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Generalized perturbed contractions with related fixed point results

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ABSTRACT. In this paper, we introduce the concept of (F,G)-perturbed contraction in a metric space and we establish some fixed point theorems which extends the results from Branga, A. N. Olaru, I. M. Some Fixed Point Results in Spaces with Perturbed Metrics. *Carpathian J. Math.* **38** (2022), no. 3, 641–654 and Olaru, I. M.; Secelean, N. A. A new approach of some contractive mappings on metric spaces. *Mathematics* **9** (2021), 1433. Also, an application to an integral equation and some illustrative examples are given throughout the paper.

1. INTRODUCTION

Wardowski [15] introduced a new type of contractive self-mapping T defined on a metric space (X,d), the so called F-contraction. This is defined by the inequality

$$\tau + F(d(Tx,Ty)) \leq F(d(x,y)), \text{ for all x, y} \in \text{X, Tx} \neq \text{Ty},$$

where $\tau > 0$ and $F:(0,\infty) \to \mathbb{R}$ satisfies the conditions $(F1)$-$(F3)$ defined bellow:

- (F_1) F is a strictly increasing function;
- (F_2) for each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers, we have $\lim\limits_{n\to\infty} \alpha_n = 0$ if and only if $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$;
- (F_3) there exists $k \in (0,1)$ such that $\lim\limits_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Wardowski proved that, whenever (X,d) is complete, every F-contraction has a unique fixed point, which is the limit of the Picard iterations. This result has been expanded by weakening the condition (F_1)-(F_3) or by defining new contractive conditions. A survey regarding the extensions of F-contractions is given in Karapinar et al. [4]. Later on, Olaru and Secelean [8] generalized the concept of F-contractions and established a fixed point theorem that expands some known results in the literature. Related to the fixed point theory for an operator $T: X \to X$ on altered metric space we mention that it has been developed firstly by Delbosco [3], Skof [13], M. S. Khan, M. Swaleh and S. Sessa [5] by altering the metric with some distance control function $\phi: \mathbb{R}_+ \to \mathbb{R}_+$. A survey related to fixed point theorems by altering distances between points in metric space can be found on Jha et al. [6]. Next by using the approach from Nussbaum [7], Rus and Şerban [12] we introduce the concept of (F,G)-perturbed contraction in a metric space and we establish some fixed point theorems in less conditions than those used by Olaru and Secelean [8], which extends some fixed point results in the literature. Also, some illustrative examples are given throughout the paper, together with an application to an integral equation.

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2. Results

Definition 2.1. Let us consider $F, G : (0, \infty) \to \mathbb{R}$. We say that $(F, G) \in \mathcal{G}_0$ if the following conditions are satisfied:

(H_1) for each $r \geq t > 0$ we have $F(r) > G(t)$;
(H_2) $\liminf_{s \searrow t} \big(F(s) - G(s)\big) > 0$ for each $t > 0$.

Remark 2.1. ([8]) Under hypothesis (H_1), condition (H_2) is equivalent to

(H_2') for each sequence $\{t_n\}_{n\in\mathbb{N}} \subset (0, \infty)$, such that $t_n \searrow t > 0$, we have

$$\sum_{n\geq 1}(F(t_n) - G(t_n)) = \infty.$$

Definition 2.2. ([8]) We say that a function $F : (0, \infty) \to \mathbb{R}$ *satisfies property (P)* if, for every monotonically decreasing sequence $\{t_k\}_{k\in\mathbb{N}}$ of positive numbers such that $F(t_k) \underset{k}{\longrightarrow} -\infty$, one has $t_k \underset{k}{\longrightarrow} 0$.

Example 2.1. Let us consider $F, G : (0, \infty) \to \mathbb{R}$ defined by

$$F(t) = \ln \frac{3t^2 + 4t}{t + 1},$$
$$G(t) = \ln 2t.$$

Then, F is monotone, satisfies property (P) and $(F, G) \in \mathcal{G}_0$.

Proof. We deduce:

(H_1) If $r \geq t > 0$, then $F(r) = \ln \frac{3r^2+4r}{r+1} \geq \ln \frac{3t^2+4t}{t+1} > \ln 2t = G(t)$.
(H_2) Let any $t > 0$. Then,

$$F(t) - G(t) = \ln \frac{3t + 4}{2t + 2} > \ln \frac{3}{2}$$

and hence, $\liminf_{s\searrow t} \big(F(s) - G(s)\big) \geq \ln \frac{3}{2} > 0$.

□

In the following definition we introduce a new type of contraction self-mappings.

Definition 2.3. Let us consider (X, d) a metric space and $T : X \to X$ an operator. We say that T is a (F, G)-perturbed contraction if there exist $(F, G) \in \mathcal{G}_0$ and $g : (0, \infty) \to (0, \infty)$ a strictly increasing and continuous function, such that

$$F(g(\delta(T(B)))) \leq G(g(\delta(B))), \text{ for all } B \in P_b(X), \delta(T(B)) \neq 0, \tag{2.1}$$

where $\delta : \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ is defined by

$$\delta(A) := \sup\{d(a, b) \mid a, b \in A\} \qquad \text{and} \qquad P_b(X) := \{Y \subseteq X \mid Y \textit{ is bounded}\}.$$

Remark 2.2. If $T : X \to X$ is an operator and $B \in P_b(X)$ satisfies the condition $\delta(T(B)) \neq 0$, then $\delta(B) \neq 0$.

Proof. Let us suppose that $\delta(B) = 0$. It follows that B has a single element. Therefore, $T(B)$ contains a unique point, hence $\delta(T(B)) = 0$, which is in contradiction with our hypothesis. Consequently, $\delta(B) \neq 0$. □

Our first main result is the next one

Theorem 2.1. *Let us consider (X, d) a complete metric space, $(F, G) \in \mathcal{G}_0$ and $T : X \to X$ an operator satisfying the conditions:*

(i) *there exists* $x_0 \in X$ *such that the sequence* $\{T^n(x_0)\}_{n\in\mathbb{N}}$ *is bounded;*
(ii) F *verifies the property (P) and* G *is a monotonically increasing function;*
(iii) T *is a* (F,G)*-perturbed contraction.*

Then T *has an unique fixed point* $x^\star \in X$ *and the sequence* $\{T^n(x_0)\}_{n\in\mathbb{N}}$ *is convergent to* $x^\star \in X$.

Proof. First, we will prove that T is a contractive operator. Let us consider $x, y \in X, T(x) \neq T(y)$, be arbitrary elements. We deduce that $x \neq y$. Defining the set $B := \{x, y\}$, we obtain: $B \in P_b(X)$, $\delta(B) = d(x,y) \neq 0$, $T(B) = \{T(x), T(y)\}$, $\delta(T(B)) = d(T(x), T(y)) \neq 0$. By using the relation (2.1) we find $F(g(d(T(x),T(y)))) \leq G(g(d(x,y)))$. Considering the hypothesis (H_1), the previous inequality implies that $g(d(T(x),T(y))) < g(d(x,y))$. Taking into account that $g : (0,\infty) \to (0,\infty)$ is a strictly increasing function, we get $d(T(x),T(y)) < d(x,y)$. Since the elements $x, y \in X, T(x) \neq T(y)$, are chosen arbitrarily, it follows that

$$d(T(x),T(y)) < d(x,y), \text{ for all } x, y \in X, T(x) \neq T(y), \tag{2.2}$$

hence the operator T is contractive.

Further, we will show that T has at most one fixed point. Let us suppose that T has two distinct fixed points, i.e. there exist $x^\star, y^\star \in X$ such that $x^\star \neq y^\star$, $x^\star = T(x^\star)$, $y^\star = T(y^\star)$. By using the relation (2.2) we deduce

$$d(x^\star, y^\star) = d(T(x^\star), T(y^\star)) < d(x^\star, y^\star),$$

which is a contradiction. Therefore, T has at most one fixed point.

Since the sequence $\{T^n(x_0)\}_{n\in\mathbb{N}}$ is bounded, it follows that there exists a bounded set $A \subset X$ such that $\{T^n(x_0)\}_{n\in\mathbb{N}} \subseteq A$. Next, we will prove that $T(A)$ is a bounded set. The following cases may occur:

a) if $\delta(T(A)) \neq 0$, then by using the relation (2.1) we deduce that $F(g(\delta(T(A)))) \leq G(g(\delta(A)))$. Considering the hypothesis (H_1), we obtain $g(\delta(T(A))) < g(\delta(A))$, and taking into account that $g : (0,\infty) \to (0,\infty)$ is a strictly increasing function, we find $\delta(T(A)) < \delta(A)$. Since A is a bounded set, we get $\delta(A) < \infty$, hence $\delta(T(A)) < \infty$, thus $T(A)$ is a bounded set.
b) if $\delta(T(A)) = 0$, then $T(A)$ has a single point, hence $T(A)$ is a bounded set.

Let us define the sequence $\{A_n\}_{n\in\mathbb{N}^*}$ by

$$A_1 = T(A), A_2 = T(A_1 \cap A), \cdots, A_n = T(A_{n-1} \cap A),\ n \in \mathbb{N}^* \setminus \{1\}.$$

Applying the mathematical induction method we deduce the following properties of the sequence $\{A_n\}_{n\in\mathbb{N}^*}$:

1) $A_n \subseteq A_{n-1}$ for all $n \in \mathbb{N}^* \setminus \{1\}$;
2) $T^n(x_0) \in A_n$ for all $n \in \mathbb{N}^*$, hence $A_n \neq \emptyset$ for all $n \in \mathbb{N}^*$;
3) $A_n \subseteq T(A_{n-1})$ for all $n \in \mathbb{N}^* \setminus \{1\}$;
4) $A_n \subseteq T(A)$ for all $n \in \mathbb{N}^*$;
5) A_n is a bounded set for all $n \in \mathbb{N}^*$.

We distinguish the following cases:

1. If there exists $k \in \mathbb{N}^*$ such that $\delta(T(A_k)) = 0$, then the set $T(A_k)$ contains a single point. On the other hand, according to 2), $T^k(x_0) \in A_k$. Therefore, $T(A_k) = \{T^k(x_0)\}$. By using 2), 3) and the above equality we obtain $T^{k+1}(x_0) \in A_{k+1} \subseteq T(A_k) = \{T^k(x_0)\}$. It follows that $T^{k+1}(x_0) = T^k(x_0)$, hence $T(T^k(x_0)) = T^k(x_0)$, thus $T^k(x_0)$ is a fixed point of the operator T. Since T has at most one fixed point, we find that $T^k(x_0)$ is the unique fixed point of the operator T. Moreover, according to 2), 3) and 1), for every $n \geq k+1$ we get $T^n(x_0) \in A_n \subseteq T(A_{n-1}) \subseteq$

$T(A_k) = \{T^k(x_0)\}$, thus $T^n(x_0) = T^k(x_0)$ for all $n \geq k+1$. Therefore, the sequence $\{T^n(x_0)\}_{n\in\mathbb{N}}$ is convergent to $x^\star := T^k(x_0)$. The proof is finished in this case.

2. If $\delta(T(A_n)) \neq 0$ for all $n \in \mathbb{N}^*$. According to Remark 2.2 we deduce $\delta(A_n) \neq 0$ for all $n \in \mathbb{N}^*$. Let us define the sequence $d_n := \delta(A_n) > 0$ for all $n \in \mathbb{N}^*$. By using 1) we find $\delta(A_n) \leq \delta(A_{n-1})$ for all $n \in \mathbb{N}^* \setminus \{1\}$, hence $d_n \leq d_{n-1}$ for all $n \in \mathbb{N}^* \setminus \{1\}$, i.e. $\{d_n\}_{n\in\mathbb{N}^*}$ is a monotonically decreasing sequence. Because the sequence $\{d_n\}_{n\in\mathbb{N}^*}$ is bounded from below by 0, it follows that $\{d_n\}_{n\in\mathbb{N}^*}$ is convergent to a point $d \geq 0$. In the sequel, we will prove that $d = 0$. Let us suppose that $d > 0$. As $g : (0, \infty) \to (0, \infty)$ is a strictly increasing and continuous function, we get: $g(d_n) > 0$ for all $n \in \mathbb{N}^*$, $\{g(d_n)\}_{n\in\mathbb{N}^*}$ is a monotonically decreasing sequence and $\{g(d_n)\}_{n\in\mathbb{N}^*}$ is convergent to the point $g(d) > 0$. From the relation (2.1) and considering that G is a monotonically increasing function, g is a strictly increasing function, we deduce that

$$F(g(d_n)) = F(g(\delta(A_n))) = F(g(\delta(T(A_{n-1} \cap A)))) \leq$$
$$G(g(\delta(A_{n-1} \cap A))) \leq G(g(\delta(A_{n-1}))) = G(g(d_{n-1})) \text{ for all } n \in \mathbb{N}^* \setminus \{1\},$$

hence

$$F(g(d_n)) - F(g(d_{n-1})) \leq G(g(d_{n-1})) - F(g(d_{n-1})) \text{ for all } n \in \mathbb{N}^* \setminus \{1\}.$$

Taking into account the previous inequality and the hypothesis (H_2') we obtain

$$F(g(d_n)) - F(g(d_0)) = \sum_{k=1}^{n} (F(g(d_k)) - F(g(d_{k-1}))) \leq$$
$$\sum_{k=1}^{n} (G(g(d_{k-1})) - F(g(d_{k-1}))) =$$
$$-\sum_{k=1}^{n} (F(g(d_{k-1})) - G(g(d_{k-1}))) \to -\infty \text{ as } n \to \infty.$$

It follows that $\lim_{n\to\infty} F(g(d_n)) = -\infty$ and considering that F verifies the property (P), we find $\lim_{n\to\infty} g(d_n) = 0$. As $g : (0, \infty) \to (0, \infty)$ is a strictly increasing and continuous function, we get $\lim_{n\to\infty} d_n = 0$, hence $d = 0$, which is in contradiction with our assumption $d > 0$. Therefore, $d = 0$, i.e. $\lim_{n\to\infty} d_n = 0$.

Further, we show that $\{T^n(x_0)\}_{n\in\mathbb{N}^*}$ is a Cauchy sequence. By using 2) and 1) we deduce $T^{n+p}(x_0) \in A_{n+p} \subseteq A_n$, $T^n(x_0) \in A_n$ for all $n, p \in \mathbb{N}^*$. Hence, $d(T^{n+p}(x_0), T^n(x_0)) \leq \delta(A_n) = d_n$ for all $n, p \in \mathbb{N}^*$. Because $d_n \to 0$ as $n \to \infty$, it follows that $\{T^n(x_0)\}_{n\in\mathbb{N}^*}$ is a Cauchy sequence. From the completeness of X we obtain that there exists $x^\star \in X$ such that $T^n(x_0) \to x^\star$ as $n \to \infty$.

Finally, we will prove that $x^\star$ is the unique fixed point of T. As the operator T is contractive, it is continuous. Considering 2) and 1) we find

$$0 \leq d(Tx^\star, x^\star) = \lim_{n\to\infty} d(T(T^n(x_0)), T^n(x_0)) = \lim_{n\to\infty} d(T^{n+1}(x_0), T^n(x_0))$$
$$\leq \lim_{n\to\infty} \delta(A_n) = \lim_{n\to\infty} d_n = 0.$$

Therefore, $T(x^\star) = x^\star$, i.e. $x^\star$ is a fixed point of T. Since T has at most one fixed point, we find that $x^\star$ is the unique fixed point of the operator T. Moreover, the sequence $\{T^n(x_0)\}_{n\in\mathbb{N}}$ is convergent to $x^\star$. The proof is also complete in this case.

□

A direct consequence of the previous theorem is the following

Corollary 2.1. *Let us consider (X, d) a complete metric space, $(F, G) \in \mathcal{G}_0$ and the operators $\tilde{T} : X \times \cdots \times X \to X$, $T : X \to X$, $T(x) = \tilde{T}(x, x, \cdots, x)$ satisfying the conditions:*

(i) *there exists* $x_0 \in X$ *such that the sequence* $\{T^n(x_0)\}_{n\in\mathbb{N}}$ *is bounded;*

(ii) F *verifies the property (P) and* G *is a monotonically increasing function;*

(iii) T *is a* (F,G)*-perturbed contraction.*

Then there exists a unique point $x^\star \in X$ *such that* $x^* = \tilde{T}(x^*,\cdots,x^*)$ *and the sequence* $\{T^n(x_0)\}_{n\in\mathbb{N}}$ *is convergent to* $x^\star \in X$.

Proof. The conclusion follows from Theorem 2.1. □

Lemma 2.1. *Let us consider* $g:(0,\infty)\to(0,\infty)$ *a increasing and continuous function and* $A\subset(0,\infty)$ *a bounded set. Then* $g(\sup A)=\sup g(A)$.

Proof. Let us denote $M=\sup A$. Then, for each $x\in A$ we have $g(x)\le g(M)$, which implies that $\sup g(A)\le g(M)=g(\sup A)$. On the other hand, from definition of M we have: for $\varepsilon>0$ there exists $x_\varepsilon\in A$ such that $M-\varepsilon<x_\varepsilon$. Therefore $g(M-\varepsilon)<g(x_\varepsilon)\le \sup g(A)$. Since g is continuous, we get $g(\sup A)=g(M)\le\sup g(A)$. □

Example 2.2. Let us consider $\tau>0$, $F,G:(0,\infty)\to\mathbb{R}$ defined by:

$$F(t)=\begin{cases}\tau-\frac{1}{t} & , \quad t\in(0,\frac{1}{\tau}]\\ \tau-\frac{2}{t} & , \quad t\in(\frac{1}{\tau},\infty),\end{cases}$$

$$G(t)=-\frac{2}{t},$$

Then:

(i) F is not increasing, satisfies the property (P) and $(F,G)\in\mathcal{G}_0$;

Proof. (i) We remark that the function F is not increasing and it satisfies the property (P). Also, $(F,G)\in\mathcal{G}_0$ because the hypotheses (H_1), (H_2) are fulfilled.

(H_1) Let us consider $r\ge t>0$. The following cases can occur:
Case 1: $r,t\in(0,\frac{1}{\tau}]\Rightarrow F(r)=\tau-\frac{1}{r}>-\frac{1}{r}>-\frac{2}{t}=G(t)$;
Case 2: $r,t\in(\frac{1}{\tau},\infty)\Rightarrow F(r)=\tau-\frac{2}{r}>-\frac{2}{r}\ge-\frac{2}{t}=G(t)$;
Case 3: $t\in(0,\frac{1}{\tau}]$ and $r\in(\frac{1}{\tau},\infty)\Rightarrow F(r)=\tau-\frac{2}{r}>-\frac{2}{r}>-\frac{2}{t}=G(t)$.

(H_2) Choose $t>0$ be arbitrary. Then

$$F(s)-G(s)=\begin{cases}\tau+\frac{1}{s} & , \quad s\in(0,\frac{1}{\tau}]\\ \tau & , \quad s\in(\frac{1}{\tau},\infty)\end{cases}$$

hence, $\liminf_{s\searrow t}\big(F(s)-G(s)\big)\ge\tau>0$.

□

Example 2.3. Let us consider the linear space $C([0,1],\mathbb{R})=\{f:[0,1]\to\mathbb{R}\mid f \text{ is continuous on } [0,1]\}$, endowed with the infinity norm $\|\cdot\|_\infty:C([0,1],\mathbb{R})\to\mathbb{R}_+$, $\|x\|_\infty=\sup_{t\in[0,1]}|x(t)|$ and the induced metric $d:C([0,1],\mathbb{R})\times C([0,1],\mathbb{R})\to\mathbb{R}_+$, $d(x,y)=\|x-y\|_\infty$. It is well known that the metric space $(C([0,1],\mathbb{R}),d)$ is complete. We choose $\tau>0$, $F,G:(0,\infty)\to\mathbb{R}$, $K\in C([0,1]\times[0,1]\times\mathbb{R},\mathbb{R})$, $f\in C([0,1],\mathbb{R})$ and the operator $T:C([0,1],\mathbb{R})\to C([0,1],\mathbb{R})$, that are defined by:

$$F(t)=\tau-\frac{1}{t}$$

$$G(t)=-\frac{1}{t},$$

$$T(x)(t)=f(t)+\int_0^t K(t,s,x(s))ds,\ t\in[0,1].$$

We suppose that, for all $t,s\in[0,1]$ and $u,v\in\mathbb{R}$, we have

$$|K(t,s,u)-K(t,s,v)|^2\le\frac{|u-v|^2}{\tau|u-v|^2+1}. \tag{2.3}$$

Then:

(i) F satisfies the property (P) and $(F,G)\in\mathcal{G}_0$;

(ii) T is a (F,G)-perturbed contraction.

Proof. (i) We remark that the function F satisfies the property (P). Also, $(F,G)\in\mathcal{G}_0$ because the hypotheses (H_1), (H_2) are fulfilled.

(H_1) Let us consider $r\ge t>0$. Then $F(r)=\tau-\frac{1}{r}>-\frac{1}{r}>-\frac{1}{t}=G(t)$;

(H_2) Choose $t>0$ be arbitrary. Then $F(s)-G(s)=\tau$ and therefore,

$$\liminf_{s\searrow t}\big(F(s)-G(s)\big)=\tau>0$$

(ii) Let us consider $B\subset C([0,1],\mathbb{R})$ an arbitrary bounded set satisfying $\delta(T(B))\neq 0$. Then, for all $x,y\in B$ and $t\in[0,1]$, via Cauchy-Schwarz-Buniakowski inequality, we deduce

$$|T(x)(t)-T(y)(t)|^2\le\int_0^t|K(t,s,x(s))-K(t,s,y(s))|^2ds\le$$
$$\int_0^t\frac{|x(s)-y(s)|^2}{\tau|x(s)-y(s)|^2+1}ds\le\frac{\|x-y\|_\infty^2}{\tau\|x-y\|_\infty^2+1}.$$

Passing to $\sup\limits_{t\in[0,1]}$ in the above inequality and using Lemma 2.1 we obtain

$$\|T(x)-T(y)\|_\infty^2\le\frac{\|x-y\|_\infty^2}{\tau\|x-y\|_\infty^2+1}.$$

Considering $\sup\limits_{x,y\in B}$ and applying again Lemma 2.1, from the previous relation we find

$$\delta^2(T(B))\le\frac{\delta^2(B)}{\tau\delta^2(B)+1}\Rightarrow\tau-\frac{1}{\delta^2(T(B))}\le-\frac{1}{\delta^2(B)}$$

thus

$$F(\delta^2(T(B)))\le G(\delta^2(B)).$$

Therefore, there exist $(F,G)\in\mathcal{G}_0$ and $g:(0,\infty)\to(0,\infty)$, $g(t)=t^2$ a strictly increasing and continuous function, such that

$$F(g(\delta(T(B))))\le G(g(\delta(B))),\ \text{for all } B\in P_b(C([0,1],\mathbb{R})),\delta(T(B))\neq 0,$$

hence T is a (F,G)-perturbed contraction. □

Lemma 2.2. [14] *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of elements from a metric space (X,d) and Δ be a subset of $(0,\nu)$, $\nu\in\overline{\mathbb{R}}_+$, such that $(0,\nu)\backslash\Delta$ is dense in $(0,\nu)$. If $d(x_n,x_{n+1})\overset{n}{\to}0$ and $\{x_n\}_{n\in\mathbb{N}}$ is not a Cauchy sequence, then there exists $\eta\in(0,\nu)\backslash\Delta$ and the sequences of natural numbers $\{m_k\}_{k\in\mathbb{N}}$, $\{n_k\}_{k\in\mathbb{N}}$ such that*

(1) *$d(x_{m_k},x_{n_k})\searrow\eta$, $k\to\infty$,*

(2) *$d(x_{m_k+p},x_{n_k+q})\to\eta$, $k\to\infty$, where $p,q\in\{0,1\}$.*

Theorem 2.2. *Let us consider (X,d) a complete metric space, $(F,G)\in\mathcal{G}_0$, $g:(0,\infty)\to(0,\infty)$ a strictly increasing and continuous function and $T:X\to X$ such that:*

(i) *the set of continuity points of $F\circ g$ is dense in $(0,\infty)$;*

(ii) *F satisfies property (P);*

(iii) *$F(g(d(Tx,Ty)))\le G(g(d(x,y)))$ for all $x,y\in X, x\neq y$.*

Then T has a unique fixed point $x^\star \in X$ and for every $x_0 \in X$ the sequence $\{T^n(x_0)\}_{n\in\mathbb{N}}$ is convergent to $x^\star \in X$.

Proof. First of all, we remark that the hypothesis (H_2) and the condition (iii) lead us to the fact that the operator T is contractive, i.e

(2.4) $$d(Tx,Ty) < d(x,y), \text{ for all } x,y \in X, x \neq y.$$

The relation (2.4) implies that T has at most one fixed point.

In order to show that T has a fixed point, let $x_0 \in X$ be an arbitrary element. We define a sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_n = Tx_{n-1}$, $n \geq 1$, and denote $d_n = d(x_{n+1}, x_n)$, $n \geq 0$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$ then x_{n_0} is a fixed point of T. Next, we suppose that $x_{n+1} \neq x_n$ for each $n \in \mathbb{N}$. Then $d_n > 0$ for all $n \in \mathbb{N}$ and taking into account the relation (2.4) we conclude that there is $d \geq 0$ such that $d_n \searrow d$.

Next, we state that $d = 0$. Suppose that $d > 0$. Since g is a strictly increasing and continuous function, we deduce $g(d_n) \searrow g(d)$. Indeed, relation (iii) implies that $F(g(d_n)) \leq G(g(d_{n-1}))$ for all $n \geq 1$. From the above inequality we obtain that

$$F(g(d_n)) - F(g(d_{n-1})) \leq G(g(d_{n-1})) - F(g(d_{n-1})),$$

for each $n \geq 1$. Therefore, using the hypothesis (H_2') we deduce

$$F(g(d_n)) - F(g(d_0)) = \sum_{k=1}^{n}(F(g(d_k)) - F(g(d_{k-1}))) \leq$$

$$\sum_{k=1}^{n}(G(g(d_{k-1})) - F(g(d_{k-1}))) = \sum_{k=1}^{n}(G(g(d_{k-1})) - F(g(d_{k-1}))) \to -\infty.$$

It follows that $\lim_{n\to\infty} F(g(d_n)) = -\infty$ and considering the condition (ii) and the fact that g is a strictly increasing and continuous function, we get $\lim_{n\to\infty} d_n = 0$.

Now, we assume that $\{x_n\}_{n\in\mathbb{N}}$ is not Cauchy sequence. Let us consider Δ the set of discontinuities of $F \circ g$. According to Lemma 2.2 applied for $(0,\infty)\backslash\Delta$, one can find $\eta \in (0,\infty)\backslash\Delta$ and the sequences $\{m_k\}_{k\in\mathbb{N}}, \{n_k\}_{k\in\mathbb{N}}$ such that

$$d(x_{m_k}, x_{n_k}) \searrow \eta, d(x_{m_k+1}, x_{n_k+1}) \to \eta, k \to \infty.$$

Since $\eta > 0$, there is $K \in \mathbb{N}$ such that $d(x_{m_k+1}, x_{n_k+1}) > 0$ for all $k \geq K$. Therefore, for all $k \geq K$ we get

$$F(g(d(x_{m_k+1}, x_{n_k+1}))) \leq G(g(d(x_{m_k}, x_{n_k}))),$$

hence

$$-G(g(d(x_{m_k}, x_{n_k}))) \leq -F(g(d(x_{m_k+1}, x_{n_k+1})),$$

thus

$$F(g(d(x_{m_k}, x_{n_k}))) - G(g(d(x_{m_k}, x_{n_k}))) \leq$$
$$F(g(d(x_{m_k}, x_{n_k}))) - F(g(d(x_{m_k+1}, x_{n_k+1})))).$$

It follows that

$$F(g(d(x_{m_k}, x_{n_k}))) - G(g(d(x_{m_k}, x_{n_k}))) \leq$$
$$\limsup_{k\to\infty}[F(g(d(x_{m_k}, x_{n_k}))) - G(g(d(x_{m_k}, x_{n_k})))] \leq$$
$$\limsup_{k\to\infty}[F(g(d(x_{m_k}, x_{n_k}))) - F(g(d(x_{m_k+1}, x_{n_k+1}))))] = F(g(\eta)) - F(g(\eta)) = 0,$$

which is a contradiction with (H_2). Therefore, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and from the completeness of X there exists $x^\star \in X$ such that $x_n \to x^\star$ as $n \to \infty$. Finally, the condition (2.4) yields that T is continuous and

$$d(Tx^\star, x^\star) = \lim_{n\to\infty} d(Tx_n, x_n) = \lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

Thus, $Tx^\star = x^\star$, i.e. $x^\star$ is the unique fixed point of T. □

3. CONCLUSIONS

In this paper we have extended the results from [8] and [2] by proving an existence and uniqueness result of fixed point for an operator $T : X \to X$ which satisfies a general contractive condition of type

$$F(g(\delta(T(B)))) \leq G(g(\delta(B))), \text{ for all } B \in P_b(X), \delta(T(B)) \neq 0,$$

where F, G, g, δ are given by Definition 2.3. The above result has been applied to study the existence and uniqueness of fixed point for operators defined on cartesian product of perturbed metric spaces. Further we provide an existence and uniqueness result of the fixed point for the operator $T : X \to X$ which satisfies a contractive condition of type $F(g(d(Tx,Ty))) \leq G(g(d(x,y)))$ for all $x, y \in X, x \neq y$. Also, we highlighted an operator which is an $(F, G) - contraction$ that is not a contraction. As further research direction, by following Akkouchi [1] and Pant et al. [9], [10], [11] we would like to extend the main results to common fixed point theory.

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