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Solving split inverse problems

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ABSTRACT. In this paper, we study several classes of split inverse problems. We start with the split common fixed point problem with multiple output sets for multivalued demicontractive mappings. A relaxed inertial iterative method for solving this problem is presented and analysed. Furthermore, the method is applied to a system of split variational inequalities, system of split equilibrium problem and other related split problems. Several numerical experiments illustrate and validate the applicability of our proposed method.

1. INTRODUCTION

The *split inverse problem* (SIP) is a mathematical model that allows much flexibility and thus attracts mush interest since its introductory, see [3, 7, 8, 13, 17, 32, 42]. It was applied successfully in many fields, such as in signal processing, phase retrieval, image recovery, data compression, intensity-modulated radiation therapy, e.g. see [9, 10]). The SIP model is formulated as follows:

$$\text{Find } \hat{x} \in H_1 \quad \text{that solves IP}_1 \tag{1.1}$$

such that

$$\hat{y} := A\hat{x} \in H_2 \quad \text{solves IP}_2, \tag{1.2}$$

where H_1 and H_2 are real Hilbert spaces, IP_1 denotes an inverse problem formulated in H_1 and IP_2 denotes an inverse problem formulated in H_2, and $A : H_1 \to H_2$ is a bounded linear operator.

In 1994, Censor and Elfving in [10] introduced the first instance of the SIP called the *split convex feasibility problem* (SCFP) for modelling inverse problems that arise from medical image reconstruction. The SCFP finds application in control theory, approximation theory, signal processing, geophysics, communications, biomedical engineering, etc. [9, 25]. Let C and Q be nonempty, closed and convex subsets of Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. The SCFP is defined as follows:

$$\text{Find } \hat{x} \in C \text{ such that } \hat{y} = A\hat{x} \in Q. \tag{1.3}$$

Researchers have developed and studied several iterative methods for solving the SCFP (1.3) and related optimization problems in Hilbert and Banach spaces, (see, e.g. [9, 23, 25, 28, 41] and the references therein).

Let H be a Hilbert space and $T : H \to H$ a mapping. We denote the fixed point set of T by $F(T)$; $F(T) := \{x \in H : Tx = x\}$. The *Fixed Point Problem* has application in several areas, such as nonlinear optimization and split inverse problems and so on, (see [4, 33]).

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One of the important generalizations of the SCFP is the *split common fixed point problem* (SCFPP), which was first introduced and studied by Censor and Segal [11] in the Euclidean spaces. Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator and $S^k : H_1 \to H_1 (k = 1, 2, \dots, t)$ and $T^j : H_2 \to H_2 (j = 1, 2, \dots, s)$ be mappings with fixed point sets $F(S^k)$ and $F(T^j)$, respectively. The SCFPP is formulated as follows:

$$\text{Find} \quad \hat{x} \in \cap_{k=1}^{t} F(S^k) \quad \text{such that} \quad \hat{y} = A\hat{x} \in \cap_{j=1}^{s} F(T^j). \tag{1.4}$$

When $t = s = 1$, the SCFPP (1.4) is reduced to the *split fixed point problem* (SFPP), that is,

$$\text{Find} \quad \hat{x} \in F(S^1) \quad \text{such that} \quad \hat{y} = A\hat{x} \in F(T^1). \tag{1.5}$$

The SCFPP also includes several other optimization problems as special cases, such as the split equilibrium problem (SEP), the split variational inequality problem (SVIP), the split common null point problem (SCNPP) and the split monotone variational inclusion problem (SMVIP) (e.g., see [20, 29, 35, 40]). Several iterative methods have been introduced and studied by researchers for approximating the solutions of the SCFPP for various classes of mappings (e.g., see [5, 21, 24, 39, 45]).

Very recently, Jailoka and Suantai [22] studied the SFPP (1.5) for the class of multivalued demicontractive mappings. They proposed a new iterative method, which combines the viscosity technique with the self-adaptive step size strategy for approximating the solution of the problem in the framework of Hilbert spaces. Moreover, they proved a strong convergence theorem for the proposed algorithm without prior knowledge of the operator norm.

More recently, Reich *et al.* [36] introduced and studied the concept of *split common fixed point problem with multiple output sets* (SCFPPMOS). Let $H, H_i, i = 1, 2, \dots, N$ be real Hilbert spaces and $A_i : H \to H_i,\ i = 1, 2, ..., N$, be bounded linear operators. Let $S^k : H \to H,\ k = 1, 2, \dots, L,\ T_i^j : H_i \to H_i,\ j = 1, 2, ..., M$ be nonexpansive mappings. The SCFPPMOS is formulated as follows: Find a point $u^\dagger \in H$ such that

$$u^\dagger \in \Gamma := \cap_{k=1}^{L} F(S^k) \cap \Big(\cap_{i=1}^{N} A_i^{-1} \big(\cap_{j=1}^{M} F(T_i^j) \big) \Big) \neq \emptyset. \tag{1.6}$$

Moreover, the authors proposed the following viscosity iterative method for approximating the solution of SCFPPMOS (1.6) for the class of single-valued nonexpansive mappings in the framework of Hilbert spaces:

Algorithm 1.1.

Step 0. *For any $x_0 \in H$, let the sequence $\{x_n\}$ be defined as follows:*

Step 1. *Compute*

$$y_n^k = S^k x_n \quad \textit{for all } k = 1, 2, \dots, L \textit{ and let}$$
$$d_n = \max_{k=1,2\dots,L} \{\|y_n^k - x_n\|\},$$
$$D_n = \{k \in \{1, 2, \dots, L\} : \|y_n^k - x_n\| = d_n\}.$$

Step 2. *Compute*

$$z_{n,i}^j = T_i^j(A_i x_n) \quad \textit{for all } i = 1, 2, \dots, N \textit{ and } j = 1, 2, \dots, M, \textit{ and let}$$
$$d_{n,i} = \max_{j=1,2,\dots,M} \{\|z_{n,i}^j - A_i x_n\|\}, \quad i = 1, 2, \dots, N,$$
$$D_{n,i} = \{j \in \{1, 2, \dots, M\} : \|z_{n,i}^j - A_i x_n\| = d_{n,i}\},\ i = 1, 2, \dots, N.$$

Step 3. *Let* $f_n := \max\{d_n,\ \max_{i=1,2,\ldots,N}\{d_{n,i}\}\}$.
If $d_n = f_n$, *then choose* $k_n \in D_n$ *and let* $t_n = y_{n,k_n}$, *and let* $\Theta = I$.
Else, choose $j_n \in D_{n,i_n}$, *where* $d_{n,i_n} = f_n$, *and let* $t_n = z^{j_n}_{n,i_n}$ *and* $\Theta = A_{i_n}$.

Step 4. *Compute*

$$u_n = x_n - \delta_n \Theta^*(\Theta x_n - t_n), \quad \textit{where}$$
$$\delta_n = \rho_n \frac{\|\Theta x_n - t_n\|^2}{\|\Theta^*(\Theta x_n - t_n)\|^2 + a_n},$$

$\{\rho_n\} \subset [c,d] \subset (0,1)$ *and* $\{a_n\}$ *is a sequence of positive real numbers.*

Step 5. *Compute*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)u_n, \quad n \geq 0,$$

where $\{\alpha_n\} \subset (0,1)$ *and* $f : H \to H$ *is a contraction with coefficient* $c \in [0,1)$.

Under certain conditions on the control parameters, the authors obtained strong convergence result for the proposed Algorithm 1.1. However, we need to point out that the proposed Algorithm 1.1 has some drawbacks. First, we observe that the algorithm requires solving several maximum distance problems per iteration, which will result in huge cost of implementation of the proposed method. Moreover, the proposed algorithm is only applicable for the class of single-valued nonexpansive mappings.

In recent times, developing algorithms with high rate of convergence for solving optimization problems has become of great interest to researchers. There are generally two important techniques employed by authors to improve the rate of convergence of iterative methods, which are the *inertial technique* and the *relaxation technique*. The inertial algorithm is based on a discrete version of the second-order dissipative dynamical system, which was first introduced by Polyak [34]. The main feature of the inertial-algorithm is that it uses the previous two iterates to generate the next iterate. It is worthy of note that this small change can greatly improve the rate of convergence of an iterative method (e.g., see [2, 1, 12, 14, 15, 30, 31, 44]). The relaxation method is another popular method authors employ to improve the speed of iterative algorithms (see, e.g. [6, 18]). Both techniques naturally arise from an explicit time discretization of a dynamical system (see, e.g., [6, 46]).

In view of the above discourse, it is natural to ask the following research questions:

Is it possible to construct a new iterative method for approximating the solution of SCFPPMOS such that the algorithm does not involve any maximum distance problem? Can the algorithm be designed such that it solves the SCFPPMOS for a larger class of mappings than the single-valued nonexpansive mappings?

In this study, we provide affirmative answers to the above questions. More precisely, we propose a new relaxed inertial iterative method that does not involve any maximum distance problem for approximating the solution of SCFPPMOS for a class of multivalued demicontractive mappings. Mathematically, the problem we consider is formulated as follows: Let $H, H_i, i = 1,2,\ldots,N$ be real Hilbert spaces and $A_i : H \to H_i,\ i = 1,2,...,N,$ be bounded linear operators. Let $T^j : H \to H,\ j = 1,2,\ldots,M,\ T^j_i : H_i \to H_i,\ i = 1,2,...,N,\ j = 1,2,...,M$ be multivalued demicontractive mappings. We study the following SCFPPMOS: Find an element $u^\dagger \in H$ such that

$$u^\dagger \in \Omega := \cap_{j=1}^M F(T^J) \cap \Big(\cap_{i=1}^N A_i^{-1}\big(\cap_{j=1}^M F(T^j_i)\big)\Big) \neq \emptyset. \tag{1.7}$$

Under some mild conditions on the control parameters and without the knowledge of the operators' norms, we obtain strong convergence result for our proposed method. Moreover, we apply our result to study and to approximate the solutions of certain classes of split inverse problems. Our proposed method employs the relaxation and inertial techniques with a very efficient self-adaptive step size method to further improve its rate of convergence. Finally, we present several numerical experiments to demonstrate the implementability and computational advantage of our proposed method.

The paper is outlined as follows: In Section 2, some definitions and results employed in the convergence analysis are recalled. In Section 3, the proposed method is presented and in Section 4 we analyze the convergence of the proposed method. In Section 5, our result is applied to study certain classes of split inverse problems while in Section 6 we present several numerical experiments with graphical illustrations. Finally, we give some concluding remarks in Section 7.

2. Preliminaries

Definition 2.1. Let H be a real Hilbert space. A mapping $T : H \to H$ is said to be

(1) *L- Lipschitz continuous* on H, if there exists a constant $L > 0$ such that
$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in H.$$
If $L \in [0, 1)$, then T is called a *contraction* with coefficient L.

(2) *nonexpansive* on H, if T is 1-Lipschitz continuous.

(3) *averaged*, if it can be written as
$$T = (1 - \alpha)I + \alpha S,$$
where $\alpha \in (0, 1)$, $S : H \to H$ is nonexpansive and I is the identity mapping on H.

(4) *monotone* on H, if
$$\langle Tx - Ty, x - y\rangle \geq 0, \qquad \forall x, y \in H.$$

(5) *k-inverse strongly monotone* (k-ism) on H, if there exists a constant $k > 0$ such that
$$\langle Tx - Ty, x - y\rangle \geq k\|Tx - Ty\|^2, \quad \forall x, y \in H.$$

(6) *firmly nonexpansive* on H, if
$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \qquad \forall x, y \in H.$$

Definition 2.2. A subset K of H is called proximinal if for each $x \in H$, there exists $y \in K$ such that
$$\|x - y\| = d(x, K) = \inf\{\|x - z\| : z \in K\}$$
In this study, we denote the families of all nonempty closed bounded subsets, nonempty closed convex subsets, nonempty compact subsets, and nonempty proximinal bounded subsets of H by $CB(H)$, $CC(H)$, $KC(H)$, and $P(H)$, respectively.

The *Pompeiu-Hausdorff* metric on $CB(H)$ is defined by
$$H(A, B) := \max\Big\{\sup_{x\in A} d(x, B),\ \sup_{y\in B}(y, A)\Big\}.$$
for all $A, B \in CB(H)$, where $d(x, B) = \inf_{b\in B} \|x - b\|$.

Let $T : H \to 2^H$ be a multivalued mapping. We say that T satisfies the endpoint condition if $Tp = \{p\}$ for all $p \in F(T)$. For multivalued mappings $T_i : H \to 2^H (i \in \mathbb{N})$ with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$, we say T_i satisfies the common endpoint condition if $T_i(p) = \{p\}$ for all $i \in \mathbb{N}, p \in \cap_{i=1}^{\infty} F(T_i)$. Next, we recall some definitions on multivalued mappings.

Definition 2.3. A multivalued mapping $T : H \to CB(H)$ is said to be

(i) *nonexpansive*, if
$$H(Tx, Ty) \leq ||x - y||, \text{ for all } x, y \in H,$$

(ii) *quasi-nonexpansive*, if $F(T) \neq \emptyset$ and

$$H(Tx, Tp) \leq ||x - p||, \text{ for all } x \in H, p \in F(T),$$

(iii) *k-demicontractive* for $0 \leq k < 1$, if $F(T) \neq \emptyset$ and

$$H(Tx, Tp)^2 \leq ||x - p||^2 + kd(x, Tx)^2, \text{ for all } x \in H, p \in F(T).$$

We note that the class of k-demicontractive mappings includes several other types of classes of nonlinear mappings such as nonexpansive and quasi-nonexpansive mappings.

Definition 2.4. Let $T : H \to CB(H)$ be a multivalued mapping. The multivalued mapping $I - T$ is said to be demiclosed at zero if for any sequence $\{x_n\} \subset H$ which converges weakly to q and the sequence $\{||x_n - u_n||\}$ converges strongly to 0, where $u_n \in Tx_n$, then $q \in F(T)$.

We present the following useful results on multivalued demicontractive mappings.

Lemma 2.1. [21] *Let $A : H \to H$ be a bounded linear operator, and suppose $S : H \to CB(H)$ and $T : H \to CB(H)$ are two multivalued demicontractive mappings. Let $\Gamma := F(S) \cap A^{-1}(F(T)) \neq \emptyset$. Then, we have*

(i) *Γ is closed;*

(ii) *if $Sp = \{p\}$ and $T(Ap) = \{Ap\}$ for all $p \in \Gamma$, then Γ is convex.*

Lemma 2.2. [21] *Let H be a Hilbert space and $T : H \to CB(H)$ be a k-demicontractive multivalued mapping. If $p \in F(T)$ such that $Tp = \{p\}$, then the following inequalities hold for all $x \in H, y \in Sx$:*

(i.) $\langle x - y, p - y\rangle \leq \frac{1+k}{2}\|x - y\|^2$;

(ii.) $\langle x - y, x - p\rangle \geq \frac{1-k}{2}\|x - y\|^2$.

Definition 2.5. Let H be a real Hilbert space. A function $c : H \to \mathbb{R} \cup \{+\infty\}$ is said to be weakly lower semi-continuous (w-lsc) at $x \in H$, if

$$c(x) \leq \liminf_{n\to\infty} c(x_n)$$

holds for every sequence $\{x_n\}$ in H satisfying $x_n \rightharpoonup x$.

Definition 2.6. Let H be a real Hilbert space and let $B : H \to 2^H$ be a multivalued operator. The *effective domain* of B denoted by dom(B) is given as dom$(B) = \{x \in H : Bx \neq \emptyset\}$. The multivalued operator $B : H \to 2^H$ is said to be

- *monotone*, if

$$\langle u - v, x - y\rangle \geq 0 \text{ for all } u \in B(x), v \in B(y).$$

- *maximal monotone*, if the graph Gr(B) of B,

$$\text{Gr}(B) := \{(x, u) \in H \times H | u \in B(x)\},$$

is not properly contained in the graph of any other monotone mapping. In other words, B is maximal if and only if for $x \in$ dom(B) and $u \in Bx$ such that $\langle u - v, x - y\rangle \geq 0$ implies $(y, v) \in$ Gr(B).

Lemma 2.3. [16, 19] *Let H be a real Hilbert space. Then the following results hold for all $x, y \in H$ and $\delta \in (0, 1)$:*

(i) $||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle$;

(ii) $||x + y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2$;

(iii) $||\delta x + (1 - \delta)y||^2 = \delta||x||^2 + (1 - \delta)||y||^2 - \delta(1 - \delta)||x - y||^2$.

Lemma 2.4. ([38]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence in $(0,1)$ with $\sum_{n=1}^{\infty}\alpha_n=\infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1}\leq(1-\alpha_n)a_n+\alpha_n b_n \quad \textit{for all } n\geq 1.$$

If $\limsup_{k\to\infty} b_{n_k}\leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_{k+1}}-a_{n_k})\geq 0$, then $\lim_{n\to\infty} a_n=0$.

3. MAIN RESULTS

In this section, we present our proposed algorithm for solving the SCFPPMOS (1.7). We establish our strong convergent result under the following conditions:
Let $H, H_i, i=1,2,...,N$, be real Hilbert spaces and let $A_i: H\to H_i$ be bounded linear operators with adjoints A_i^*. Let $T^j: H\to H$ and $T_i^j: H_i\to H_i, j=1,2,...,M$, be multivalued k^j-demicontractive and k_i^j-demicontractive mappings, respectively such that $T^j(p)=\{p\}, T_i^j(A_ip)=\{A_ip\}$ for all $p\in\Omega$. Moreover, we require that the control parameters satisfy the following conditions.
Assumption A:

- (A1) $\{\alpha_n\}\subset(0,1), \lim_{n\to\infty}\alpha_n=0, \sum_{n=1}^{\infty}\alpha_n=+\infty, \lim_{n\to\infty}\frac{\epsilon_n}{\alpha_n}=0, \{\xi_n\}\subset[a,b]\subset(0,1), \theta>0$;
- (A2) $0<\phi_i^j<\phi_i'^j<1-k_i^j, \{\phi_{n,i}^j\}\subset\mathbb{R}_+, \lim_{n\to\infty}\phi_{n,i}^j=0, 0<a_i\leq\delta_{n,i}\leq b_i<1, 0<c_i^j\leq\beta_{n,i}^j,\leq d_i^j<1$,
  $\sum_{i=0}^{N}\delta_{n,i}=1$ and $\sum_{j=1}^{M}\beta_{n,i}^j=1$ for each $i=0,1,\ldots,N$.

Now, the algorithm is presented as follows:

Algorithm 3.2.

Step 0. *Select initial data $x_0,x_1\in H$. Let $H_0=H, T_0^j=T^j, A_0=I^H$ and set $n=1$.*
Step 1. *Given the $(n-1)$th and nth iterates, choose θ_n such that $0\leq\theta_n\leq\hat{\theta}_n$ with $\hat{\theta}_n$ defined by*

$$\hat{\theta}_n=\begin{cases}\min\left\{\theta,\ \frac{\epsilon_n}{\|x_n-x_{n-1}\|}\right\}, & \textit{if } x_n\neq x_{n-1},\\ \theta, & \textit{otherwise.}\end{cases} \tag{3.8}$$

Step 2. *Compute*

$$w_n=(1-\alpha_n)(x_n+\theta_n(x_n-x_{n-1})).$$

Step 3. *Compute*

$$y_n=\sum_{i=0}^{N}\delta_{n,i}\sum_{j=1}^{M}\beta_{n,i}^j\big(w_n-\gamma_{n,i}^jA_i^*(A_iw_n-v_{n,i}^j)\big),$$

where $v_{n,i}^j\in T_i^j(A_iw_n)$ and

$$\gamma_{n,i}^j=\begin{cases}\frac{(\phi_{n,i}^j+\phi_i^j)\|A_iw_n-v_{n,i}^j\|^2}{\|A_i^*(A_iw_n-v_{n,i}^j)\|^2}, & \textit{if } \ A_iw_n\neq v_{n,i}^j,\\ 0, & \textit{otherwise.}\end{cases} \tag{3.9}$$

Step 4. *Compute*

$$x_{n+1}=\xi_nw_n+(1-\xi_n)y_n.$$

Set $n:=n+1$ and return to ***Step 1****.*

Remark 3.1. We highlight below some of the features of our proposed method.

- Unlike the result in [36] (Algorithm 1.1), we observe that our proposed algorithm has a very simple structure and does not involve any maximum distance problem.
- Our proposed algorithm does not require knowledge of the operators' norms for its implementation, rather it uses a simple but very efficient self-adaptive step size technique. Some of the control parameters are relaxed to enlarge the range of values of the step sizes for the algorithm.
- The method combines the relaxation and the inertial techniques to speed up its rate of convergence.
- Our method solves SCFPPMOS for a larger class of mappings (multivalued demi-contractive mappings) than the result in [36] (Algorithm 1.1).
- The sequence generated by our proposed method converges strongly to the minimum-norm solution of the SCFPPMOS (1.7). In several practical problems, finding the minimum-norm solution of a problem is desirable and useful.

Remark 3.2. By condition (A1), it follows from (3.8) that

$$\lim_{n\to\infty} \theta_n ||x_n - x_{n-1}|| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\theta_n}{\alpha_n} ||x_n - x_{n-1}|| = 0.$$

4. CONVERGENCE ANALYSIS

First, we establish some lemmas required to prove the strong convergence theorem for the proposed algorithm.

Lemma 4.5. *The step size $\gamma_{n,i}^j$ defined by* (3.9) *is well defined for each $i = 0, 1, 2, \ldots, N,\ j = 1, 2, \ldots, M$.*

Proof. Let $p \in \Omega$, then $A_i p \in \cap_{j=1}^M F(T_i^j),\ i = 0, 1, 2, \ldots, N$. If $A_i w_n \neq v_{n,i}^j$, we claim that $A_i^*(A_i w_n - v_{n,i}^j) \neq 0$. Now, we prove by contradiction by supposing $A_i^*(A_i w_n - v_{n,i}^j) = 0$. Then, by Lemma 2.2(ii) we have

$$\begin{aligned} \|A_i w_n - v_{n,i}^j\|^2 &\leq \frac{2}{1-k_i^j} \langle A_i w_n - v_{n,i}^j, A_i w_n - A_i p\rangle \\ &= \frac{2}{1-k_i^j} \langle A_i^*(A_i w_n - v_{n,i}^j), w_n - p\rangle = 0, \end{aligned}$$

which implies that $A_i w_n = v_{n,i}^j$, $i = 0, 1, 2, \ldots, N, j = 1, 2, \ldots, M$. This is a contradiction. Hence, $A_i^*(A_i w_n - v_{n,i}^j) \neq 0$.

□

Lemma 4.6. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2 under Assumption A. Then $\{x_n\}$ is bounded.*

Proof. Let $p \in \Omega$. Then, from the definition of w_n and by applying the triangle inequality we have

$$\begin{aligned} \|w_n - p\| &= \|(1-\alpha_n)(x_n + \theta_n(x_n - x_{n-1})) - p\| \\ &= \|(1-\alpha_n)(x_n - p) + (1-\alpha_n)\theta_n(x_n - x_{n-1}) - \alpha_n p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + (1-\alpha_n)\theta_n\|x_n - x_{n-1}\| + \alpha_n\|p\| \\ &= (1-\alpha_n)\|x_n - p\| + \alpha_n\Big[(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \|p\|\Big]. \end{aligned}$$

By Remark (3.2), we have

$$\lim_{n\to\infty}\Big[(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+\|p\|\Big]=\|p\|.$$

Thus, there exists $M_1>0$ such that $(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+\|p\|\le M_1$ for all $n\in\mathbb{N}$. It follows that

$$\|w_n-p\|\le(1-\alpha_n)\|x_n-p\|+\alpha_n M_1. \tag{4.10}$$

Since $p\in\Omega$, then $A_ip\in\cap_{j=1}^M F(T_i^j),\ i=0,1,2,\dots,N$. Moreover, since $\|\cdot\|^2$ is convex, then we have

$$\begin{aligned}\|y_n-p\|^2&=\|\sum_{i=0}^N\delta_{n,i}\sum_{j=1}^N\beta_{n,i}^j\big(w_n-\gamma_{n,i}^jA_i^*(A_iw_n-v_{n,i}^j)\big)-p\|^2\\&\le\sum_{i=0}^N\delta_{n,i}\sum_{j=1}^N\beta_{n,i}^j\|w_n-\gamma_{n,i}^jA_i^*(A_iw_n-v_{n,i}^j)-p\|^2.\end{aligned}\tag{4.11}$$

From the last inequality, and by applying Lemma 2.2, Lemma 2.3(ii) together with the definition of $\gamma_{n,i}^j$, we have

$$\begin{aligned}\|w_n-\gamma_{n,i}^jA_i^*(A_iw_n-v_{n,i}^j)-p\|^2&=\|w_n-p\|^2-2\gamma_{n,i}^j\langle w_n-p,A_i^*(A_iw_n-v_{n,i}^j)\rangle\\&\quad+(\gamma_{n,i}^j)^2\|A_i^*(A_iw_n-v_{n,i}^j)\|^2\\&=\|w_n-p\|^2-2\gamma_{n,i}^j\langle A_iw_n-A_ip,A_iw_n-v_{n,i}^j\rangle\\&\quad+(\gamma_{n,i}^j)^2\|A_i^*(A_iw_n-v_{n,i}^j)\|^2\\&\le\|w_n-p\|^2-(1-k_i^j)\gamma_{n,i}^j\|A_iw_n-v_{n,i}^j\|^2\\&\quad+(\gamma_{n,i}^j)^2\|A_i^*(A_iw_n-v_{n,i}^j)\|^2\\&=\|w_n-p\|^2\\&\quad-\big(1-k_i^j-(\phi_{n,i}^j+\phi_i^j)\big)(\phi_{n,i}^j+\phi_i^j)\frac{\|A_iw_n-v_{n,i}^j\|^4}{\|A_i^*(A_iw_n-v_{n,i}^j)\|^2}.\end{aligned}\tag{4.12}$$

By the conditions on $\phi_{n,i}^j$ and ϕ_i^j, there exists a positive integer N_0 such that $\big(1-k_i^j-(\phi_{n,i}^j+\phi_i^j)\big)>0$ for all $i=0,1,2,\dots,N, j=1,2,\dots,N, n\ge N_0$. Now, by applying (4.12) in (4.11) we have

$$\begin{aligned}\|y_n-p\|^2&\le\|w_n-p\|^2\\&\quad-\sum_{i=0}^N\delta_{n,i}\sum_{j=1}^N\beta_{n,i}^j\big(1-k_i^j-(\phi_{n,i}^j+\phi_i^j)\big)(\phi_{n,i}^j+\phi_i^j)\frac{\|A_iw_n-v_{n,i}^j\|^4}{\|A_i^*(A_iw_n-v_{n,i}^j)\|^2}\\&\le\|w_n-p\|^2.\end{aligned}\tag{4.13}$$

Next, from the definition of x_{n+1} and by applying (4.10) and (4.13) we have

$$\begin{aligned}\|x_{n+1}-p\|&=\|\xi_nw_n+(1-\xi_n)y_n-p\|\\&\le\xi_n\|w_n-p\|+(1-\xi_n)\|y_n-p\|\\&\le\xi_n\|w_n-p\|+(1-\xi_n)\|w_n-p\|\\&=\|w_n-p\|\end{aligned}$$

$$
\begin{aligned}
&\le (1-\alpha_n)\|x_n - p\| + \alpha_n M_1\\
&\le \max\{\|x_n - p\|, M_1\}\\
&\ \vdots\\
&\le \max\{\|x_{N_0} - p\|, M_1\},
\end{aligned}
$$

which implies that $\{x_n\}$ is bounded. Consequently, both $\{w_n\}$ and $\{y_n\}$ are bounded. □

Lemma 4.7. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2 such that Assumption A holds. Then, the following inequality holds for all $p \in \Omega$:*

$$
\begin{aligned}
\|x_{n+1} - p\|^2 \le{}& (1-\alpha_n)\|x_n - p\|^2 + \alpha_n d_n - \xi_n(1-\xi_n)\|w_n - y_n\|^2\\
&- (1-\xi_n)\sum_{i=0}^{N}\delta_{n,i}\sum_{j=1}^{N}\beta_{n,i}^j\big(1-k_i^j-(\phi_{n,i}^j+\phi_i^j)\big)(\phi_{n,i}^j+\phi_i^j)\frac{\|A_iw_n - v_{n,i}^j\|^4}{\|A_i^*(A_iw_n - v_{n,i}^j)\|^2}.
\end{aligned}
$$

Proof. Let $p \in \Omega$. Then, by applying Lemma 2.3 together with the definition of w_n we obtain

$$
\begin{aligned}
\|w_n - p\|^2 ={}& \|(1-\alpha_n)(x_n - p) + (1-\alpha_n)\theta_n(x_n - x_{n-1}) - \alpha_n p\|^2\\
\le{}& \|(1-\alpha_n)(x_n - p) + (1-\alpha_n)\theta_n(x_n - x_{n-1})\|^2 + 2\alpha_n\langle -p,\ w_n - p\rangle\\
\le{}& (1-\alpha_n)^2\|x_n - p\|^2 + 2(1-\alpha_n)\theta_n\|x_n - p\|\|x_n - x_{n-1}\|\\
&+ (1-\alpha_n)^2\theta_n^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\langle -p,\ w_n - x_{n+1}\rangle + 2\alpha_n\langle -p,\ x_{n+1} - p\rangle\\
\le{}& (1-\alpha_n)\|x_n - p\|^2 + 2\theta_n\|x_n - p\|\|x_n - x_{n-1}\| + \theta_n^2\|x_n - x_{n-1}\|^2\\
&+ 2\alpha_n\|p\|\|w_n - x_{n+1}\| + 2\alpha_n\langle p,\ p - x_{n+1}\rangle. \qquad (4.14)
\end{aligned}
$$

Now, applying the definition of x_{n+1} together with (4.13), (4.14) and Lemma 2.3 we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 ={}& \|\xi_n w_n + (1-\xi_n)y_n - p\|^2\\
={}& \xi_n\|w_n - p\|^2 + (1-\xi_n)\|y_n - p\|^2 - \xi_n(1-\xi_n)\|w_n - y_n\|^2\\
\le{}& \xi_n\|w_n - p\|^2 + (1-\xi_n)\Big[\|w_n - p\|^2\\
&- \sum_{i=0}^{N}\delta_{n,i}\sum_{j=1}^{N}\beta_{n,i}^j\big(1-k_i^j-(\phi_{n,i}^j+\phi_i^j)\big)(\phi_{n,i}^j+\phi_i^j)\frac{\|A_iw_n - v_{n,i}^j\|^4}{\|A_i^*(A_iw_n - v_{n,i}^j)\|^2}\Big]\\
&- \xi_n(1-\xi_n)\|w_n - y_n\|^2\\
={}& \|w_n - p\|^2 - \xi_n(1-\xi_n)\|w_n - y_n\|^2\\
&- (1-\xi_n)\sum_{i=0}^{N}\delta_{n,i}\sum_{j=1}^{N}\beta_{n,i}^j\big(1-k_i^j-(\phi_{n,i}^j+\phi_i^j)\big)(\phi_{n,i}^j+\phi_i^j)\frac{\|A_iw_n - v_{n,i}^j\|^4}{\|A_i^*(A_iw_n - v_{n,i}^j)\|^2}\\
\le{}& (1-\alpha_n)\|x_n - p\|^2 + 2\theta_n\|x_n - p\|\|x_n - x_{n-1}\| + \theta_n^2\|x_n - x_{n-1}\|^2\\
&+ 2\alpha_n\|p\|\|w_n - x_{n+1}\| + 2\alpha_n\langle p,\ p - x_{n+1}\rangle - \xi_n(1-\xi_n)\|w_n - y_n\|^2\\
&- (1-\xi_n)\sum_{i=0}^{N}\delta_{n,i}\sum_{j=1}^{N}\beta_{n,i}^j\big(1-k_i^j-(\phi_{n,i}^j+\phi_i^j)\big)(\phi_{n,i}^j+\phi_i^j)\frac{\|A_iw_n - v_{n,i}^j\|^4}{\|A_i^*(A_iw_n - v_{n,i}^j)\|^2}
\end{aligned}
$$

$$
\begin{aligned}
&= (1-\alpha_n)\|x_n-p\|^2 + \alpha_n\Big[2\|x_n-p\|\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\| + \theta_n\|x_n-x_{n-1}\| \\
&\quad \times \frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\| + 2\|p\|\|w_n-x_{n+1}\| + 2\langle p,\ p-x_{n+1}\rangle\Big] - \xi_n(1-\xi_n)\|w_n-y_n\|^2 \\
&\quad - (1-\xi_n)\sum_{i=0}^{N}\delta_{n,i}\sum_{j=1}^{N}\beta_{n,i}^j\big(1-k_i^j-(\phi_{n,i}^j+\phi_i^j)\big)(\phi_{n,i}^j+\phi_i^j)\frac{\|A_iw_n-v_{n,i}^j\|^4}{\|A_i^*(A_iw_n-v_{n,i}^j)\|^2} \\
&= (1-\alpha_n)\|x_n-p\|^2 + \alpha_n d_n - \xi_n(1-\xi_n)\|w_n-y_n\|^2 \\
&\quad - (1-\xi_n)\sum_{i=0}^{N}\delta_{n,i}\sum_{j=1}^{N}\beta_{n,i}^j\big(1-k_i^j-(\phi_{n,i}^j+\phi_i^j)\big)(\phi_{n,i}^j+\phi_i^j)\frac{\|A_iw_n-v_{n,i}^j\|^4}{\|A_i^*(A_iw_n-v_{n,i}^j)\|^2},
\end{aligned}
$$

where $d_n = 2\|x_n-p\|\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\| + \theta_n\|x_n-x_{n-1}\|\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\| + 2\|p\|\|w_n-x_{n+1}\| + 2\langle p,\ p-x_{n+1}\rangle$. Thus, we have the required inequality.

□

Theorem 4.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2 such that Assumption A holds. Then, $\{x_n\}$ converges strongly to $\hat{x}\in\Omega$, where $\|\hat{x}\| = \min\{\|p\| : p\in\Omega\}$.*

Proof. Let $\|\hat{x}\| = \min\{\|p\| : p\in\Omega\}$, that is, $\hat{x} = P_\Omega(0)$. Then, from Lemma 4.7 we obtain

$$
\|x_{n+1}-p\|^2 \le (1-\alpha_n)\|x_n-\hat{x}\|^2 + \alpha_n\hat{d}_n, \tag{4.15}
$$

where $\hat{d}_n = 2\|x_n-\hat{x}\|\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\| + \theta_n\|x_n-x_{n-1}\|\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\| + 2\|\hat{x}\|\|w_n-x_{n+1}\| + 2\langle\hat{x},\ \hat{x}-x_{n+1}\rangle$.

Now, we claim that the sequence $\{\|x_n-\hat{x}\|\}$ converges to zero. To establish this, by Lemma 2.4 it is sufficient to show that $\limsup_{k\to\infty}\hat{d}_{n_k}\le 0$ for every subsequence $\{\|x_{n_k}-\hat{x}\|\}$ of $\{\|x_n-\hat{x}\|\}$ satisfying

$$
\liminf_{k\to\infty}\left(\|x_{n_k+1}-\hat{x}\| - \|x_{n_k}-\hat{x}\|\right)\ge 0. \tag{4.16}
$$

Suppose that $\{\|x_{n_k}-\hat{x}\|\}$ is a subsequence of $\{\|x_n-\hat{x}\|\}$ such that (4.16) holds. Again, from Lemma 4.7, we obtain

$$
\begin{aligned}
&(1-\xi_{n_k})\sum_{i=0}^{N}\delta_{n_k,i}\sum_{j=1}^{N}\beta_{n_k,i}^j\big(1-k_i^j-(\phi_{n_k,i}^j+\phi_i^j)\big)(\phi_{n_k,i}^j+\phi_i^j)\frac{\|A_iw_{n_k}-v_{n_k,i}^j\|^4}{\|A_i^*(A_iw_{n_k}-v_{n_k,i}^j)\|^2} \\
&\qquad +\xi_{n_k}(1-\xi_{n_k})\|w_{n_k}-y_{n_k}\|^2 \\
&\qquad \le (1-\alpha_{n_k})\|x_{n_k}-\hat{x}\|^2 - \|x_{n_k+1}-\hat{x}\|^2 + \alpha_{n_k}\hat{d}_{n_k}.
\end{aligned}
$$

By (4.16), Remark 3.2 and the fact that $\lim_{k\to\infty}\alpha_{n_k} = 0$, we have

$$
\begin{aligned}
&(1-\xi_{n_k})\sum_{i=0}^{N}\delta_{n_k,i}\sum_{j=1}^{N}\beta_{n_k,i}^j\big(1-k_i^j-(\phi_{n_k,i}^j+\phi_i^j)\big)(\phi_{n_k,i}^j+\phi_i^j)\frac{\|A_iw_{n_k}-v_{n_k,i}^j\|^4}{\|A_i^*(A_iw_{n_k}-v_{n_k,i}^j)\|^2} \\
&\qquad + \xi_{n_k}(1-\xi_{n_k})\|w_{n_k}-y_{n_k}\|^2 \to 0, \quad k\to\infty.
\end{aligned}
$$

Consequently, by the conditions on the control parameters we get

$$\text{(4.17)}\qquad \lim_{k\to\infty}\|w_{n_k}-y_{n_k}\|=0;$$
$$\lim_{k\to\infty}\frac{\|A_iw_{n_k}-v^j_{n_k,i}\|^4}{\|A_i^*(A_iw_{n_k}-v^j_{n_k,i})\|^2}=0,\quad \forall i=0,1,2,\ldots,N, j=1,2,\ldots,M,$$

which implies that

$$\lim_{k\to\infty}\frac{\|A_iw_{n_k}-v^j_{n_k,i}\|^2}{\|A_i^*(A_iw_{n_k}-v^j_{n_k,i})\|}=0,\quad \forall i=0,1,2,\ldots,N, j=1,2,\ldots,M.$$

Since $\{\|A_i^*(A_iw_{n_k}-v^j_{n_k,i})\|\}$ is bounded, it follows that

$$\text{(4.18)}\qquad \lim_{k\to\infty}\|A_iw_{n_k}-v^j_{n_k,i}\|=0,\quad \forall i=0,1,2,\ldots,N, j=1,2,\ldots,M.$$

Thus, we have

$$\begin{aligned}\|A_i^*(A_iw_{n_k}-v^j_{n_k,i})\| &\le \|A_i^*\|\|(A_iw_{n_k}-v^j_{n_k,i})\|\\ &= \|A_i\|\|(A_iw_{n_k}-v^j_{n_k,i})\|\to 0,\ k\to\infty,\ \forall i=0,1,\ldots,N, j=1,\ldots,M.\end{aligned}$$

From the definition of w_n and by Remark 3.2, we obtain

$$\begin{aligned}\|w_{n_k}-x_{n_k}\| &= \|(1-\alpha_{n_k})(x_{n_k}+\theta_{n_k}(x_{n_k}-x_{n_k-1}))-x_{n_k}\|\\ &= \|(1-\alpha_{n_k})(x_{n_k}-x_{n_k})+(1-\alpha_{n_k})\theta_{n_k}(x_{n_k}-x_{n_k-1})-\alpha_{n_k}x_{n_k}\|\\ \text{(4.19)}\qquad &\le (1-\alpha_{n_k})\|x_{n_k}-x_{n_k}\|+(1-\alpha_{n_k})\theta_{n_k}\|x_{n_k}-x_{n_k-1}\|+\alpha_{n_k}\|x_{n_k}\|\to 0,\quad k\to\infty.\end{aligned}$$

Now, from (4.17) and (4.19) we obtain

$$\text{(4.20)}\qquad \|x_{n_k}-y_{n_k}\|\to 0,\quad k\to\infty.$$

From the definition of x_{n+1} and by applying (4.19) and (4.20), we obtain

$$\begin{aligned}\|x_{n_k+1}-x_{n_k}\| &= \|\xi_{n_k}w_{n_k}+(1-\xi_{n_k})y_{n_k}-x_{n_k}\|\\ \text{(4.21)}\qquad &\le \xi_{n_k}\|w_{n_k}-x_{n_k}\|+(1-\xi_{n_k})\|y_{n_k}-x_{n_k}\|\to 0,\quad k\to\infty.\end{aligned}$$

It follows from (4.19) and (4.21) that

$$\text{(4.22)}\qquad \|w_{n_k}-x_{n_k+1}\|\to 0,\quad k\to\infty.$$

Since $\{x_n\}$ is bounded, $w_\omega(x_n)\neq\emptyset$. Let $x^*\in w_\omega(x_n)$ be an arbitrary element. Then, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup x^*$. By (4.19), it follows that $w_{n_k}\rightharpoonup x^*$. Since $A_i,\ i=0,1,2,\ldots,N$ are bounded linear operators, then we have $A_iw_{n_k}\rightharpoonup A_ix^*,\ i=0,1,2,\ldots,N$. By the demiclosed property of $T_i^j,\ i=0,1,2,\ldots,N, j=1,2,\ldots,M$, it follows from (4.18) that $A_ix^*\in F(T_i^j),\ i=0,1,2,\ldots,N, j=1,2,\ldots,M$. This implies that $A_ix^*\in\cap_{j=1}^M F(T_i^j),\ i=0,1,2,\ldots,N$. Thus, we have $x^*\in A_i^{-1}\big(\cap_{j=1}^M F(T_i^j)\big),\ i=0,1,2,\ldots,N$, which implies that $x^*\in\cap_{i=0}^N\big(A_i^{-1}\big(\cap_{j=1}^M F(T_i^j)\big)\big)$. Hence, we have $x^*\in\Omega$. Since $x^*\in w_\omega(x_n)$ was picked arbitrarily, it follows that $w_\omega(x_n)\subset\Omega$.
Next, by the boundedness of $\{x_{n_k}\}$, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}}\rightharpoonup q$ and

$$\limsup_{k\to\infty}\langle \hat{x}, \hat{x}-x_{n_k}\rangle = \lim_{j\to\infty}\langle \hat{x}, \hat{x}-x_{n_{k_j}}\rangle.$$

Since $\hat{x} = P_\Omega(0)$, it follows from the property of the metric projection that

$$\limsup_{k\to\infty}\langle \hat{x}, \hat{x}-x_{n_k}\rangle = \lim_{j\to\infty}\langle \hat{x}, \hat{x}-x_{n_{k_j}}\rangle = \langle \hat{x}, \hat{x}-q\rangle \leq 0, \tag{4.23}$$

Hence, from (4.21) and (4.23) we obtain

$$\limsup_{k\to\infty}\langle \hat{x}, \hat{x}-x_{n_{k+1}}\rangle \leq 0. \tag{4.24}$$

Now, by Remark 3.2, (4.22) and (4.24) we have $\limsup_{k\to\infty} \hat{d}_{n_k} \leq 0$. Thus, by invoking Lemma 2.4 it follows from (4.15) that $\{\|x_n - \hat{x}\|\}$ converges to zero as required.

□

Since the class of multivalued demicontractive mappings contains the class of single-valued demicontractive mappings, we obtain the following consequent result for approximating the solution of split common fixed point problem with multiple output sets for single-valued demicontractive mappings.

Corollary 4.4. *Let $H, H_i, i = 1,2,...,N$, be real Hilbert spaces and let $A_i : H \to H_i, i = 1,2,...,N$, be bounded linear operators with adjoints A_i^*. Let $T^j : H \to H$ and $T_i^j : H_i \to H_i, j = 1,2,...,M$, be single-valued k^j-demicontractive and k_i^j-demicontractive mappings, respectively. Suppose Assumption A holds and the solution set $\Omega \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by the following algorithm converges strongly to $\hat{x} \in \Omega$, where $\|\hat{x}\| = \min\{\|p\| : p \in \Omega\}$.*

Algorithm 4.5.

Step 0. *Select initial data $x_0, x_1 \in H$. Let $H_0 = H, T_0^j = T^j, A_0 = I^H$ and set $n = 1$.*

Step 1. *Given the $(n-1)$th and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \hat{\theta}_n$ with $\hat{\theta}_n$ defined by*

$$\hat{\theta}_n = \begin{cases} \min\left\{\theta, \ \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases}$$

Step 2. *Compute*

$$w_n = (1-\alpha_n)(x_n + \theta_n(x_n - x_{n-1})).$$

Step 3. *Compute*

$$y_n = \sum_{i=0}^{N} \delta_{n,i} \sum_{j=1}^{M} \beta_{n,i}^j \big(w_n - \gamma_{n,i}^j A_i^*(A_i w_n - T_i^j(A_i w_n))\big),$$

where

$$\gamma_{n,i}^j = \begin{cases} \frac{(\phi_{n,i}^j + \phi_i^j)\|A_i w_n - T_i^j(A_i w_n)\|^2}{\|A_i^*(A_i w_n - T_i^j(A_i w_n))\|^2}, & \text{if } \quad A_i w_n \neq T_i^j(A_i w_n), \\ 0, & \text{otherwise.} \end{cases}$$

Step 4. *Compute*

$$x_{n+1} = \xi_n w_n + (1-\xi_n) y_n.$$

*Set $n := n+1$ and return to **Step 1**.*

---

5. APPLICATIONS

5.1. System of Split Variational Inequality Problem with Multiple Output Sets.

Let C be a nonempty, closed and convex subset of H, and let $F : H \to H$ be a mapping. The variational inequality problem (VIP) is formulated as finding a point $p \in C$ such that

$$\langle x - p, Fp \rangle \geq 0, \quad \forall\, x \in C. \tag{5.25}$$

We denote the solution set of the VIP (5.25) by $VI(C, F)$. It is known that

$$F(P_C(I - \lambda F)) = VI(C, F), \tag{5.26}$$

where $\lambda > 0$. Moreover, it is known that if F is δ-inverse strongly monotone, where $\delta > 0$, i.e.,

$$\langle x - y, Fx - Fy \rangle \geq \delta \|Fx - Fy\|^2, \quad \forall x, y \in H,$$

and $\lambda \in (0, 2\delta)$, then $P_C(I - \lambda F)$ is nonexpansive.
Let $H, H_i, i = 1, 2, ..., N$, be real Hilbert spaces and let $C^j \subset H, C_i^j \subset H_i, i = 1, 2, ..., N, j = 1, 2, ..., M$, be nonempty, closed and convex subsets such that $\cap_{j=1}^M C^j \neq \emptyset$. Let $A_i : H \to H_i, i = 1, 2, ..., N$, be bounded linear operators and let $F^j : H \to H, F_i^j : H_i \to H_i, i = 1, 2, ..., N, j = 1, 2, ..., M$, be single-valued operators. In this subsection, we apply our result to approximate the solution of the following *system of split variational inequality problem with multiple output sets* (SSVIPMOS): Find $x^* \in \cap_{j=1}^M C^j$ such that

$$x^* \in \Omega_1 := \cap_{j=1}^M VI(C^j, F^j) \cap \Big(\cap_{i=1}^N A_i^{-1}\big(\cap_{j=1}^M VI(C_i^j, F_i^j)\big)\Big) \neq \emptyset. \tag{5.27}$$

Since the class of nonexpansive mappings is properly contained in the class of demicontractive mappings, then for $i = 0, 1, 2, ..., N, j = 1, 2, ..., M$, if we set $T_i^j = P_{C_i^j}(I^{H_i} - \lambda_i^j F_i^j)$ in Corollary 4.4 and apply (5.26), we obtain the following result for approximating the solution of SSVIPMOS (5.27).

Theorem 5.6. *Let $H, H_i, i = 1, 2, ..., N$, be real Hilbert spaces and let $A_i : H \to H_i, i = 1, 2, ..., N$, be bounded linear operators with adjoints A_i^*. Let $F^j : H \to H$ and $F_i^j : H_i \to H_i, j = 1, 2, ..., M$, be δ^j-inverse strongly monotone and δ_i^j-inverse strongly monotone mappings, respectively, where $\delta_i, \delta_i^j > 0$. Let $T_i^j = P_{C_i^j}(I^{H_i} - \lambda_i^j F_i^j)$ in Corollary 4.4, where $\lambda_i^j \in (0, 2\delta_i^j), i = 0, 1, 2, \ldots, N, j = 1, 2, \ldots, M$, ($C_0^j = C^j, F_0^j = F^j, \delta_0^j = \delta^j$) and suppose Assumption A holds. Then, the sequence $\{x_n\}$ generated by Algorithm 4.5 converges strongly to $\hat{x} \in \Omega_1$, where $\|\hat{x}\| = \min\{\|p\| : p \in \Omega_1\}$.*

5.2. System of Split Equilibrium Problem with Multiple Output Sets.

Let C be a nonempty, closed and convex subset of a real Hilbert space H, and let $F : C \times C \to \mathbb{R}$ be a bifunction. The *equilibrium problem* (EP) for the bifunction F on C is formulated as finding a point $\hat{x} \in C$ such that

$$F(\hat{x}, y) \geq 0, \quad \forall y \in C. \tag{5.28}$$

The solution of the EP (5.28) is denoted by $EP(F, C)$.
In solving the EP (5.28), we assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

- (C1) $F(x, x) = 0$ for all $x \in C$;
- (C2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
- (C3) F is upper hemicontinuous, that is, for all $x, y, z \in C$, $\lim_{t \downarrow 0} F\big(tz + (1 - t)x, y\big) \leq F(x, y)$;

(C4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

We need the following lemma to establish our next result.

Lemma 5.8. [26] *Let C be a nonempty closed convex subset of a Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (C1)-(C4). For $r > 0$ and $x \in H$, define a mapping $T_r^F : H \to C$ as follows:*

$$T_r^F(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall\, y \in C\}. \tag{5.29}$$

Then T_r^F is well defined and the following hold:

(1) for each $x \in H, T_r^F(x) \neq \emptyset$;
(2) T_r^F is single-valued;
(3) T_r^F is firmly nonexpansive, that is, for any $x, y \in H$,

$$\|T_r^F x - T_r^F y\|^2 \leq \langle T_r^F x - T_r^F y, x - y\rangle;$$

(4) $F(T_r^F) = EP(F, C)$;
(5) $EP(F, C)$ is closed and convex.

Let $H, H_i, i = 1, 2, \ldots, N$ be real Hilbert spaces, and let $C^j \subset H, C_i^j \subset H_i, i = 1, 2, ..., N, j = 1, 2, ..., M,$ be nonempty, closed and convex subsets. Let $F^j : C^j \times C^j \to \mathbb{R}, F_i^j : C_i^j \times C_i^j \to \mathbb{R}$ be bifunctions satisfying conditions (C1)-(C4), and let $A_i : H \to H_i$ be bounded linear operators. Here, we apply our result to study the following *system of split equilibrium problem with multiple output sets* (SSEPMOS): Find $x^* \in H$ such that

$$x^* \in \Omega_2 := \cap_{j=1}^M EP(F^j, C^j) \cap \Big(\cap_{i=1}^N A_i^{-1}\big(\cap_{j=1}^M EP(F_i^j, C_i^j)\big)\Big) \neq \emptyset. \tag{5.30}$$

Since by Lemma 5.8, T_r^F is nonexpansive and $F(T_r^F) = EP(F, C)$, then for $i = 0, 1, 2, ..., N$, $j = 1, 2, ..., M$, if we set $T_i^j = T_{r_i^j}^{F_i^j}$ in Corollary 4.4, we obtain the following result for approximating the solution of SSEPMOS (5.30).

Theorem 5.7. *Let $H, H_i, i = 1, 2, ..., N$, be real Hilbert spaces and let $A_i : H \to H_i, i = 1, 2, ..., N$, be bounded linear operators with adjoints A_i^*. Let F^j, F_i^j be as defined above and set $T_i^j = T_{r_i^j}^{F_i^j}, i = 0, 1, 2, \ldots, N, j = 1, 2, \ldots, M$, in Corollary 4.4 ($F_0^j = F^j$). Moreover, suppose Assumption A holds. Then, the sequence $\{x_n\}$ generated by Algorithm 4.5 converges strongly to $\hat{x} \in \Omega_2$, where $\|\hat{x}\| = \min\{\|p\| : p \in \Omega_2\}$.*

5.3. System of Split Monotone Variational Inclusion Problem with Multiple Output Sets.

Moudafi in [27] introduced a split inverse problem known as the *split monotone variational inclusion problem* (SMVIP). Let H_1, H_2 be real Hilbert spaces, $f_1 : H_1 \to H_1, f_2 : H_2 \to H_2$, are single-valued mappings, $A : H_1 \to H_2$ is a bounded linear operator, $B_1 : H_1 \to 2^{H_1}, B_2 : H_2 \to 2^{H_2}$ are multivalued maximal monotone mappings. The SMVIP is formulated as follows:

$$\text{find a point } \hat{x} \in H_1 \text{ such that } 0 \in f_1(\hat{x}) + B_1(\hat{x}) \tag{5.31}$$

and

$$\hat{y} = A\hat{x} \in H_2 \text{ such that } 0 \in f_2(\hat{y}) + B_2(\hat{y}). \tag{5.32}$$

We point out that if (5.31) and (5.32) are considered separately, then each of (5.31) and (5.32) is a monotone variational inclusion problem (MVIP) with solution set $(B_1 + f_1)^{-1}(0)$

and $(B_2 + f_2)^{-1}(0)$, respectively. The mapping $J_\lambda^{B_1} : H_1 \to H_1$ is called the resolvent operator associated with B_1 and λ, and is defined by

$$J_\lambda^{B_1}(x) = (I^{H_1} + \lambda B_1)^{-1}x, \quad x \in H_1,\ \lambda > 0. \tag{5.33}$$

It is known that B is maximal monotone if and only if J_r^B is single-valued, firmly nonexpansive and $\mathrm{dom}(J_r^B) = H$.

Lemma 5.9. [43] *Let H be a real Hilbert space, $r > 0, f : H \to H$ be a μ-inverse strongly monotone mapping and $B : H \to 2^H$ be a maximal monotone mapping. Then, the following hold:*

(i) $F(J_r^B(I^H - rf)) = (B + f)^{-1}(0)$;

(ii) *if $r \in (0, 2\mu)$, then $J_r^B(I^H - rf)$ is averaged.*

Let $H, H_i, i = 1, 2, ..., N$, be real Hilbert spaces and let $A_i : H \to H_i, i = 1, 2, ..., N$, be bounded linear operators. Let $B^j : H \to 2^H, B_i^j : H_i \to 2^{H_i}, i = 1, 2, ..., N$, be multivalued operators, and $f^j : H \to H, f_i^j : H_i \to H_i, i = 1, 2, ..., N, j = 1, 2, ..., M$, be single-valued operators. Here, we apply our result to study the following *system of split monotone variational inclusion problem with multiple output sets* (SSMVIPMOS): Find $x^* \in H$ such that

$$x^* \in \Omega_3 := \cap_{j=1}^M (f^j + B^j)^{-1}(0) \cap \Big(\cap_{i=1}^N A_i^{-1}\big(\cap_{j=1}^M (f_i^j + B_i^j)^{-1}(0)\big)\Big) \neq \emptyset. \tag{5.34}$$

Since every averaged mapping is nonexpansive, and thus demicontractive, then for $i = 0, 1, 2, ..., N, j = 1, 2, ..., M$, if we set $T_i^j = J_{r_i^j}^{B_i^j}(I^{H_i} - r_i^j f_i^j)$ in Corollary 4.4 and apply Lemma 5.9, we obtain the following result for approximating the solution of SSMVIPMOS (5.34).

Theorem 5.8. *Let $H, H_i, i = 1, 2, ..., N$, be real Hilbert spaces and let $A_i : H \to H_i, i = 1, 2, ..., N$, be bounded linear operators with adjoints A_i^*. Let $B^j : H \to 2^H, B_i^j : H_i \to 2^{H_i}, i = 1, 2, ..., N$, be multivalued operators, and $f^j : H \to H, f_i^j : H_i \to H_i, i = 1, 2, ..., N, j = 1, 2, ..., M$, be δ^j-inverse strongly monotone and δ_i^j-inverse strongly monotone mappings, respectively, where $\delta_i, \delta_i^j > 0$. Let $T_i^j = J_{r_i^j}^{B_i^j}(I^{H_i} - r_i^j f_i^j)$ in Corollary 4.4, where $r_i^j \in (0, 2\delta_i^j), i = 0, 1, 2, \dots, N, j = 1, 2, \dots, M$, $(B_0^j = B^j, f_0^j = f^j, \delta_0^j = \delta^j)$ and suppose Assumption A holds. Then, the sequence $\{x_n\}$ generated by Algorithm 4.5 converges strongly to $\hat{x} \in \Omega_3$, where $\|\hat{x}\| = \min\{\|p\| : p \in \Omega_3\}$.*

5.4. System of Split Convex Minimization Problem with Multiple Output Sets.

Let $g : H \to \mathbb{R}$ be a convex and differentiable function, and let $G : H \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous function. It is well known that if ∇g is $\frac{1}{\mu}$-Lipschitz continuous, then it is μ-inverse strongly monotone (and thus monotone), where ∇g is the gradient of g. Furthermore, the subdifferential ∂G of G is maximal monotone (see [37]). Moreover,

$$g(x^*) + G(x^*) = \min_{x \in H}\{g(x) + G(x)\} \iff 0 \in \nabla g(x^*) + \partial G(x^*). \tag{5.35}$$

Let $H, H_i, i = 1, 2, \dots, N$ be real Hilbert spaces, and let $A_i : H \to H_i$ be bounded linear operators. Let $g : H \to \mathbb{R}, g_i : H_i \to \mathbb{R}$ be convex and differentiable functions, and let $G : H \to (-\infty, +\infty], G_i : H_i \to (-\infty, +\infty]$ be proper convex and lower semi-continuous functions. Here, we apply our result to approximate the solution of the following *system split convex minimization problem with multiple output sets* (SSCMPMOS): Find $x^* \in H$ such that

(5.36)

$$x^* \in \Omega_4 := \cap_{j=1}^M \big(\arg\min_H \{g^j(x) + G^j(x)\}\big) \cap \Big(\cap_{i=1}^N A_i^{-1}\Big(\cap_{j=1}^M \big(\arg\min_{H_i} \{g_i^j(x) + G_i^j(x)\}\big)\Big)\Big) \neq \emptyset.$$

Observe that by Lemma 5.9, $F(J_r^{\partial G}(I^H - r\nabla g)) = (\partial G + \nabla g)^{-1}(0)$.
Hence, for $i = 0,1,2,...,N, j = 1,2,...,M$, if we set $T_i^j = J_{r_i^j}^{\partial G_i^j}(I^{H_i} - r_i^j \nabla g_i^j)$ in Corollary 4.4 and apply (5.35) together with Lemma 5.9, we obtain the following result for approximating the solution of SSCMPMOS (5.36).

Theorem 5.9. *Let $H, H_i, i = 1,2,...,N$, be real Hilbert spaces and let $A_i : H \to H_i, i = 1,2,...,N$, be bounded linear operators with adjoints A_i^*. Let $G^j, G_i^j, g^j, g_i^j, j = 1,2,...,M$, be as defined above and such that $\nabla g^j, \nabla g_i^j$ are $\frac{1}{\delta^j}$-Lipschitz continuous and $\frac{1}{\delta_i^j}$-Lipschitz continuous, respectively, where $\delta_i, \delta_i^j > 0$. Let $T_i^j = J_{r_i^j}^{\partial G_i^j}(I^{H_i} - r_i^j \nabla g_i^j)$ in Corollary 4.4, where $r_i^j \in (0, 2\delta_i^j), i = 0,1,2,\dots,N, j = 1,2,\dots,M$, ($\partial G_0^j = \partial G^j, \nabla g_0^j = \nabla g^j, \delta_0^j = \delta^j$) and suppose Assumption A holds. Then, the sequence $\{x_n\}$ generated by Algorithm 4.5 converges strongly to $\hat{x} \in \Omega_4$, where $\|\hat{x}\| = \min\{\|p\| : p \in \Omega_4\}$.*

6. NUMERICAL EXAMPLES

In this section, we present some numerical experiments to demonstrate the efficiency of our proposed method Algorithm 3.2 in comparison with Algorithm 1.1. For simplicity, in all the experiments we consider the case when $L = M = N = 5$. All numerical computations were carried out using Matlab version R2021(b).
In our proposed method Algorithm 3.2, we choose $\alpha_n = \frac{1}{2n+3}, \epsilon_n = \frac{50}{(2n+3)^2}, \theta = 1.8, \phi_i^j = 0.98, \phi_{n,i}^j = \frac{100}{n^{0.01}}, \delta_{n,i} = \frac{1}{6}, \beta_{n,i}^j = \frac{1}{5}, \xi_n = \frac{n+1}{2n+1}, \rho_n = \frac{n+1}{3n+2}, a_n = \frac{3n+2}{4n+3}, f(x) = \frac{2}{3}x$.

Example 6.1. Let $H_i = \mathbb{R}, i = 0,1,\dots,5$, with the inner product defined by $\langle x,y\rangle = xy$, for all $x, y \in \mathbb{R}$, and the induced usual norm $|\cdot|$. For $i = 0,1,\dots,5$, the mappings $A_i, S^k, T^j, T_i^j : \mathbb{R} \to \mathbb{R}$ are defined by $A_i(x) = \frac{3}{i+4}x, S^k(x) = \frac{2}{k+2}x, T^j(x) = \frac{2}{j+2}x, T_i^j(x) = \frac{2}{k+2}x, \ \forall x \in \mathbb{R}$. Then, $A_i^*(y) = \frac{3}{i+4}y, \ \forall y \in \mathbb{R}$. Clearly, all the conditions of Theorem 4.3 are satisfied.
We use $|x_{n+1} - x_n| < 10^{-6}$ as the stopping criterion and choose different starting points as follows:
Case 1: $x_0 = 32, \ x_1 = 11$;
Case 2: $x_0 = 9, \ x_1 = -9$;
Case 3: $x_0 = -33, \ x_1 = -10$;
Case 4: $x_0 = -15, \ x_1 = 10$.
The numerical results are reported in Figures 1-4 and Table 1.

TABLE 1. Numerical Results for Example 6.1

	Case 1		Case 2		Case 3		Case 4	
	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
Reich *et al. Alg.*	102	0.0099	97	0.0069	102	0.0095	99	0.0082
Proposed (non-inertial)	28	0.0052	28	0.0047	28	0.0055	28	0.0074
Proposed Alg. 3.2	25	0.0041	25	0.0042	25	0.0039	25	0.0051

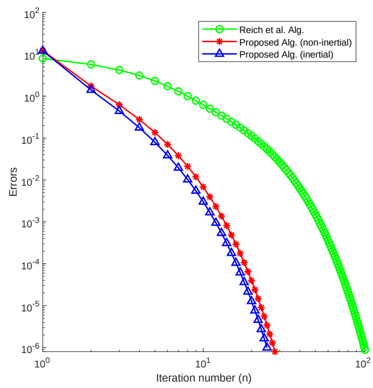

FIGURE 1. Example 6.1: Case 1

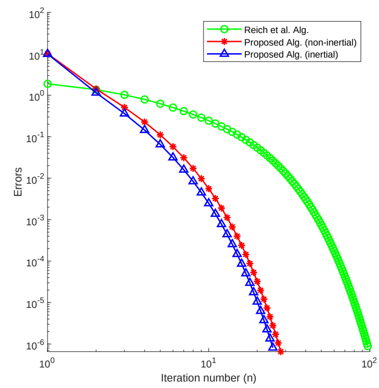

FIGURE 2. Example 6.1: Case 2

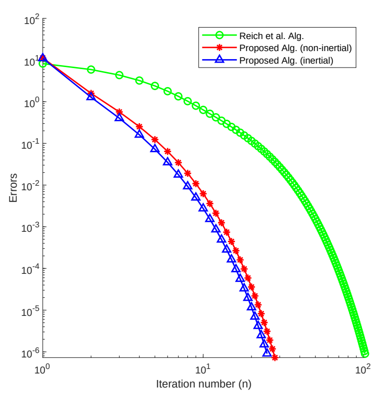

FIGURE 3. Example 6.1: Case 3

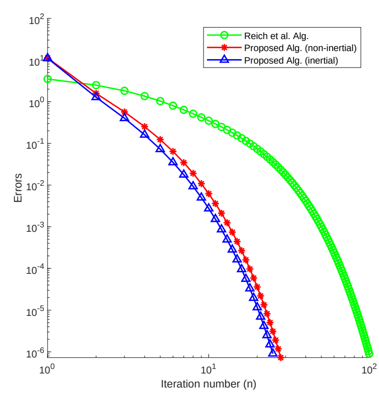

FIGURE 4. Example 6.1: Case 4

Example 6.2. Let $H_i = (\ell_2(\mathbb{R}), \|\cdot\|_2), i = 0, 1, \ldots, 4$, where $\ell_2(\mathbb{R}) \coloneqq \{x = (x_1, x_2, \ldots, x_j, \ldots), x_j \in \mathbb{R} : \sum_{j=1}^{\infty} |x_j|^2 < \infty\}, ||x||_2 = (\sum_{j=1}^{\infty} |x_j|^2)^{\frac{1}{2}}$ for all $x \in \ell_2(\mathbb{R})$. For $i = 0, 1, \ldots, 5$, the mappings $A_i, S^k, T^j, T_i^j : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ are defined by $A_i(x) = \frac{2}{i+3}x, S^k(x) = \frac{3}{k+5}x, T^j(x) = \frac{3}{j+5}x, T_i^j(x) = \frac{3}{k+5}x, \forall x \in \ell_2(\mathbb{R})$. Then, $A_i^*(y) = \frac{2}{i+3}y, \forall y \in \ell_2(\mathbb{R})$. It is clear that all the conditions of Theorem 4.3 are satisfied.
We use $\|x_{n+1} - x_n\| < 10^{-6}$ as the stopping criterion and choose different starting points as follows:
Case 1: $x_0 = (-1, 0.1, -0.01, \ldots), \ x_1 = (3, 1, \frac{1}{3}, \ldots)$,
Case 2: $x_0 = (5, 0.5, 0.05, \ldots), \ x_1 = (2, 1, \frac{1}{2}, \ldots)$,
Case 3: $x_0 = (4, 1, \frac{1}{4}, \ldots), \ x_1 = (-3, 0.3, -0.03, , \ldots)$,
Case 4: $x_0 = (-3, 1, -\frac{1}{3}, \ldots), \ x_1 = (-2, 1, -\frac{1}{2}, \ldots)$.
The numerical results are reported in Figures 5-8 and Table 2.

TABLE 2. Numerical Results for Example 6.2

	Case 1		Case 2		Case 3		Case 4	
	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
Reich *et al. Alg.*	84	0.0093	94	0.0113	93	0.0084	92	0.0067
Proposed (non-inertial)	23	0.0053	23	0.0059	23	0.0055	23	0.0047
Proposed Alg. 3.2	21	0.0032	21	0.0040	21	0.0035	21	0.0028

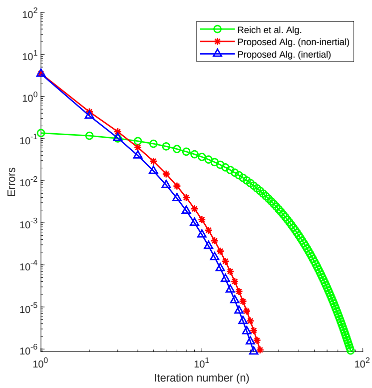

FIGURE 5. Example 6.1: Case 1

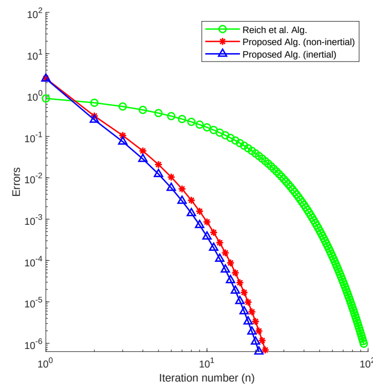

FIGURE 6. Example 6.1: Case 2

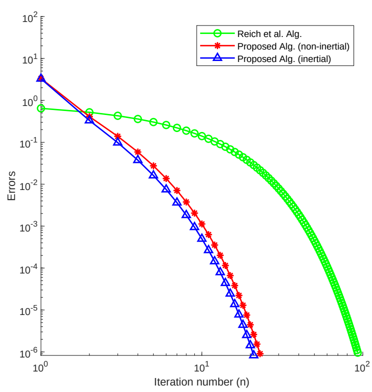

FIGURE 7. Example 6.1: Case 3

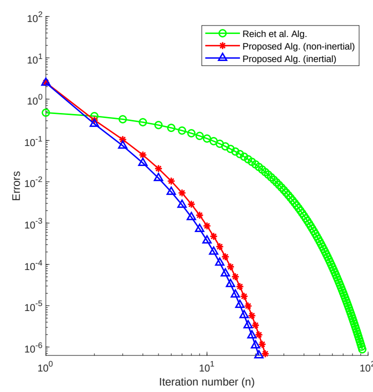

FIGURE 8. Example 6.1: Case 4

7. Conclusion

In this paper, we studied certain classes of split inverse problems. We proposed a novel relaxed inertial iterative method for approximating the solutions of these split inverse problems in the framework of Hilbert spaces. Moreover, we obtained strong convergence result for the proposed algorithm without the knowledge of the operators' norms. Finally, we carried out several numerical experiments to demonstrate the applicability and efficiency of our proposed method.

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