CARPATHIAN J. MATH.
Volume **39** (2023), No. 3,
Pages 621 - 632

Online version at https://www.carpathian.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2023.03.04

Best proximity results for p-proximal contractions on topological spaces

A. BERA[1,5], L. K. DEY[1], A. PETRUŞEL[2,3] and A. CHANDA[4]

ABSTRACT. In this article, we investigate some sufficient conditions for the existence and uniqueness of best proximity points for the topological p-proximal contractions and p-proximal contractive mappings on arbitrary topological spaces. Moreover, our results are illustrated by a few numerical examples and they generalize some known results in the literature.

1. INTRODUCTION AND PRELIMINARIES

Fixed point theory in complete metric spaces or in Banach spaces serves as a consequential tool to get to the bottom of a huge number of applications in many scientific fields, to name a few, mathematics, computer science and economics. One can note that numerous problems related to preceding domains of research can be expressed as equations of the form $Tx = x$, where T is a self-mapping defined in a suitable underlying structure. For excellent books on these topics see [4, 6, 13, 17, 20] and many others. Whenever T is not a self-mapping, the equation $Tx = x$ does not necessarily possess a solution. In these circumstances, it is worthy to search for an optimal approximate solution which minimizes the error due to approximation. In other words, for a non-self-mapping $T : A \to B$ defined on a metric space, one enquires for an approximate solution x^* in A so that the error $d(x^*, Tx^*)$ is minimum. This solution x^* is said to be a best proximity point of the mapping. The concerning theory dealing with existence of best proximity points is rich enough and for some interesting findings keen readers are referred to [3, 7, 8, 10, 14, 16, 27–29] and references therein.

Very recently, Raj and Piramatchi [21] extended the notion of best proximity points from usual metric spaces to arbitrary topological spaces and affirmed certain related results alongside a generalized version of the well-known Edelstein fixed point theorem. Following this direction of research, Som et al. [24] proposed couple of new concepts namely topologically weak proximal contractions and topologically proximal weakly contractive mappings with respect to a real-valued continuous function g defined on $X \times X$ and investigated sufficient conditions for the existence and uniqueness of best proximity points for the preceding class of mappings.

It can be very fascinating to note that the concept of best proximity points in the setting of Banach spaces for various non-expansive mappings is investigated by many mathematicians. In the article [5], the author gives some characterizations of nearly strongly convex and very convex spaces in terms of best approximation theoretic properties of Banach spaces. Of late, Shukla and Panicker [26] established several weak and strong convergence theorems for Kirk iterative method in the Banach spaces context. In [15], the author proposes a self-adaptive projection method to obtain a common element in

Received: 14.03.2022. In revised form: 11.02.2023. Accepted: 18.02.2023
2020 *Mathematics Subject Classification.* 47H10, 47H09.
Key words and phrases. *Best proximity points, topological spaces, p-proximal contractions.*
Corresponding author: A. Petruşel, petrusel@math.ubbcluj.ro

the solution set of variational inequalities and fixed point set for relatively non-expansive mappings in 2-uniformly convex and uniformly smooth real Banach spaces. More results in this direction of research can be found in [2, 11, 18, 25].

On the other hand, in 2020, Altun [1] proposed the notions of p-proximal contraction and p-proximal contractive non-self-mapping on metric spaces and proved best proximity point theorems for these type of mappings. Very recently, in [12], it is observed that the main result for p-proximal contractions, given in [1], is a straightforward consequence of a fixed point theorem for p-contractions, given by Popescu [19].

Firstly, we recollect the definition of a p-proximal contraction.

Definition 1.1. Let $T : A \to B$ be a mapping defined on two non-empty subsets A and B of a metric space (X, d). Then T is said to be a p-proximal contraction if there exists $k \in (0, 1)$ such that

$$d(u_1, Tx_1) = dist(A, B),\ d(u_2, Tx_2) = dist(A, B)$$

imply

$$d(u_1, u_2) \le k\left(d(x_1, x_2) + |d(u_1, x_1) - d(u_2, x_2)|\right),$$

for all $x_1, x_2, u_1, u_2 \in A$.

As a particular case, whenever $A = B = X$, then the previous definition reduces to the collection of self-mappings known as p-contractions introduced by Popescu [19] beforehand. In the following, we recall some definitions and notations which are playing crucial roles in this article. For a detailed reading, one is referred to [24].

Definition 1.2. Let Ξ be a topological space and $\Phi : \Xi \times \Xi \to \mathbb{R}$ be a continuous mapping. Let (ξ_n) be a sequence in Ξ. Then (ξ_n) is said to be Φ-convergent to $\xi \in \Xi$ if $|\Phi(\xi_n, \xi)| \to 0$ as $n \to \infty$.

Definition 1.3. Let Ξ be a topological space and $\Phi : \Xi \times \Xi \to \mathbb{R}$ be a continuous mapping. Let (ξ_n) be a sequence in Ξ. Then (ξ_n) is said to be Φ-Cauchy to if $|\Phi(\xi_n, \xi_m)| \to 0$ as $n, m \to \infty$.

Definition 1.4. Let Ξ be a topological space and $\Phi : \Xi \times \Xi \to \mathbb{R}$ be a continuous mapping. Then Ξ is said to be Φ-complete if every Φ-Cauchy sequence (ξ_n) in Ξ is Φ-convergent to an element in Ξ.

Definition 1.5. [21] Let $\Re, \Omega$ be two non-empty subsets of a topological space Ξ and $\Phi : \Xi \times \Xi \to \mathbb{R}$ be a continuous mapping. Consider

$$D_\Phi(\Re, \Omega) = \inf\left\{|\Phi(\alpha, \beta)| : \alpha \in \Re, \beta \in \Omega\right\}.$$

It is clear that if Ξ is a metric space and Φ is a distance function, then $D_\Phi(\Re, \Omega)$ is the distance between the two aforementioned sets. Here we consider the following two sets which will be important in establishing our findings:

$$\Re_\Phi = \left\{\alpha \in \Re : |\Phi(\alpha, \beta)| = D_\Phi(\Re, \Omega) \text{ for some } \beta \in \Omega\right\},$$
$$\Omega_\Phi = \left\{\alpha \in \Omega : |\Phi(\alpha, \beta)| = D_\Phi(\Re, \Omega) \text{ for some } \beta \in \Re\right\}.$$

Definition 1.6. [21] Let $\Re, \Omega$ be two non-empty subsets of a topological space Ξ with $\Re_\Phi \neq \phi$ and $\Phi : \Xi \times \Xi \to \mathbb{R}$ be a continuous mapping. Then $(\Re, \Omega)$ is said to satisfy topologically p-property if

$$\left.\begin{array}{l}|\Phi(\alpha_1, \beta_1)| = D_\Phi(\Re, \Omega),\\ |\Phi(\alpha_2, \beta_2)| = D_\Phi(\Re, \Omega)\end{array}\right\} \Rightarrow |\Phi(\alpha_1, \alpha_2)| = |\Phi(\beta_1, \beta_2)| \tag{1.1}$$

for $\alpha_1, \alpha_2 \in \Re$ and $\beta_1, \beta_2 \in \Omega$.

One can note that if Ξ is a metric space and Φ is a distance function on Ξ, then it is the usual p-property, which is the fundamental property to obtain the best proximity points for non-self mappings, introduced in [22]. In our next example, we show that a pair of subsets $(\Re, \Omega)$ of a topological space Ξ may satisfy p-property with respect to one mapping but may not so for another mapping.

Example 1.1. Consider $\mathbb{R}^2$ with usual topology. Let $\Re = \{2\} \times [-2, 0]$ and $\Omega = \{2\} \times [0, 2]$. Let $\mathcal{F}_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ and $\mathcal{F}_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$\mathcal{F}_1\left((\alpha_1, \alpha_2), (\beta_1, \beta_2)\right) = \alpha_2^2 - \beta_2^2 \text{ for all } (\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \mathbb{R}^2$$

and

$$\mathcal{F}_2\left((\alpha_1, \alpha_2), (\beta_1, \beta_2)\right) = \alpha_2\beta_2 \text{ for all } (\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \mathbb{R}^2$$

respectively. Then $\mathcal{F}_1$ is a continuous function and also $D_{\mathcal{F}_1}(\Re, \Omega) = 0$. Now it can be easily verified that the pair $(\Re, \Omega)$ of subsets of a topological space Ξ satisfies p-property with respect to the mapping $\mathcal{F}_1$ but does not so for the mapping $\mathcal{F}_2$ though $\mathcal{F}_2$ is continuous function and $D_{\mathcal{F}_2}(\Re, \Omega) = 0$. As $\mathcal{F}_2\left((1, -\frac{1}{2}), (1, 0)\right) = 0 = D_{\mathcal{F}_2}(\Re, \Omega)$ and $\mathcal{F}_2\left((1, -\frac{1}{3}), (1, 0)\right) = 0 = D_{\mathcal{F}_2}(\Re, \Omega)$ but $\mathcal{F}_2\left((1, -\frac{1}{2}), (1, -\frac{1}{3})\right) \neq \mathcal{F}_2\left((1, 0), (1, 0)\right)$.

In the following, we extend the notions of proximal contractions and modified proximal contractions [1] in the context of a topological space.

Definition 1.7. Let $\Re, \Omega$ be two non-empty subsets of a topological space Ξ and $\Phi : \Xi \times \Xi \to \mathbb{R}$ be a continuous mapping. A mapping $\mathcal{F} : \Re \to \Omega$ is said to be a topologically proximal contraction with respect to Φ if there exists a real number $c \in (0, 1)$ such that

$$\left.\begin{array}{l} |\Phi(\alpha_1, \mathcal{F}(\beta_1))| = D_\Phi(\Re, \Omega), \\ |\Phi(\alpha_2, \mathcal{F}(\beta_2))| = D_\Phi(\Re, \Omega) \end{array}\right\} \Rightarrow |\Phi(\alpha_1, \alpha_2)| \leq c|\Phi(\beta_1, \beta_2)| \tag{1.2}$$

for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Re$.

Now for distinct β_1 and β_2, we give a modified definition of topological proximal contraction, which as follows.

Definition 1.8. Let $\Re, \Omega$ be two non-empty subsets of a topological space Ξ and $\Phi : \Xi \times \Xi \to \mathbb{R}$ be a continuous mapping. A mapping $\mathcal{F}$ is said to be a modified topologically proximal contraction with respect to Φ if there exists a real number $c \in (0, 1)$ such that

$$\left.\begin{array}{l} |\Phi(\alpha_1, \mathcal{F}(\beta_1))| = D_\Phi(\Re, \Omega), \\ |\Phi(\alpha_2, \mathcal{F}(\beta_2))| = D_\Phi(\Re, \Omega) \end{array}\right\} \Rightarrow |\Phi(\alpha_1, \alpha_2)| \leq c|\Phi(\beta_1, \beta_2)| \tag{1.3}$$

for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Re$ with $\beta_1 \neq \beta_2$.

In our next example, we show that every topological proximal contraction is a modified topological proximal contraction but the converse may not hold.

Example 1.2. Consider $\mathbb{R}^2$ with usual topology. We consider the sets $\Re = \{(0, -\frac{1}{2}), (0, \frac{1}{2})\}$ and $\Omega = \{(0, 0), (1, \frac{3}{4}), (1, 5)\}$. Let $\Phi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous mapping defined as

$$\Phi\left((\alpha_1, \alpha_2), (\beta_1, \beta_2)\right) = (\alpha_1 - \beta_1) + (\alpha_2 - \beta_2) \text{ for all } (\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \mathbb{R}^2.$$

Then $D_\Phi(\Re, \Omega) = \frac{1}{2}$ and define the mapping $\mathcal{F} : \Re \to \Omega$ as follows:

$$\mathcal{F}(x, y) = \begin{cases} \left(1, 1 + \frac{y}{2}\right), & \text{at } (0, -\frac{1}{2}); \\ (0, 1 - 2y), & \text{at } (0, \frac{1}{2}). \end{cases}$$

Then for $\beta_1 = \beta_2 = (0, \frac{1}{2})$ and $\alpha_1 = (0, \frac{1}{2}), \alpha_2 = (0, -\frac{1}{2})$, we get

$$\begin{array}{l} |\Phi(\alpha_1, \mathcal{F}(\beta_1))| = D_\Phi(\Re, \Omega) = \frac{1}{2}, \\ |\Phi(\alpha_2, \mathcal{F}(\beta_2))| = D_\Phi(\Re, \Omega) = \frac{1}{2} \end{array}$$

but $|\Phi(\alpha_1, \alpha_2)| = 1 > c|\Phi(\beta_1, \beta_2)|$ for any $c \in (0, 1)$. Therefore $\mathcal{F}$ is not a topological proximal contraction. However, we can not find two distinct β_1 and β_2 such that $|\Phi(\alpha_1, \mathcal{F}(\beta_1))| = D_\Phi(\Re, \Omega)$ and $|\Phi(\alpha_2, \mathcal{F}(\beta_2))| = D_\Phi(\Re, \Omega)$ hold but the implication (1.3) holds for all $\alpha_1, \alpha_2 \in \Re$. Therefore $\mathcal{F}$ is a modified topological proximal contraction.

2. **Best proximity point theorem on topological spaces**

To begin with, we introduce the notion of topological p-proximal contractions.

Definition 2.9. Let $\Re, \Omega$ be two non-empty subsets of a topological space Ξ and $\Phi : \Xi \times \Xi \to \mathbb{R}$ be a continuous mapping. A mapping $\mathcal{F} : \Re \to \Omega$ is said to be a topological p-proximal contraction if there exists a real number $c \in (0, 1)$ such that

(2.4)
$$\left.\begin{array}{l}|\Phi(\alpha_1, \mathcal{F}(\beta_1))| = D_\Phi(\Re, \Omega),\\ |\Phi(\alpha_2, \mathcal{F}(\beta_2))| = D_\Phi(\Re, \Omega)\end{array}\right\} \Rightarrow |\Phi(\alpha_1, \alpha_2)| \leq c\left(|\Phi(\beta_1, \beta_2)| + ||\Phi(\alpha_1, \beta_1)| - |\Phi(\alpha_2, \beta_2)||\right)$$

for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Re$ with $\beta_1 \neq \beta_2$.

Note that if Ξ is a metric space and Φ is a distance function on Ξ, then $\mathcal{F}$ becomes the usual p-proximal contraction, introduced in [1]. Additionally, if we consider $\Re = \Omega = \Xi$, then the above definition of a topological p-proximal contraction can be written as

$$|\Phi(\mathcal{F}(\beta_1), \mathcal{F}(\beta_2))| \leq c\left(|\Phi(\beta_1, \beta_2)| + ||\Phi(\beta_1, \mathcal{F}(\beta_2))| - |\Phi(\beta_2, \mathcal{F}(\beta_2))||\right)$$

for all $\beta_1, \beta_2 \in \Xi$. In such cases, the mapping satisfying the previous contraction is called a p-contraction. The following example shows that a mapping $\mathcal{F} : \Re \to \Omega$, where $\Re, \Omega$ are non-empty subsets of a topological space Ξ, may be a topological p-proximal contraction with respect to a mapping $\Phi : \Xi \times \Xi \to \mathbb{R}$ and may not be such with respect to another mapping $g : \Xi \times \Xi \to \mathbb{R}$.

Example 2.3. Let us consider the topological space $\mathbb{R}$ with usual topology and the sets $\Re = \{(x, y) : 3 \leq x \leq 4, 4 \leq y \leq 6\}$ and $\Omega = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Also consider that $\Phi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous mapping defined as $\Phi((x_1, y_1), (x_2, y_2)) = y_1 - y_2$ for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Then $D_\Phi(\Re, \Omega) = 3$. Let us define $\mathcal{F} : \Re \to \Omega$ by

$$\mathcal{F}(x, y) = \left(\frac{x}{3} - 1, \frac{y}{8}\right) \text{ for all } (x, y) \in \Re.$$

Let $\alpha_1 = (x_1, y_1), \alpha_2 = (x'_1, y'_1), \beta_1 = (x_2, y_2), \beta_2 = (x'_2, y'_2) \in \mathbb{R}^2$ be such that

$$\begin{array}{l}|\Phi(\alpha_1, \mathcal{F}(\beta_1))| = D_\Phi(\Re, \Omega)\\ |\Phi(\alpha_2, \mathcal{F}(\beta_2))| = D_\Phi(\Re, \Omega).\end{array}$$

Then

$$\begin{aligned}|\Phi(\alpha_1, \mathcal{F}(\beta_1))| &= 3\\ \Rightarrow \left|\left((x_1, y_1), \left(\frac{x'_1}{3} - 1, \frac{y'_1}{8}\right)\right)\right| &= 3\\ \Rightarrow \left|y_1 - \frac{y'_1}{8}\right| &= 3.\end{aligned}$$

Similarly,

$$\begin{aligned}|\Phi(\alpha_2, \mathcal{F}(\beta_2))| = D_\Phi(\Re, \Omega) &= 3\\ \Rightarrow \left|y_2 - \frac{y'_2}{8}\right| &= 3.\end{aligned}$$

Now,

$$|\Phi(\alpha_1, \alpha_2)| = |\Phi((x_1, y_1), (x_2, y_2))| = |y_2 - y_1|. \tag{2.5}$$

Again,

$$|\Phi(\beta_1, \beta_2)| + ||\Phi(\alpha_1, \beta_1)| - |\Phi(\alpha_2, \beta_2)|| = |y_1' - y_2'| + ||y_1 - y_1'| - |y_2 - y_2'|| . \tag{2.6}$$

Then the following cases occur:
Case 1: Let

$$y_1 - \frac{y_1'}{8} = 3 \Rightarrow y_1 = \frac{y_1'}{8} + 3$$

and

$$y_2 - \frac{y_2'}{8} = 3 \Rightarrow y_2 = \frac{y_2'}{8} + 3.$$

Then by using (2.5) and (2.6), we get

$$|y_2 - y_1| = \left| \frac{y_1'}{8} - \frac{y_2'}{8} \right| = \frac{1}{8}|y_2' - y_1'|$$

and subsequently,

$$\begin{aligned}|\Phi(\alpha_1, \alpha_2)| &\leq c\left(|\Phi(\beta_1, \beta_2)| + ||\Phi(\alpha_1, \beta_1)| - |\Phi(\alpha_2, \beta_2)||\right) \\ &= c\left(|y_1' - y_2'| + ||y_1 - y_1'| - |y_2 - y_2'||\right), \text{ where } c = \frac{1}{2}.\end{aligned}$$

Case 2: Let

$$y_1 - \frac{y_1'}{8} = -3 \Rightarrow y_1 = \frac{y_1'}{8} - 3$$

and

$$y_2 - \frac{y_2'}{8} = -3 \Rightarrow y_2 = \frac{y_2'}{8} - 3.$$

Then by using (2.5) and (2.6), we get

$$|y_2 - y_1| = \left| \frac{y_1'}{8} - \frac{y_2'}{8} \right| = \frac{1}{8}|y_2' - y_1'|$$

and subsequently,

$$\begin{aligned}|\Phi(\alpha_1, \alpha_2)| &\leq c\left(|\Phi(\beta_1, \beta_2)| + ||\Phi(\alpha_1, \beta_1)| - |\Phi(\alpha_2, \beta_2)||\right) \\ &= c\left(|y_1' - y_2'| + ||y_1 - y_1'| - |y_2 - y_2'||\right), \text{ where } c = \frac{1}{2}.\end{aligned}$$

Then it can be verified that $|\Phi(\alpha_1, \alpha_2)| \leq c\left(|\Phi(\beta_1, \beta_2)| + ||\Phi(\alpha_1, \beta_1)| - |\Phi(\alpha_2, \beta_2)||\right)$ for some $c \in [0, 1)$. Therefore $\mathcal{F}$ is a topological p-proximal contraction with respect to Φ.

On the other hand, if we define $\Phi_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by $|\Phi_1(x_1, y_1), (x_2, y_2)| = 1$, then it can be easily verified that $\mathcal{F}$ is not a topological p-proximal contraction with respect to Φ_1.

In the subsequent example, we show that there exists a topological space Ξ and a mapping $\mathcal{F} : \Re \to \Omega$, where $\Re, \Omega$ are non-empty subsets of Ξ, such that $\mathcal{F}$ is a topological p-proximal contraction with respect to a continuous real-valued function Φ. Notice that if the topological space is metrizable with respect to a metric d, then the mapping $\mathcal{F}$ is not a p-proximal contraction with respect to the metric d.

Example 2.4. Let us consider $\mathbb{R}$ with the usual topology. Let $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\Phi(u, v) = u^2 - v^2, \text{ for all } u, v \in \mathbb{R}.$$

Then Φ is a continuous function. Let $\Re = \{0, 1, 2, 3, 5\}$ and $\Omega = \{-1, -2, 4\}$, and $\mathcal{F} : \Re \to \Omega$ be defined as $\mathcal{F}(0) = \mathcal{F}(5) = 4, \mathcal{F}(2) = \mathcal{F}(3) = -2, \mathcal{F}(1) = -1$. Then it can be easily verified that $D_\Phi(\Re, \Omega) = 0$. Let $\alpha_1 = \alpha_2 = \beta_1 = 2$ and $\beta_2 = 3$, then

$$|\Phi(\alpha_1, \mathcal{F}(\beta_1))| = D_\Phi(\Re, \Omega) = 0,$$
$$|\Phi(\alpha_2, \mathcal{F}(\beta_2))| = D_\Phi(\Re, \Omega) = 0.$$

Therefore it can be verified that for $c = \frac{1}{10}$,

$$0 = |\Phi(\alpha_1, \alpha_2)| \leq c\,(|\Phi(\beta_1, \beta_2)| + ||\Phi(\alpha_1, \beta_1)| - |\Phi(\alpha_2, \beta_2)||)$$

for all $\alpha_1, \alpha_2, \beta_1, \beta_2$ with $\beta_1 \neq \beta_2$. Thus $\mathcal{F}$ is a topological p-proximal contraction with respect to Φ. On the other hand, let d be the usual metric on $\mathbb{R}$ and $D_d(\Re, \Omega) = \inf\{d(u, v) : u \in \Re, v \in \Omega\}$. Therefore $D_d(\Re, \Omega) = 1$. Now

$$|d(5, \mathcal{F}(0))| = D_d(\Re, \Omega) = 1,$$
$$|d(0, \mathcal{F}(1))| = D_d(\Re, \Omega) = 1.$$

But $5 = d(5, 0) > c\,(|d(0, 1)| + ||d(5, 0)| - |d(1, 0)||)$ for $c = \frac{1}{10}$. Hence $\mathcal{F}$ is not a p-proximal contraction with respect to d for $c = \frac{1}{10}$.

Now we are in a position to present a best proximity point theorem concerning the newly introduced topological p-proximal contractions on an arbitrary topological space.

Theorem 2.1. *Let Ξ be a Φ-complete topological space, where $\Phi : \Xi \times \Xi \to \mathbb{R}$ is a continuous function such that $\Phi(x, y) = 0$ if and only if $x = y$, $|\Phi(x, y)| = |\Phi(y, x)|$ and $|\Phi(x, z)| \leq |\Phi(x, y)| + |\Phi(y, z)|$ for all $x, y, z \in \Xi$. Let $\Re, \Omega$ be non-empty subsets where $\Re$ is a Φ-closed subset of Ξ such that Ω is approximately Φ-compact with respect to $\Re$. Let $\mathcal{F} : \Re \to \Omega$ be a topological p-proximal contraction with respect to Φ such that $\mathcal{F}(\Re_\Phi) \subseteq \Omega_\Phi$ and $\Re_\Phi$ is non-empty. Then $\mathcal{F}$ has a unique best proximity point.*

Proof. Let $\xi_0 \in \Re_\Phi$ be an arbitrary element. Since $\mathcal{F}(\xi_0) \in \mathcal{F}(\Re_\Phi) \subseteq \Omega_\Phi$, there exists $\xi_1 \in \Re_\Phi$ such that $|\Phi(\xi_1, \mathcal{F}(\xi_0))| = D_\Phi(\Re, \Omega)$. Similarly as $\mathcal{F}(\mathcal{F}(\xi_1)) \in \mathcal{F}(\Re_\Phi) \subseteq \Omega_\Phi$, there exists $\xi_2 \in \Re_\Phi$ such that $|\Phi(\xi_2, \mathcal{F}(\xi_1))| = D_\Phi(\Re, \Omega)$. Proceeding in this way, we can construct a sequence (ξ_n) in $\Re_\Phi$ such that

$$|\Phi(\xi_{n+1}, \mathcal{F}(\xi_n))| = D_\Phi(\Re, \Omega) \text{ for all } n \in \mathbb{N}. \tag{2.7}$$

Now, if there exists an $n \in \mathbb{N}$ such that $\xi_n = \xi_{n+1}$, then it is clear that ξ_n is the best proximity point of the mapping $\mathcal{F}$. Let we assume that $\xi_n \neq \xi_{n+1}$ for all $n \in \mathbb{N}$. From the construction of the sequence, we have

$$|\Phi(\xi_n, \mathcal{F}(\xi_{n-1}))| = D_\Phi(\Re, \Omega),$$
$$|\Phi(\xi_{n+1}, \mathcal{F}(\xi_n))| = D_\Phi(\Re, \Omega),$$

for all $n \in \mathbb{N}$. As $\mathcal{F}$ is a topological p-proximal contraction with respect to Φ, we have

$$|\Phi(\xi_n, \xi_{n+1})| \leq c\,(|\Phi(\xi_{n-1}, \xi_n)| + ||\Phi(\xi_{n-1}, \xi_n)| - |\Phi(\xi_{n+1}, \xi_n)||)$$

for all $n \in \mathbb{N}$ and $c \in (0, 1)$. Let us suppose that $|\Phi(\xi_{n-1}, \xi_n)| \leq |\Phi(\xi_{n+1}, \xi_n)|$ for some $n \in \mathbb{N}$. Then

$$\begin{aligned}|\Phi(\xi_n, \xi_{n+1})| &\leq c\,(|\Phi(\xi_{n-1}, \xi_n)| + |\Phi(\xi_{n+1}, \xi_n)| - \Phi(\xi_{n-1}, \xi_n)|) \\ &\leq c|\Phi(\xi_{n+1}, \xi_n)|,\end{aligned}$$

which is a contradiction. Thus the only possibility is $|\Phi(\xi_n, \xi_{n+1})| \leq |\Phi(\xi_{n-1}, \xi_n)|$ for some $n \in \mathbb{N}$. Then

$$\begin{aligned}
|\Phi(\xi_n, \xi_{n+1})| &\leq c\left(|\Phi(\xi_{n-1}, \xi_n)| + |\Phi(\xi_{n-1}, \xi_n)| - |\Phi(\xi_{n+1}, \xi_n)|\right) \\
&\leq c\left(2|\Phi(\xi_{n-1}, \xi_n)| - |\Phi(\xi_{n+1}, \xi_n)|\right) \\
(1+c)|\Phi(\xi_n, \xi_{n+1})| &\leq c\left(2|\Phi(\xi_{n-1}, \xi_n)|\right) \\
|\Phi(\xi_n, \xi_{n+1})| &\leq \frac{2c}{1+c}\,|\Phi(\xi_{n-1}, \xi_n)| \\
&\leq \left(\frac{2c}{1+c}\right)^2 |\Phi(\xi_{n-2}, \xi_{n-1})| \leq \cdots \leq \left(\frac{2c}{1+c}\right)^n |\Phi(\xi_0, \xi_1)|\,.
\end{aligned}$$

Suppose that $m > n$ and $n \in \mathbb{N}$. Let $m = n + p$ where $p \geq 1$. Then by the given condition and the above inequality, we have

$$\begin{aligned}
|\Phi(\xi_n, \xi_m)| &= |\Phi(\xi_n, \xi_{n+p})| \\
&\leq |\Phi(\xi_n, \xi_{n+1})| + |\Phi(\xi_{n+1}, \xi_{n+2})| + |\Phi(\xi_{n+1}, \xi_{n+2})| + \cdots + |\Phi(\xi_{n+p-1}, \xi_{n+p})| \\
&\leq \left(\frac{2c}{1+c}\right)^n |\Phi(\xi_0, \xi_1)| + \left(\frac{2c}{1+c}\right)^{n+1} |\Phi(\xi_0, \xi_1)| \\
&+ \left(\frac{2c}{1+c}\right)^{n+2} |\Phi(\xi_0, \xi_1)| + \cdots + \left(\frac{2c}{1+c}\right)^{n+p-1} |\Phi(\xi_0, \xi_1)| \\
&= \left\{1 + \frac{2c}{1+c} + \left(\frac{2c}{1+c}\right)^2 + \cdots + \left(\frac{2c}{1+c}\right)^{p-1}\right\} \left(\frac{2c}{1+c}\right)^n |\Phi(\xi_0, \xi_1)| \\
&= \left(\frac{2c}{1+c}\right)^n \frac{1 - \left(\frac{2c}{1+c}\right)^p}{1 - \frac{2c}{1+c}} |\Phi(\xi_0, \xi_1)| \\
&\leq \left(\frac{2c}{1+c}\right)^n \frac{1}{1 - \frac{2c}{1+c}} |\Phi(\xi_0, \xi_1)|.
\end{aligned}$$

Taking limit on both sides of the above equation, we get $|\Phi(\xi_n, \xi_m)| \to 0$ as $n, m \to \infty$. Therefore, the sequence (ξ_n) is a Φ-Cauchy sequence. Since Ξ is Φ-complete and $\Re$ is Φ-closed subset of Ξ, there exists a point $\xi^* \in \Re$ such that $|\Phi(\xi_n, \xi^*)| \to 0$ for all $n \in \mathbb{N}$. Again from (2.7) and given condition, we get

$$\begin{aligned}
|\Phi(\xi^*, \mathcal{F}(\xi_n))| &\leq |\Phi(\xi^*, \xi_{n+1})| + |\Phi(\xi_{n+1}, \mathcal{F}(\xi_n))| \\
&\leq |\Phi(\xi^*, \xi_{n+1})| + D_\Phi(\Re, \Omega) \\
&\leq |\Phi(\xi^*, \xi_{n+1})| + D_\Phi(\xi^*, \Omega).
\end{aligned}$$

Taking limit on both sides, we have $|\Phi(\xi^*, \mathcal{F}(\xi_n))| \to |\Phi(\xi^*, \Omega)|$ as $n \to \infty$. Since Ω is approximately Φ-compact with respect to $\Re$, there exists a subsequence $(\mathcal{F}(\xi_{n_k}))$ of $(\mathcal{F}(\xi_n))$ such that $|\Phi\left(\mathcal{F}(\xi_{n_k}), \gamma\right)| \to 0$ as $k \to \infty$ for some $\gamma \in \Omega$. Then

$$\begin{aligned}
|\Phi(\xi^*, \gamma)| &\leq \lim_{k\to\infty} \left[|\Phi(\xi^*, \xi_{n_{k+1}})| + |\Phi(\xi_{n_{k+1}}, \mathcal{F}(\xi_{n_k}))| + |\Phi(\mathcal{F}(\xi_{n_k}), \gamma)|\right] \\
&= \lim_{k\to\infty} |\Phi(\xi_{n_{k+1}}, \mathcal{F}(\xi_{n_k}))| = D_\Phi(\Re, \Omega).
\end{aligned}$$

Therefore, $|\Phi(\xi^*, \gamma)| = D_\Phi(\Re, \Omega)$ implies $\gamma \in \Re_\Phi$. Also since $\mathcal{F}(\Re_\Phi) \subseteq \Omega_\Phi$, there exists an element $\lambda \in \Re_\Phi$ such that

$$|\Phi(\lambda, \mathcal{F}(\xi^*))| = D_\Phi(\Re, \Omega). \tag{2.8}$$

Now we can assume without loss of generality that $\xi^* \neq \xi_n$ for all $n \in \mathbb{N}$. Then from (2.7), (2.8) and the definition of topological p-proximal contraction, we get

$$|\Phi(\xi_{n+1}, \lambda)| \leq c\left(|\Phi(\xi_n, \xi^*)| + ||\Phi(\xi_{n+1}, \xi_n)| - |\Phi(\lambda, \xi^*)||\right)$$

for all $n \in \mathbb{N}$. Taking limit both sides, we obtain

$$|\Phi(\xi^*, \lambda)| \leq c|\Phi(\lambda, \xi^*)|,$$

which gives that $\xi^* = \lambda$. Therefore from (2.8) we conclude that ξ^* is the best proximity point of the mapping $\mathcal{F}$. For uniqueness, let us assume that there are two different best proximity points ξ^* and λ^* of the mapping $\mathcal{F}$ on $\Re$. Then we get

$$\begin{aligned} |\Phi(\xi^*, \mathcal{F}(\xi^*))| &= D_\Phi(\Re, \Omega), \\ |\Phi(\lambda^*, \mathcal{F}(\lambda^*))| &= D_\Phi(\Re, \Omega). \end{aligned}$$

Since $\mathcal{F}$ is a topological p-proximal contraction, then we have

$$\begin{aligned} |\Phi(\xi^*, \lambda^*)| &\leq c\left(|\Phi(\xi^*, \lambda^*)| + ||\Phi(\xi^*, \xi^*)| - |\Phi(\lambda^*, \lambda^*)||\right) \\ &= c|\Phi(\xi^*, \lambda^*)|, \end{aligned}$$

which is a contradiction. Therefore $\lambda^* = \xi^*$. This completes the proof. □

Next we illustrate the previously proved Theorem 2.1 by an example.

Example 2.5. Let us consider the topological space $\mathbb{R}$ with usual topology and $\Xi = [-1, 1] \times [-1, 1]$. Let $\Re = \{(x, y) : \frac{2}{3} \leq x \leq 1, \frac{2}{3} \leq y \leq 1\}$ and $\Omega = \{(x, y) : 0 \leq x \leq \frac{1}{2}, 0 \leq y \leq \frac{1}{2}\}$. Let us define $\mathcal{F} : \Re \to \Omega$ by

$$\mathcal{F}(x, y) = \left(\frac{x}{2}, \frac{y}{3}\right) \text{ for all } (x, y) \in \Re.$$

Let $\Phi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a mapping defined as $\Phi\left((x_1, y_1), (x_2, y_2)\right) = y_1 - y_2$ for all (x_1, y_1), $(x_2, y_2) \in \mathbb{R}$. Then Φ is a continuous mapping on $\Xi \times \Xi$ and $D_\Phi(\Re, \Omega) = \frac{1}{6}$. Now, it can be easily seen that Ξ is Φ-complete. Also

$$\Re_\Phi = \left\{\left(x_1, \frac{2}{3}\right) : \frac{2}{3} \leq x_1 \leq 1\right\}$$

and

$$\Omega_\Phi = \left\{\left(x_2, \frac{1}{2}\right) : 0 \leq x_2 \leq \frac{1}{2}\right\}.$$

Then

$$\begin{aligned} \mathcal{F}(\Re_\Phi) &= \mathcal{F}\left(x_1, \frac{2}{3}\right), \text{ where } \frac{2}{3} \leq x_1 \leq 1 \\ &= \left(\frac{x_1}{2}, \frac{2}{9}\right) \subseteq \Omega_\Phi. \end{aligned}$$

Thus $\Re_\Phi$ is non-empty, Φ-closed and also $\mathcal{F}(\Re_\Phi) \subseteq \Omega_\Phi$. Let $\alpha_1 = (x_1, y_1), \alpha_2 = (x_1', y_1'), \beta_1 = (x_2, y_2), \beta_2 = (x_2', y_2') \in \Re$ be such that

$$\begin{aligned} |\Phi(\alpha_1, \mathcal{F}(\beta_1))| &= D_\Phi(\Re, \Omega), \\ |\Phi(\alpha_2, \mathcal{F}(\beta_2))| &= D_\Phi(\Re, \Omega). \end{aligned}$$

Then

$$|\Phi(\alpha_1, \mathcal{F}(\beta_1))| = \frac{1}{6}$$
$$\Rightarrow \left| \left((x_1, y_1), \left(\frac{x'_1}{2}, \frac{y'_1}{3} \right) \right) \right| = \frac{1}{6}$$
$$\Rightarrow \left| y_1 - \frac{y'_1}{3} \right| = \frac{1}{6}.$$

Similarly,

$$|\Phi(\alpha_2, \mathcal{F}(\beta_2))| = D_\Phi(\Re, \Omega) = \frac{1}{6}$$
$$\Rightarrow \left| y_2 - \frac{y'_2}{3} \right| = \frac{1}{6}.$$

Now,

$$(2.9) \qquad |\Phi(\alpha_1, \alpha_2)| = |\Phi((x_1, y_1), (x_2, y_2))| = |y_2 - y_1|.$$

Again,

$$(2.10) \qquad |\Phi(\beta_1, \beta_2)| + ||\Phi(\alpha_1, \beta_1)| - |\Phi(\alpha_2, \beta_2)|| = |y'_1 - y'_2| + ||y_1 - y'_1| - |y_2 - y'_2||\,.$$

Then the following cases occur:

Case 1: Let

$$y_1 - \frac{y'_1}{3} = \frac{1}{6} \Rightarrow y_1 = \frac{y'_1}{3} + \frac{1}{6}$$

and

$$y_2 - \frac{y'_2}{3} = \frac{1}{6} \Rightarrow y_2 = \frac{y'_2}{3} + \frac{1}{6}.$$

Then by using (2.9) and (2.10), we get

$$|y_2 - y_1| = \left| \frac{y'_1}{3} - \frac{y'_2}{3} \right| = \frac{1}{3}|y'_2 - y'_1|,$$

and subsequently,

$$|\Phi(\alpha_1, \alpha_2)| \leq c\left(|\Phi(\beta_1, \beta_2)| + ||\Phi(\alpha_1, \beta_1)| - |\Phi(\alpha_2, \beta_2)||\right)$$
$$= c\left(|y'_1 - y'_2| + ||y_1 - y'_1| - |y_2 - y'_2||\right), \text{ where } c = \frac{2}{3}.$$

Case 2: Let

$$y_1 - \frac{y'_1}{3} = -\frac{1}{6} \Rightarrow y_1 = \frac{y'_1}{3} - \frac{1}{6}$$

and

$$y_2 - \frac{y'_2}{3} = -\frac{1}{6} \Rightarrow y_2 = \frac{y'_2}{3} - \frac{1}{6}.$$

Then by using (2.9) and (2.10), we get

$$|y_2 - y_1| = \left| \frac{y'_1}{3} - \frac{y'_2}{3} \right| = \frac{1}{3}|y'_2 - y'_1|$$

and subsequently,

$$|\Phi(\alpha_1, \alpha_2)| \leq c\left(|\Phi(\beta_1, \beta_2)| + ||\Phi(\alpha_1, \beta_1)| - |\Phi(\alpha_2, \beta_2)||\right)$$
$$= c\left(|y'_1 - y'_2| + ||y_1 - y'_1| - |y_2 - y'_2||\right), \text{ where } c = \frac{2}{3}.$$

Then it can be verified that $|\Phi(\alpha_1, \alpha_2)| \leq c\left(|\Phi(\beta_1, \beta_2)| + ||\Phi(\alpha_1, \beta_1)| - |\Phi(\alpha_2, \beta_2)||\right)$ for some $c \in [0, 1)$. This assures that the mapping $\mathcal{F}$ is a topological p-proximal contraction with respect to Φ. Also all the hypotheses of the Theorem 2.1 are satisfied. Therefore, we can conclude that the mapping $\mathcal{F}$ has a unique best proximity point $(\frac{2}{3}, \frac{1}{2})$.

In the subsequent discussion, we first introduce the concept of topological p-contractive mappings and further, confirm a result of best proximity point theorem related to topological p-proximal contractive mappings on topological spaces.

Definition 2.10. Let $\Re, \Omega$ be two non-empty subsets of a topological space Ξ. A mapping $\mathcal{F} : \Re \to \Omega$ is said to be a topological p-proximal contractive mapping if

$$\left.\begin{array}{l} |\Phi(\alpha_1, \mathcal{F}(\beta_1))| = D_\Phi(\Re, \Omega), \\ |\Phi(\alpha_2, \mathcal{F}(\beta_2))| = D_\Phi(\Re, \Omega) \end{array}\right\} \Rightarrow |\Phi(\alpha_1, \alpha_2)| < |\Phi(\beta_1, \beta_2)| + ||\Phi(\alpha_1, \beta_1)| - |\Phi(\alpha_2, \beta_2)||$$

for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Re$ with $\beta_1 \neq \beta_2$.

Note that if Ξ be a metric space and Φ be a metric on Ξ, then this reduces to the notion of usual p-proximal contractive mappings. Also if $\Re = \Omega = \Xi$, then the above inequality can be written as

$$|\Phi(\mathcal{F}(\beta_1), \mathcal{F}(\beta_2))| < |\Phi(\beta_1, \beta_2)| + ||\Phi(\beta_1, \mathcal{F}(\beta_2))| - |\Phi(\beta_2, \mathcal{F}(\beta_2))||$$

for all $\beta_1, \beta_2 \in \Xi$.

Now, we are in a position to prove a best proximity point theorem for topological p-proximal contractive mappings. Further, we give an example to validate our finding.

Theorem 2.2. *Let Ξ be a topological space, where $\Phi : \Xi \times \Xi \to \mathbb{R}$ is a continuous function satisfying the following $|\Phi(x, y)| = |\Phi(y, x)|$ and $|\Phi(x, z)| \leq |\Phi(x, y)| + |\Phi(y, z)|$ for all $x, y, z \in \Xi$. Let $\Re, \Omega$ be non-empty subsets of Ξ and $\mathcal{F} : \Re \to \Omega$ be a topological p-proximal contractive mapping with respect to Φ. Assume that $(\Re, \Omega)$ has the topological p-property with respect to Φ and $\mathcal{F}(\Re_\Phi) \subseteq \Omega_\Phi$. If there exists $\xi, \gamma \in \Re_\Phi$ such that*

$$|\Phi(\xi, \mathcal{F}(\lambda))| = D_\Phi(\Re, \Omega) \tag{2.11}$$

and

$$|\Phi(\xi, \lambda)| \leq |\Phi(\mathcal{F}(\xi), \mathcal{F}(\lambda))|, \tag{2.12}$$

then $\mathcal{F}$ has a unique best proximity point.

Proof. Let $\xi, \lambda \in \Re_\Phi$. Since $\mathcal{F}(\Re_\Phi) \subseteq \Omega_\Phi$, then there exists a point $\gamma \in \Re_\Phi$ such that

$$|\Phi(\gamma, \mathcal{F}(\xi))| = D_\Phi(\Re, \Omega). \tag{2.13}$$

Employing the fact that $(\Re, \Omega)$ has the topological p-property and using (2.11)-(2.13), we have

$$|\Phi(\xi, \gamma)| = |\Phi\left(\mathcal{F}(\lambda), \mathcal{F}(\xi)\right)|. \tag{2.14}$$

Let us assume $\xi \neq \lambda$. As $\mathcal{F}$ is a topological p-proximal contractive mapping and applying (2.11)-(2.13), we have

$$|\Phi(\xi, \gamma)| < |\Phi(\xi, \lambda)| + ||\Phi(\xi, \lambda)| - |\Phi(\gamma, \xi)||\,. \tag{2.15}$$

Then from (2.15), by using (2.12) and (2.14), we get

$$|\Phi(\xi, \gamma)| < |\Phi(\xi, \lambda)| + |\Phi(\gamma, \xi)| - |\Phi(\xi, \lambda)| = |\Phi(\gamma, \xi)|,$$

which is a contradiction. Therefore $\xi = \lambda$. That is $\mathcal{F}$ has a best proximity point. For uniqueness, let us assume that there are two different best proximity points ξ^* and λ^* of the mapping $\mathcal{F}$ on $\Re$. Then we get

$$\begin{array}{l} |\Phi(\xi^*, \mathcal{F}(\xi^*))| = D_\Phi(\Re, \Omega), \\ |\Phi(\lambda^*, \mathcal{F}(\lambda^*))| = D_\Phi(\Re, \Omega). \end{array}$$

Since $\mathcal{F}$ is a topological p-proximal contractive mapping, we have

$$|\Phi(\xi^*, \lambda^*)| < |\Phi(\xi^*, \lambda^*)| + ||\Phi(\xi^*, \xi^*| - |\Phi(\lambda^*, \lambda^*|| = |\Phi(\xi^*, \lambda^*)|,$$

which is a contradiction. Therefore $\lambda^* = \xi^*$ and $\mathcal{F}$ has a unique best proximity point. □

Here we present one corollary of our obtained theorem which is the generalization of Edelstein fixed point theorem [9] on metric spaces.

Corollary 2.1. *If we consider, $|\Phi(\beta_1, \mathcal{F}(\beta_2))| = |\Phi(\beta_2, \mathcal{F}(\beta_2))|$ in Theorem 2.2, then we get the Edelstein fixed point theorem in the topological space Ξ.*

Next we furnish a supporting example of the above Theorem.

Example 2.6. Consider $\mathbb{R}^2$ with usual topology and $\Xi = [-1, 1] \times [-1, 1]$ with subspace topology. Let $\Re = \{0\} \times [-1, 0]$ and $\Omega = \{1\} \times [0, 1]$. Let $\Phi : \Xi \times \Xi \to \mathbb{R}$ defined by $\Phi((\xi_1, \xi_2), (\lambda_1, \lambda_2)) = \xi_2^2 - \lambda_2^2$. Then Φ is a continuous mapping on $\Xi \times \Xi$ and $D_\Phi(\Re, \Omega) = 0$. We define the mapping $\mathcal{F} : \Re \to \Omega$ as $\mathcal{F}(0, t) = \left(1, -\frac{t}{5}\right)$, for all $t \in [-1, 0]$. Now it can be easily verified that Ξ is Φ-complete. Now, let $(0, \xi) \in \Re_\Phi$. Then there exists $(0, \lambda) \in \Omega$ such that $|\Phi((0, \xi), (0, \lambda))| = 0$ implies $|\xi^2 - \lambda^2| = 0$. This is satisfied only when $\xi = \lambda = 0$. Therefore $\Re_\Phi = \{(0, 0)\}$ and also $\Omega_\Phi = \{(0, 0)\}$. Thus $\Re_\Phi$ is non-empty, Φ-closed and also $\mathcal{F}(\Re_\Phi) \subseteq \Omega_\Phi$. Then the mapping $\mathcal{F}$ is topological p-proximal contractive with respect to Φ and $(\Re, \Omega)$ satisfies the p-property. Additionally for $\xi = \lambda = (0, 0)$, the conditions (2.11) and (2.12) hold and further, all the hypotheses of Theorem 2.2 are satisfied. Therefore, we can conclude that the mapping $\mathcal{F}$ has a unique best proximity point.

Acknowledgement. The authors would like to thank the anonymous referees for their careful reading of the manuscript and many useful comments which have undoubtedly improved the previous versions of the article. The Research is funded by the Ministry of Education, Government of India. For the third author, this research is funded by the 2022 Development Fund of Babeş-Bolyai University Cluj-Napoca.

References

[1] Altun, I.; Aslantas, M; Sahin, H. Best proximity point results for p-proximal contractions. *Acta Math. Hungar.* **162** (2020), no. 2, 393–402.

[2] Basha, S. S. Best proximity points: optimal solutions. *J. Optim. Theory Appl.* **151** (2011), no. 1, 210–216.

[3] Basha, S. S.; Shahzad, N.; Jeyaraj, R. Best proximity point theorems: exposition of a significant non-linear programming problem. *J. Glob. Optim.* **56** (2013), no. 4, 1699–1705.

[4] Berinde, V. *Iterative Approximation of Fixed Points.* Springer, Berlin, 2007.

[5] Chakraborty, U. S. Convexity and best approximation in Banach spaces. *Rend. Circ. Mat. Palermo* **(2)**. **71** (2022), no. 1, 247–258.

[6] Chidume, C. *Geometric Properties of Banach Spaces and Nonlinear Iterations.* Springer-Verlag, London, 2009.

[7] Das. P.; Dey, L. K. Fixed point of contractive mappings in generalized metric spaces. *Math. Slovaca*. **59** (2009), no. 4, 499–504.

[8] Dey, L. K.; Mondal, S. Best proximity point of F-contraction in complete metric space. *Bull. Allahabad Math. Soc.* **30** (2015), no. 2, 173–189.

[9] Edelstein, M. On fixed and periodic points under contractive mappings. *J. Lond. Math. Soc.* **37** (1962), no. 1, 74–79.

[10] Eldred, A. A.; Veeramani, P. Existence and convergence of best proximity points. *J. Math. Anal. Appl.* **323** (2006), no. 2, 1001–1006.

[11] Gabeleh, M. Best proximity point theorems via proximal non-self mapping. *J. Optim. Theory Appl.* **164** (2015), no. 2, 565–576.

[12] Gabeleh, M.; Markin, J. A note on the paper "Best proximity point results for p-proximal contractions". *Acta Math. Hungar.* **164** (2021), no. 1, 326–329.

[13] Granas, A.; Dugundji, J. *Fixed Point Theory.* Springer-Verlag, New York, 2003.

[14] Ilchev, A.; Zlatanov, B. Fixed and best proximity points for Kannan cyclic contractions in modular function spaces. *J. Fixed Point Theory Appl.* **19** (2017), no. 4, 2873–2893.

[15] Jolaoso, L. O. A self-adaptive Tseng extragradient method for solving monotone variational inequality and fixed point problems in Banach spaces. *Demonstr. Math.* **54** (2021), no. 1, 527–547.

[16] Karmakar, S.; Dey, L. K.; Kumam, P.; Chanda, A. Best proximity results for cyclic α-implicit contractions in quasi-metric spaces and its consequences. *Advances in Fixed Point Theory*. **7** (2017), no. 3, 342–358.

[17] Kirk, W. A.; Sims, B. *Handbook of Metric Fixed Point Theory.* Kluwer Acad. Publ., Dordrecht, 2001.

[18] Kirk, W. A.; Reich, S.; Veeramani, P. Proximinal retracts and best proximity pair theorems. *Numer. Funct. Anal. Optim.* **24** (2003), no. 7-8, 851–862.

[19] Popescu, O. A new type of contractive mappings in complete metric spaces. Preprint.

[20] Rus, I. A.; Petruşel, A.; Petruşel, G. *Fixed Point Theory.* Cluj University Press, Cluj-Napoca, 2008.

[21] Sankar Raj, V.; Piramatchi, T. Best proximity point theorems is topological spaces. *J. Fixed Point Theory Appl.* **22** (2020), no. 1. Article 2.

[22] Sankar Raj, V. A best proximity point theorem for weakly contractive non-self mappings. *Nonlinear Anal.* **74** (2011), no. 14, 4804–4808.

[23] Shahzad, N.; Sadiq Basha, S.; Jeyaraj, R. Common best proximity points: global optimal solutions. *J. Optim. Theory Appl.* **148** (2011), no. 1, 69–78.

[24] Som, S.; Laha, S.; Petruşel, A.; Dey, L. K. Best proximity point results on arbitrary topological spaces and the Banach contraction principle revisited. *J. Nonlinear Convex Anal.* **23** (2022), no. 1, 67–85.

[25] Suzuki, T.; Kikkawa, M.; Vetro, C. The existence of best proximity points in metric spaces with the property UC. *Nonlinear Anal.* **71** (2009), no. 7-8, 2918–2926.

[26] Shukla, R.; Panicker, R. Some fixed point theorems for generalized enriched nonexpansive mappings in Banach spaces. *Rendiconti del Circolo Matematico di Palermo Series 2*, (2022), 1–15

[27] Vetro, C. Best proximity points: Convergence and existence theorems for p-cyclic mappings. *Nonlinear Anal.* **73** (2010), no. 7, 2283–2291.

[28] Włodarczyk, K.; Plebaniak, R.; Banach, A. Best proximity points for cyclic and noncyclic set-valued relatively quasi-asymptotic contractions in uniform spaces. *Nonlinear Anal.* **70** (2009), no. 9, 3332–3341.

[29] Zlatanov, B. Best proximity points in modular function spaces. *Arab. J. Math.* **4** (2015), no. 3, 215–227.

[1] Department of Mathematics, National Institute of Technology Durgapur, India

[2] Department of Mathematics, Babeş-Bolyai University Cluj-Napoca, Romania

[3] Academy of Romanian Scientists, Bucharest, Romania

[4] Department of Mathematics, Vellore Institute of Technology, Vellore, India

[5] Department of Mathematics, Vellore Institute of Technology, Chennai, India

Email address: beraashis.math@gmail.com

Email address: lakshmikdey@yahoo.co.in

Email address: petrusel@math.ubbcluj.ro

Email address: ankushchanda8@gmail.com