t) = X_n$, if $t \in [hn, h(n+1))$, where $h = 0.5$, $0 \leq t \leq 100$ is drawn.

On the basis of the $\rho-$type functions we introduce the $\sigma-$type piecewise constant unpredictable functions such that

$$\sigma(t) = \Sigma(\rho(t)), \tag{2.5}$$

where $\Sigma(s)$ is a continuous function, which satisfies the inverse Lipschitz condition. It can be shown that $\sigma-$type functions are discontinuous unpredictable [8]. Figure 2 (b) depicts the graph of piecewise constant unpredictable function $\sigma(t) = \rho^2(t) + \rho(t)$.

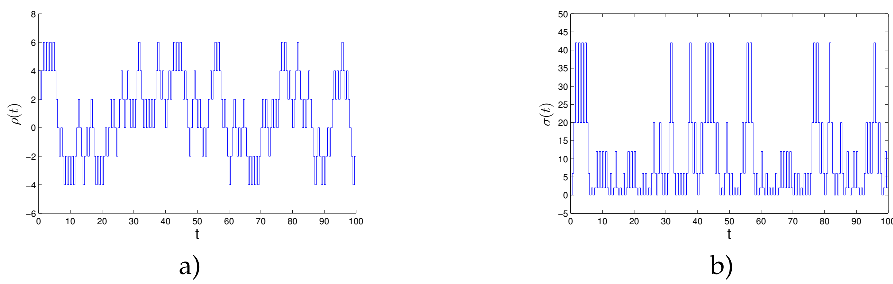


FIGURE 2. The piecewise constant functions $\rho(t)$ and $\sigma(t)$. The vertical lines are drawn for better visibility.

Now, let us define another type functions to finalize construction of continuous unpredictable functions through Markov process. Consider ordinary differential equation

$$x'(t) = \alpha x(t) + \sigma(t), \tag{2.6}$$

where α is a negative number. The equation (2.6) admits a unique exponentially stable unpredictable solution [2]. We say that the solution of the equation (2.6) is $\Theta-$type unpredictable function. It is impossible to specify the initial value of the solution, but applying the property of exponential stability one can consider any solution as arbitrary close. In Figure 3, the graph of the solution, $x(t)$, $x(0) = 0.6$ of equation (2.6), where the parameter α is equal to -3, and $\sigma(t) = \rho^2(t) + \rho(t)$ is shown. The solution exponentially approaches an unpredictable function $\Theta(t)$. Thus, the algorithm for three types of unpredictable functions, which will be applied to build the Markovian coefficients has been finalized. Next, we shall apply it for each of the coefficients in the stochastic differential equation (1.3).

Let us consider the following dissipative equations

$$p_1'(t) = \alpha_p p_1(t) + p(t), \tag{2.7}$$

$$q_1'(t) = \alpha_q q_1(t) + q(t), \tag{2.8}$$

$$r_1'(t) = \alpha_r r_1(t) + r(t), \tag{2.9}$$

$$F_1'(t) = \alpha_F F_1(t) + f(t), \tag{2.10}$$

where α_p, α_q, α_r and α_F are negative real numbers, and $p(t)$, $q(t)$, $r(t)$ and $f(t)$ are unpredictable functions of $\sigma-$type. That is, $p(t) = P(\rho(t))$, $q(t) = Q(\rho(t))$, $r(t) = R(\rho(t))$ and $f(t) = F(\rho(t))$, where $P(s)$, $Q(s)$, $R(s)$, and $F(s)$, are continuous functions with inverse Lipschitz property and the function $\rho(t)$ is determined above. The exponentially stable

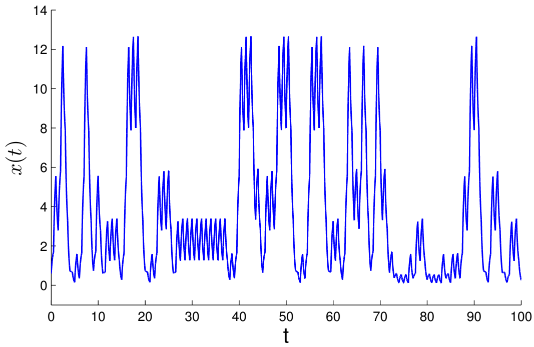


FIGURE 3. The solution $x(t)$ of equation (2.6) with initial value $x(0) = 0.6$ exponentially approaches the unpredictable Markovian function.

and bounded solutions $p_1(t)$, $q_1(t)$, $r_1(t)$ and $F_1(t)$ of equations (2.7)-(2.10) are $\Theta-$type functions. The functions are considered as Markovian components of the coefficients in the Duffing type equation (1.3).

In this paper, we utilize Markov chains without memory for the coefficients, but it is clear that one can consider chains with memories of arbitrary finite length in future studies.

3. MAIN RESULTS

In the present section, under certain conditions, it is rigorously proved that an exponentially stable unpredictable solution takes place in the dynamics of the Duffing type equation with Markovian coefficients.

We will make use of the norm $\|v\| = \max(|v_1|, |v_2|)$, for a two-dimensional vector $v = (v_1, v_2)$, and corresponding norm for square matrices will be utilized. For equation (1.3) it is provided that a solution $x(t)$ and its derivative $x'(t)$ are bounded such that $\sup_{t\in\mathbb{R}} |x(t)| < H$, $\sup_{t\in\mathbb{R}} |x'(t)| < H$, where H is a fixed positive number.

Assume that the following conditions are satisfied,

(C1) the functions $p_0(t)$, $q_0(t)$, $r_0(t)$, and $F_0(t)$ are continuous periodic with common positive period ω such that $\lambda = \frac{2\pi}{\omega}$;

(C2) the Markovian components $p_1(t)$, $q_1(t)$, $r_1(t)$ $F_1(t)$ are of $\Theta-$type with common sequences of convergence t_n, and divergence s_n such that there exist positive numbers σ, ϵ_0, which satisfy $|p_1(t+t_n)-p_1(t)| \geq \epsilon_0$, $|q_1(t+t_n)-q_1(t)| \geq \epsilon_0$, $|r_1(t+t_n)-r_1(t)| \geq \epsilon_0$, $|F_1(t+t_n) - F_1(t)| \geq \epsilon_0$, for all $t \in [s_n - \sigma; s_n + \sigma]$;

(C3) $t_n \to 0 \ (mod\omega)$ as $n \to \infty$;

(C4) $s_n \to 0 \ (mod\frac{\omega}{2})$ as $n \to \infty$.

The equation (1.3) can be written as the system

$$\begin{aligned} &x_1'(t) = x_2(t), \\ &x_2'(t) = -q_0(t)x_1(t) - p_0(t)x_2(t) - q_1(t)x_1(t) - p_1(t)x_2(t) - (r_0(t) + r_1(t))x_1^3(t) + \\ &(F_0(t) + F_1(t))cos(\lambda t). \end{aligned} \tag{3.11}$$

Consider the homogeneous system, associated with (3.11),

$$\begin{aligned} x_1'(t) &= x_2(t), \\ x_2'(t) &= -q_0(t)x_1(t) - p_0(t)x_2(t). \end{aligned} \tag{3.12}$$

Let $X(t), t \in \mathbb{R}$, is the fundamental matrix of system (3.12) such that $X(0) = I$, and I is the 2×2 identical matrix. Moreover, $X(t,s) = X(t)X^{-1}(s)$ is the transition matrix of system (3.12) such that $X(t+\omega, s+\omega) = X(t,s)$ for all $t, s \in \mathbb{R}$.

The following assumption is needed,

(C5) the multipliers of system (3.12) in modulus are less than one.

The last condition implies that there exist positive numbers $K > 1$ and μ, which satisfy

$$\|X(t,s)\| \leq Ke^{-\mu(t-s)}, \tag{3.13}$$

for $t \geq s$ [18].

For convenience, let introduce notations,

$$q_0 = \sup_{t\in\mathbb{R}} |q_0(t)|, p_0 = \sup_{t\in\mathbb{R}} |p_0(t)|, p_1 = \sup_{t\in\mathbb{R}} |p_1(t)|, q_1 = \sup_{t\in\mathbb{R}} |q_1(t)|,$$
$$r_0 = \sup_{t\in\mathbb{R}} |r_0(t)|, r_1 = \sup_{t\in\mathbb{R}} |r_1(t)|, F = \sup_{t\in\mathbb{R}} |F_0(t) + F_1(t)|.$$

Throughout the paper, the following additional conditions are required,

(C6) $\dfrac{K}{\mu}(H(p_1 + q_1) + (r_0 + r_1)H^3 + F) < H$;

(C7) $\dfrac{K}{\mu}(p_1 + q_1 + 3(r_0 + r_1)H^2) < 1.$

We will consider the system (3.11) in the matrix form

$$y' = A(t)y + B(t)y + C(t,y) + G(t), \tag{3.14}$$

where $y(t) = column(y_1(t), y_2(t))$,

$$A = \begin{pmatrix} 0 & 1 \\ -q_0(t) & -p_0(t) \end{pmatrix}, B(t) = \begin{pmatrix} 0 & 0 \\ -q_1(t) & -p_1(t) \end{pmatrix},$$
$$C(t,y) = \begin{pmatrix} 0 \\ -(r_0(t) + r_1(t))y_1^3 \end{pmatrix}, G(t) = \begin{pmatrix} 0 \\ (F_0(t) + F_1(t))cos(\lambda t) \end{pmatrix}.$$

Let us show the unpredictability of the function $G(t)$. Moreover, that the convergence and divergence sequences of the function are common with those for Markovian components. Fix a positive number ϵ, and a bounded interval $I \subset \mathbb{R}$. Duo to condition (C2), (C3), there exists a natural number n_1 such that

$$|\cos(\lambda(t + t_n)) - \cos(\lambda t)| < \frac{\epsilon}{2F},$$

for all $t \in \mathbb{R}$ and $n > n_1$. Besides it, there exists a natural number n_2 such that

$$|F_0(t + t_n) + F_1(t + t_n) - F_0(t) - F_1(t)| < \frac{\epsilon}{2}$$

for all $t \in I$ and $n > n_2$. Therefore, it is true that

$$\|G(t + t_n) - G(t)\| = |(F_0(t + t_n) + F_1(t + t_n)) \cos(\lambda(t + t_n)) -$$
$$(F_0(t) + F_1(t)) \cos(\lambda t)| \leq |F_0(t + t_n) + F_1(t + t_n)|| \cos(\lambda(t + t_n)) - \cos(\lambda t)| +$$
$$| \cos(\lambda t)||F_0(t + t_n) + F_1(t + t_n) - F_0(t) - F_1(t)| \leq F\frac{\epsilon}{2F} + \frac{\epsilon}{2} < \epsilon,$$

for all $t \in I$ and $n > \max(n_1, n_2)$. On the other hand, there exist positive numbers ϵ_0, σ and sequence u_n such that $|F_1(t + t_n) - F_1(t)| \geq \epsilon_0$ for each $t \in [s_n - \sigma, s_n + \sigma]$, $n \in \mathbb{N}$. Moreover, for sufficiently large number n one can attain that $|F_0(t + t_n) - F_0(t)| < \frac{\epsilon_0}{8}$ and $| \cos(\lambda(t + t_n)) - \cos(\lambda t)| < \frac{\epsilon_0}{8F}$, $t \in \mathbb{R}$. Applying conditions (C3),(C4), we obtain that $\cos(\lambda(s_n + t_n)) = 0 \ (mod\frac{\omega}{2})$ as $n \to \infty$. Hence, due to the uniform continuity of the cosine function, there exists positive number $\sigma_1 < \sigma$ such that $\min|\cos(\lambda(t+t_n))| > \frac{\epsilon_0}{2}$ for $t \in [s_n - \sigma_1, s_n + \sigma_1]$ and $n \to \infty$. This is why, we get that

$$\|G(t+t_n) - G(t)\| = |(F_0(t+t_n) + F_1(t+t_n))\cos(\lambda(t+t_n)) -$$
$$(F_0(t) + F_1(t))\cos(\lambda t)| = \Big|(F_1(t+t_n) - F_1(t))\cos(\lambda(t+t_n)) + (F_1(t) +$$
$$F_0(t)(\cos(\lambda(t+t_n)) - \cos(\lambda t)) + (F_0(t+t_n) - F_0(t))\cos(\lambda(t+t_n))\Big| \geq$$
$$|(F_1(t+t_n) - F_1(t))\cos(\lambda(t+t_n))| - |(F_1(t) + F_0(t))(\cos(\lambda(t+t_n)) - \cos(\lambda t))| -$$
$$|(F_0(t+t_n) - F_0(t))\cos(\lambda(t+t_n))| > \epsilon_0 \min|\cos(\lambda(t+t_n))| - F\frac{\epsilon_0}{8F} - \frac{\epsilon_0}{8} > \frac{\epsilon_0}{4},$$

for $t \in [s_n - \sigma_1, s_n + \sigma_1]$. Thus, the function $G(t)$ is unpredictable with sequences t_n, s_n, and positive numbers $\frac{\epsilon_0}{4}$, σ_1.

Condition (C5) implies that a bounded on the real axis function $z(t)$ is a solution of system (3.14) if and only if it satisfies the equation

$$z(t) = \int_{-\infty}^{t} X(t,s)[B(s)z(s) + C(s,z(s)) + G(s)]ds, \ t \in \mathbb{R}. \tag{3.15}$$

Denote by U the set of bounded and uniform continuous functions $v(t) = column(v_1(t), v_2(t))$, with common convergence sequence t_n such that $\|v(t)\|_0 < H$, where $\|v(t)\|_0 = \sup_{\mathbb{R}} \|v(t)\|$.

Define on U the operator Φ as

$$\Phi v(t) = \int_{-\infty}^{t} X(t,s)(B(s)v(s) + C(s,v(s)) + G(s))ds. \tag{3.16}$$

Lemma 3.1. *The operator Φ is invariant in U.*

Proof. Fix a function $v(t)$ that belongs to U. We have that

$$\|\Phi v(t)\| \leq \int_{-\infty}^{t} \|X(t,s)\|(\|B(s)\|\|v(s)\| + \|C(s,v(s))\| + \|G(s)\|)ds \leq$$
$$\frac{K}{\mu}((p_1+q_1)H + (r_0+r_1)H^3 + F)$$

for all $t \in \mathbb{R}$. Therefore, by the condition (C6) it is true that $\|\Phi v\|_0 < H$.

Next, the method of included intervals [4] will be utilized to prove invariance of Poisson stability in U. Let us show that $\|\Phi v(t+t_n) - \Phi v(t)\| \to 0$ on each bounded interval of $\mathbb{R}$. Fix an arbitrary positive number ϵ and a closed interval $[a,b]$, $-\infty < a < b < \infty$, of the real axis. Let us choose two numbers $c < a$, and $\xi > 0$ satisfying

$$\frac{K}{\mu}(H(p_1+q_1) + (r_0+r_1)H^3 + F)e^{-\mu(a-c)} < \frac{\epsilon}{4}, \tag{3.17}$$

$$\frac{K}{\mu}\xi(p_1+q_1+H+3(r_0+r_1)H^2+H^3+1)[1-e^{-\mu(b-c)}] < \frac{\epsilon}{2}. \tag{3.18}$$

Conditions (C3), (C4) imply that for sufficiently large n the following inequalities are valid $\|B(t+t_n) - B(t)\| < \xi$, $\|G(t+t_n) - G(t)\| < \xi$, $|r_0(t+t_n) + r_1(t+t_n) - r_0(t) - r_1(t)| < \xi$ and $\|v(t+t_n) - v(t)\| < \xi$ for $t \in [c,b]$. We obtain that

$$\|\Phi v(t+t_n)-\Phi v(t)\|=\|\int_{-\infty}^{t} X(t,s)(B(s+t_n)v(s+t_n)+C(s+t_n,v(s+t_n))+G(s+t_n))ds-$$
$$\int_{-\infty}^{t} X(t,s)(B(s)v(s)+C(s,v(s))+G(s))ds\| \le$$
$$\|\int_{-\infty}^{t} X(t,s)(B(s+t_n)v(s+t_n)-B(s)v(s)+C(s+t_n,v(s+t_n))-C(s,v(s))+$$
$$G(s+t_n)-G(s))ds\| \le \int_{-\infty}^{c} \|X(t,s)\|\Big(\|B(s+t_n)v(s+t_n)-B(s)v(s)\|+$$
$$\|C(s+t_n,v(s+t_n))-C(s,v(s))\|+\|G(s+t_n)-G(s)\|\Big)ds+$$
$$\int_{c}^{t} \|X(t,s)\|\Big(\|B(s+t_n)(v(s+t_n)-v(s))\|+\|v(s)(B(s+t_n)-B(s))\|\Big)ds+$$
$$\int_{c}^{t} \|X(t,s)\|\Big(\|C(s+t_n,v(s+t_n))-C(s+t_n,v(s))\|+$$
$$\|C(s+t_n,v(s))-C(s,v(s))\|\Big)ds+\int_{c}^{t} \|X(t,s)\|\|G(s+t_n)-G(s)\|ds \le$$
$$\frac{2K}{\mu}((p_1+q_1)H+(r_0+r_1)H^3+F)e^{-\mu(a-c)}+\frac{K}{\mu}(\xi(p_1+q_1)+H\xi)[1-e^{-\mu(b-c)}]+$$
$$\frac{K}{\mu}(3\xi(r_0+r_1)H^2+\xi H^3)[1-e^{-\mu(b-c)}]+\frac{K}{\mu}\xi[1-e^{-\mu(b-c)}],$$

is correct for all $t \in [a,b]$. From inequalities (3.17) and (3.18) it follows that $\|\Phi v(t+t_n)-\Phi v(t)\| < \epsilon$ for $t \in [a,b]$. Therefore, the sequence $\Phi v(t+t_n)$ uniformly converges to $\Phi v(t)$ on each bounded interval of $\mathbb{R}$.

The function $\Phi v(t)$ is a uniformly continuous, since its derivative is a uniformly bounded on the real axis. Thus, the set U is invariant for the operator Φ. □

Theorem 3.1. *The equation (1.3) with Markovian coefficients admits a unique exponentially stable unpredictable solution provided that the conditions (C1)-(C7) are valid. Moreover, the divergence and convergence sequences of the output stochastic dynamics are common with those, t_n and s_n, of the stochastic components of the coefficients.*

Proof. Let us prove completeness of the set U. Consider a Cauchy sequence $\phi^k(t)$ in U, which converges to a limit function $\phi(t)$ on $\mathbb{R}$. Fix a closed and bounded interval $I \subset \mathbb{R}$. We get that

$$\|\phi(t+t_n)-\phi(t)\| \le \|\phi(t+t_n)-\phi^k(t+t_n)\|+\|\phi^k(t+t_n)-\phi^k(t)\|+\|\phi^k(t)-\phi(t)\|.$$

One can choose sufficiently large n and k, such that each term on the right side of the last inequality is smaller than $\frac{\epsilon}{3}$ for an arbitrary $\epsilon > 0$ and $t \in I$. Thus, we conclude that the sequence $\phi(t+t_n)$ is uniformly converging to $\phi(t)$ on I. That is, the set U is complete.

Next, we shall show that the operator $\Phi : U \to U$ is a contraction. For any $\varphi(t)$, $\psi(t) \in U$, one can attain that

$$\|\Phi\varphi(t) - \Phi\psi(t)\| \le \int_{-\infty}^{t} \|X(t,s)\|(\|B(s)\|\|\varphi(s) - \psi(s)\| + \|C(s,\varphi(s)) - C(s,\psi(s))\|)ds \le$$
$$\frac{K}{\mu}\Big((p_1+q_1)\|\varphi(t)-\psi(t)\|_0 + (r_0+r_1)(|\varphi_1^2(t)| + |\varphi_1(t)||\psi_1(t)| + |\psi_1^2(t)|)\|\varphi(t)-\psi(t)\|_0\Big) <$$
$$\frac{K}{\mu}(p_1+q_1+3(r_0+r_1)H^2)\|\varphi(t)-\psi(t)\|_0.$$

Therefore, the inequality $\|\Phi\varphi - \Phi\psi\|_0 < \dfrac{K}{\mu}(p_1+q_1+3(r_0+r_1)H^2)\,\|\varphi-\psi\|_0$ holds, and according to the condition (C7) the operator $\Pi : U \to U$ is a contraction.

By the contraction mapping theorem there exists the unique fixed point, $z(t) \in U$ of the operator Φ, which is the unique solution of stochastic differential equation (1.3). In what follows, we will show that the solution $z(t)$ is unpredictable.

Applying the relations

$$z(t) = z(s_n) + \int_{s_n}^{t} A(s)z(s)ds + \int_{s_n}^{t} B(s)z(s)ds + \int_{s_n}^{t} C(s,z(s))ds + \int_{s_n}^{t} G(s)ds$$

and

$$z(t+t_n) = z(s_n+t_n) + \int_{s_n}^{t} A(s+t_n)z(s+t_n)ds + \int_{s_n}^{t} B(s+t_n)z(s+t_n)ds +$$
$$\int_{s_n}^{t} C(s+t_n, z(s+t_n))ds + \int_{s_n}^{t} G(s+t_n)ds$$

we obtain that

$$z(t+t_n) - z(t) = z(s_n+t_n) - z(s_n) + \int_{s_n}^{t} (A(s+t_n)z(s+t_n) - A(s)z(s))ds +$$
$$\int_{s_n}^{t} (B(s+t_n)z(s+t_n) - B(s)z(s))ds + \int_{s_n}^{t} (C(s+t_n, z(s+t_n)) - C(s,z(s)))ds +$$
$$\int_{s_n}^{t} (G(s+t_n) - G(s))ds.$$

Using conditions (C3), (C4) and uniform continuity of the entries of the matrix $A(t)$, periodic function $r_0(t)$ and solution $z(t)$, one can find a positive numbers σ_2 and integers l, k, n_0 such that the following inequalities are satisfied

$$\sigma_2 < \sigma_1; \tag{3.19}$$

$$\|A(t+t_n) - A(t)\| < \epsilon_0(\frac{1}{l} + \frac{2}{k}), \quad t \in \mathbb{R}, n > n_0; \tag{3.20}$$

$$|r_0(t+t_n) - r_0(t)| < \epsilon_0(\frac{1}{l} + \frac{2}{k}), \quad t \in \mathbb{R}, n > n_0; \tag{3.21}$$

$$\frac{2l\sigma_2}{3}\Big(\epsilon_0\Big[\frac{1}{4} - (p_1+q_1+\max(q_0+p_0,1) + H + 3H^2(r_0+r_1) + H^3)(\frac{1}{l}+\frac{2}{k})\Big] -$$
$$2(p_1+q_1)H - 2H^3 r_1\Big) > \epsilon_0; \tag{3.22}$$

$$\|z(t+s) - z(t)\| < \epsilon_0 \min(\frac{1}{k}, \frac{1}{4l}), \quad t \in \mathbb{R}, |s| < \sigma_2. \tag{3.23}$$

Let the numbers σ_2, l and k as well as numbers $n \in \mathbb{N}$, be fixed. Consider the following two alternatives: (i) $\|z(s_n+t_n) - z(s_n)\| < \epsilon_0/l$; (ii) $\|z(s_n+t_n) - z(s_n)\| \geq \epsilon_0/l$.

(i) Using (3.23) one can show that

$$\|z(t+t_n) - z(t_n)\| \leq \|z(t+t_n) - z(s_n+t_n)\| + \|z(s_n+t_n) - z(s_n)\| + \|z(s_n) - z(t)\| < \frac{\epsilon_0}{l} + \frac{\epsilon_0}{k} + \frac{\epsilon_0}{k} = \epsilon_0(\frac{1}{l} + \frac{2}{k}), \tag{3.24}$$

if $t \in [s_n, s_n + \sigma_2]$.

Therefore, the inequalities (3.19)-(3.24) imply that

$$\|z(t+t_n) - z(t)\| \geq \int_{s_n}^{t} \|G(s+t_n) - G(s)\|ds - \|z(s_n+t_n) - z(s_n)\| - \int_{s_n}^{t} \|B(s+t_n)z(s+t_n) - B(s)z(s)\|ds - \int_{s_n}^{t} \|A(s+t_n)z(s+t_n) - A(s)z(s)\|ds - \int_{s_n}^{t}\|C(s+t_n, z(s+t_n)) - C(s, z(s))\|ds \geq \int_{s_n}^{t} \|G(s+t_n) - G(s)\|ds - \|z(s_n+t_n) - z(s_n)\| - \int_{s_n}^{t} \|B(s+t_n) - B(s)\|\|z(s+t_n)\|ds - \int_{s_n}^{t} \|B(s)\|\|z(s+t_n) - z(s)\|ds - \int_{s_n}^{t} \|A(s+t_n) - A(s)\|\|z(s+t_n)\|ds - \int_{s_n}^{t} \|A(s)\|\|z(s+t_n) - z(s)\|ds - \int_{s_n}^{t} |r_0(s+t_n)z_1^3(s+t_n) - r_0(s+t_n)z_1^3(s))|ds - \int_{s_n}^{t} |r_0(s+t_n)z_1^3(s) - r_0(s)z_1^3(s))|ds - \int_{s_n}^{t} |r_1(s+t_n)z_1^3(s+t_n)) - r_1(s+t_n)z_1^3(s))|ds - \int_{s_n}^{t} |r_1(s+t_n)z_1^3(s) - r_1(s)z_1^3(s)|ds \geq \sigma_2\frac{\epsilon_0}{4} - \frac{\epsilon_0}{l} - 2\sigma_2(p_1+q_1)H - \sigma_2(p_1+q_1)\epsilon_0(\frac{1}{l} + \frac{2}{k}) - \sigma_2\epsilon_0(\frac{1}{l} + \frac{2}{k})H - \sigma_2 \max(q_0+p_0, 1)\epsilon_0(\frac{1}{l} + \frac{2}{k}) - 3\sigma_2 r_0 H^2\epsilon_0(\frac{1}{l} + \frac{2}{k}) - \sigma_2\epsilon_0(\frac{1}{l} + \frac{2}{k})H^3 - 3\sigma_2 r_1 H^2 \epsilon_0(\frac{1}{l} + \frac{2}{k}) - 2\sigma_2 r_1 H^3 > \frac{\epsilon_0}{2l}$$

for $t \in [s_n, s_n + \sigma_2]$.

(ii) If $|z(t_n+s_n) - z(s_n)| \geq \epsilon_0/l$ it is not difficult to find that (3.23) implies

$$\|z(t+t_n) - z(t)\| \geq \|z(t_n+s_n) - z(s_n)\| - \|z(s_n) - z(t)\| - \|z(t+t_n) - z(t_n+s_n)\| \geq \frac{\epsilon_0}{k} - \frac{\epsilon_0}{l} - \frac{\epsilon_0}{4l} = \frac{\epsilon_0}{2l}, \tag{3.25}$$

for $t \in [s_n - \sigma_2, s_n + \sigma_2]$ and $n \in \mathbb{N}$. Thus, it can be conclude that $z(t)$ is unpredictable solution with sequences t_n, s_n and positive numbers $\frac{\sigma_2}{2}$, $\frac{\epsilon_0}{2l}$.

Finally, let us discuss the exponential stability of the solution $z(t)$. It is true that

$$z(t) = X(t, t_0)z(t_0) + \int_{t_0}^{t} X(t,s)(B(s)z(s) + C(s, z(s)) + G(s))ds.$$

Denote by $\bar{z}(t)$ another solution of equation (1.3) such that

$$\bar{z}(t) = X(t, t_0)\bar{z}(t_0) + \int_{t_0}^{t} X(t,s)(B(s)\bar{z}(s) + C(s, \bar{z}(s)) + G(s))ds.$$

Making use of the relation

$$\bar{z}(t) - z(t) = X(t,t_0)(\bar{z}(t_0) - z(t_0)) + \int_{t_0}^{t} X(t,s)\Big(B(s)(\bar{z}(s) - z(s)) + C(s,\bar{z}(s)) - C(s,z(s))\Big)ds,$$

one can obtain

$$\begin{aligned} &\|\bar{z}(t) - z(t)\| \le \|X(t,t_0)\|\|\bar{z}(t_0) - z(t_0)\| + \\ &\int_{t_0}^{t} \|X(t,s)\|(\|B(s)\|\|\bar{z}(s) - z(s)\| + \|C(s,\bar{z}(s)) - C(s,z(s))\|)ds \le \\ &Ke^{-\mu(t-t_0)}\|\bar{z}(t_0) - z(t_0)\| + \\ &\int_{t_0}^{t} Ke^{-\mu(t-s)}\Big((p_1+q_1)\|\bar{z}(s) - z(s)\| + (r_0+r_1)(|\bar{z}_1^2(s)| + |\bar{z}_1(s)||z_1(t)| + \\ &|z_1^2(s)|)\|\bar{z}(s) - z(s)\|\Big)ds \le \frac{K}{\mu}(p_1+q_1+3(r_0+r_1)H^2)\|\bar{z}(t) - z(t)\|, \end{aligned} \tag{3.26}$$

for $t \in \mathbb{R}$. With the aid of the Gronwall-Bellman Lemma, one can verify that

$$\|\bar{z}(t) - z(t)\| \le Ke^{(K(p_1+q_1+3(r_0+r_1)H^2)-\mu)(t-t_0)}\|\bar{z}(t_0) - z(t_0)\|, \tag{3.27}$$

for all $t \ge t_0$, and condition (C7) implies that the unpredictable solution, $z(t)$, is exponentially stable solution of equation (1.3). The theorem is proved. □

The following section provides an example to confirm the theoretical results by using numerical simulations. It illustrates various unpredictable dynamics of the stochastic equation of Duffing type (1.3) for different contributions of periodic and non-periodic components of coefficients.

4. A NUMERICAL EXAMPLE AND DISCUSSIONS

Below, to visualize the exponentially stable unpredictable solution of $\Theta-$type and determine dynamics of Markov coefficients we shall apply solutions $\phi_p(t)$, $\phi_p(0) = 0.5$, $\phi_q(t)$, $\phi_q(0) = 0.6$, $\phi_r(t)$, $\phi_r(0) = 0.4$, and $\phi_F(t)$, $\phi_F(0) = 0.3$, of the dissipative equations (2.7)-(2.10), where $\alpha_p = -5, \alpha_q = -3, \alpha_r = -2, \alpha_F = -4$, and $p(t) = q(t) = r(t) = f(t) = \rho(t)$. The piecewise constant function $\rho(t)$, is constructed by Markov chain with values over intervals $[hn, h(n+1))$, $n \in \mathbb{N}$, and described in Section 2.2.

a) b)

FIGURE 4. The time series of the coordinates and trajectory of the solution $x(t)$ of equation (4.28), with Markovian components obtained for $h = 0.1\pi$. The stochastic influence is strong, since of the time step of the Markov chain is smaller than the period.

Consider the following stochastic Duffing equation

$$x''(t) + (p_0(t) + p_1(t))x'(t) + (q_0(t) + q_1(t)) + (r_0(t) + r_1(t))x^3(t) = (F_0(t) + F_1(t))\cos(\lambda t), \tag{4.28}$$

where $\lambda = 1$, $p_0(t) = 2 - 0.3\sin(4t)$, $q_0(t) = 2 - 0.2\cos(2t)$, $r_0(t) = 0.04\cos(4t)$, $F_0(t) = 0.05\sin(8t)$, $p_1(t) = -0.4\phi_p(t)$, $q_1(t) = 0.1\phi_q(t)$, $r_1(t) = 0.02\phi_r(t)$ and $F_1(t) = 0.02\phi_F(t)$. The periodic functions $p_0(t)$, $q_0(t)$, $r_0(t)$ and $F_0(t)$ with common period $\omega = 2\pi$. All conditions from (C1) to (C7) are hold with $K = 1.5$, $\mu = 2\pi$, $p_0 = 2.3$, $q_0 = 2.2$, $r_0 = 0.04$, $p_1 = 0.48$, $q_1 = 0.2$, $r_1 = 0.06$, $F = 0.03$ and $H = 0.025$. According to Theorem 3.1, the equation (4.28) admits a unique exponentially stable unpredictable solution. In Figures 4-6, the graphs of the coordinates and trajectories of solutions $x(t)$ of equation (4.28) with $h = 0.2\pi, 2\pi, 8\pi$, and initial values $x_1(0) = x_2(0) = 0$ are shown. The solutions $x(t)$ exponentially approach the unpredictable solutions, $z(t)$, as time increases.

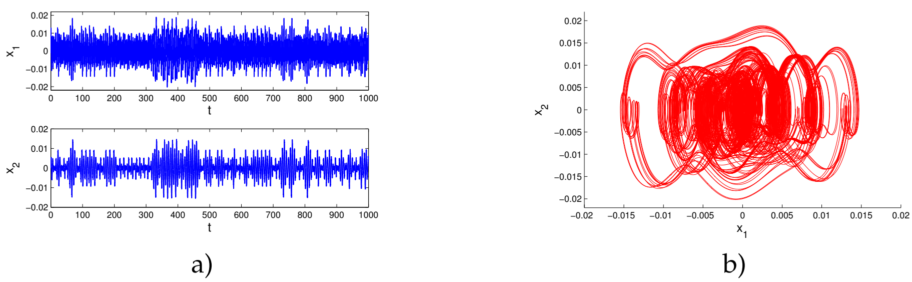


FIGURE 5. The x_1, x_2- coordinates and trajectory of the solution of (4.28), with Markovian components obtained for $h = 2\pi$. That is, the value of time steps is equal to the period 2π and our simulations show that the periodicity still cannot be seen clearly in this case.

The Theorem 3.1 can be interpreted as a result on response-driver synchronization [16] of the unpredictability in the stochastic system (2.7)-(2.10) and the stochastic Duffing equation (1.3). That is, the theorem claims, in particular, that the unpredictable solution $(p_1(t), q_1(t), r_1(t))$ of the system and the unpredictable solution, $z(t)$, admit common sequences of convergence and divergence. Delta synchronization of the unpredictability for gas discharge-semiconductor systems is considered in [7].

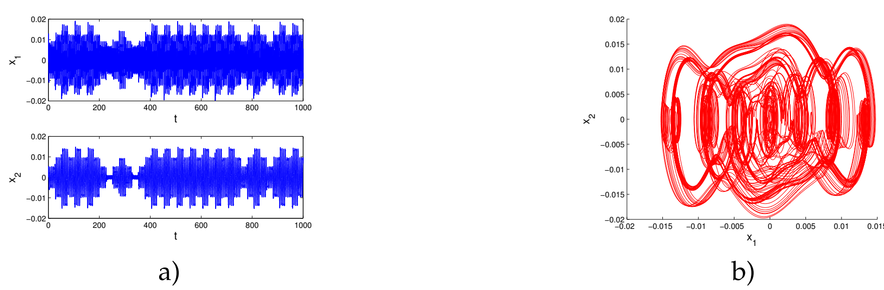


FIGURE 6. The graphs of coordinates and trajectory of the solution $x(t)$ of equation (4.28), where Markovian components obtained for $h = 8\pi$. One can see that several intervals of periodicity are placed in one step of the constancy.

We consider the various simulations for the model, since they are with different steps h of the Markov function $\rho(t)$. The choice makes qualitative difference in the stochastic dynamics. If the step is in the range $h \leq 2\pi$, Figures 4 and 5, the behavior is strongly irregular, and there is no any indication of periodicity. For $h > 2\pi$, Figure 6, one can observe that periodicity is seen locally in time, and phenomenon of intermittency [27] appears. Thus, our results demonstrate not only quantitative asymptotic characteristics, but also possibility to learn reasons for different phenomena of chaos. Possibly, the simulations may give lights on the origins of intermittency. Additionally, for $h > 2\pi$, Figure 6, the effect of local periodicity is seen in the "symmetry" of the phase portraits, which is reasoned also by the finite values of the state space. For the values of h less than 2π, Figure 4, any symmetry cannot be seen, since the stochastic dynamics dominates significantly.

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