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# Existence and uniqueness of weak periodic solutions for a coupled parabolic-elliptic system

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ABSTRACT. Based on the maximal monotone mapping theory and applying the Schauder fixed point theorem, we prove the existence and the uniqueness of weak periodic solution for nonlinear parabolic-elliptic equations in Orlicz-Sobolev spaces, with growth nonlinearity in gradient associated with some appropriate N-functions.

## 1. INTRODUCTION

In this paper, we study the existence of weak periodic solutions for the following nonlinear system

$$
\left\{
\begin{array}{lll}
\frac{\partial u}{\partial t} - Au = \rho(u)|\nabla\varphi|^2 & \text{in} & Q_T = \Omega \times (0,T), \\
\operatorname{div}(\rho(u)\nabla\varphi) = 0 & \text{in} & \Omega, \\
u = 0,\ \varphi = \varphi_0, & \text{on} & \Sigma = \partial\Omega \times (0,T), \\
u(x,0) = u(x,T),\ \varphi(x,0) = \varphi(x,T) & \text{in} & \overline{\Omega}.
\end{array}
\right.
\tag{1.1}
$$

Here $\Omega$ is an open regular bounded subset of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$, $T > 0$. $Au = -\operatorname{div}(a(x,t,u,\nabla u))$ is a Leary-lions operator and the function $\varphi_0$ is from the data. This problem is inspired by the thermistor problem.
The term "thermistor" refers to a combination of "thermal" and "resistor", it is a resistance thermometer whose strength depends on temperature. The thermistor problem takes place largely in various chemical, physical, biological and ecological phenomena. Many articles have treated the existence of periodic solutions to evolutionary equations, which are described by both ordinary differential equations and parabolic equations, in the Hilbert space or classical Sobolev spaces and under different boundary conditions. One can regard problem (1.1) as a generalization of the so-called thermistor problem, where we assume that the case of the elliptic equation is non-uniformly elliptic.
Among the first authors who investigated the thermistor problem in the classical Sobolev spaces, we cite S. N. Antontsev and M. Chipot in [8, 9, 10], where $a(x,t,u,\nabla u) = -\nabla u$ or $a(x,t,u,\nabla u) = a(u)\nabla u$ with various boundary conditions for $u$ and $\varphi$. These same problems have also been studied by G. Cimatti in [22, 23, 24] and by W. Allegretto in [5, 6]. While the periodic solution of the thermistor problem has been dealt with by M. Badii in [11, 12]. We mention also here the papers [7, 14, 19, 27, 29, 35] and the references therein.
To establish some existing results of periodic solutions of linear and quasi-linear parabolic equations, the authors have proposed various methods, including sub and upper solutions and their associated monotone iterations [19], the theory of monotone operators [12], the Mountain Pass Theorem [21], and others. As far as the uniqueness of the solution of (1.1) is concerned, we refer the reader to [1, 17, 20].

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Orlicz spaces have recently caught the attention of numerous researchers, mostly because of their applications in a variety of domains, such as image processing and electrorheological fluids. Although most of the works that have studied (1.1) in this framework have only proved the existence of capacity solutions (see [4, 13, 16, 34]). However, few authors have examined the existing results of a weak periodic solution [21].
To our best knowledge, no paper establishes such a type periodic weak solution of (1.1) in the Orlicz-Sobolev spaces, with the time-periodicity condition. This problem may be also regarded as a generalization of [11, 12, 20]. Hence, the results of the present paper are new and original.
One of the major difficulties encountered in the analysis of this kind of equation is the degeneracy problem, namely $\rho(.)$ vanish near infinity. To overcome this obstacle, we impose the following condition: There exists $\rho_* \in \mathbb{R}$ such that $0 < \rho_* \leq \rho(s)$, for all $s \in \mathbb{R}$ on the function $\rho(.)$. Another difficulty to overcome during the realization of this article is arising from the non-reflexivity of these spaces. Thus, the authors added some constraints on the $N$-function (see section 3). Finally, the last difficulty related to this problem is the lack of uniqueness of the weak solutions. For that, a certain regularity of $a(.,.,.)$ and $\rho(.)$ must be preserved to achieve uniqueness (see section 3).
Applying the maximal monotone operators theory, we begin showing the existing results of an abstract problem, in an appropriate Orlicz space of periodic functions. After, we Construct an approximate problem and prove some a priori estimates. Later, we use Schauder's fixed point theorem, to have the weak periodic solution of (1.1). Note that, all the functions, taken here, are also time-periodic.

The content of the paper is as follows. Section 2, contains some results of the setting Orlicz-Sobolev spaces and some technical lemmas which will be needed. Section 3, is devoted to specifying the assumptions on $a$, $\rho$ and $\varphi_0$. The announcement and proof of the main result (Theorem 4.3 and Theorem 4.5) will be given in section 4.

## 2. Preliminaries

The function $a : (0, \infty) \to \mathbb{R}$ is such that the mapping $m : \mathbb{R} \to \mathbb{R}$ defined by

$$m(t) = \begin{cases} \frac{a(|t|)}{t} & \text{for } t \neq 0, \\ 0 & \text{for } \quad t = 0. \end{cases}$$

is an odd, strictly increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.
For the function $m$, let us define

$$M(t) = \int_0^t m(s)ds, \; \forall t \in \mathbb{R}.$$

The function $M$ is called N-function. $M$ is continuous, convex, with $M(t) > 0$ for $t > 0$, $\dfrac{M(t)}{t} \to 0$ as $t \to 0$, and $\dfrac{M(t)}{t} \to +\infty$ as $t \to +\infty$. The N-function $\overline{M}$conjugate to $M$ is defined by $\overline{M}(t) = \displaystyle\int_0^t \overline{m}(s)ds$, where $\overline{M} : \mathbb{R}^+ \to \mathbb{R}^+$, is given by $\overline{m}(t) = \sup\limits_{s \geq 0}\{s : m(s) \leq t\}$.
Throughout this paper, we assume that

$$1 < p_* := \inf_{t>0} \frac{tm(t)}{M(t)} \leq p^* := \sup_{t>0} \frac{tm(t)}{M(t)} < \infty \tag{2.2}$$

and

$$\text{The function } t \mapsto M(\sqrt{t}) \text{ is convex for all } t \geq 0. \tag{2.3}$$

**Remark 2.1.**
The condition (2.2) implies that

- $M$ satisfies the $\Delta_2$-condition, i.e

(2.4) There exists a constant $k > 0$ such that $M(2t) \leq kM(t),\ \forall t > 0,$

- The equality $L_M(\Omega) = E_M(\Omega)$ holds, where $E_M(\Omega)$ is the closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact supports in $\Omega$.

Let $P$ and $M$ be two N-functions. $P \ll M$ means that $P$ grows essentially less rapidly than $M$, that is, for each $\epsilon > 0$, $\frac{P(t)}{M(\epsilon t)} \to 0$ as $t \to +\infty$. This is the case if and only if $\lim_{t\to+\infty} \frac{M^{-1}(t)}{P^{-1}(t)} = 0.$

**Proposition 2.1.** ([2])
*$P \ll M$ if and only if, for all $\epsilon > 0$ there exists a constant $c_\epsilon$ such that,*

(2.5) $$P(t) \leq M(\epsilon t) + c_\epsilon, \forall t \geq 0.$$

The Orlicz space $L_M(\Omega)$, is defined as the set of equivalence classes of real-valued measurable functions $u$ on $\Omega$ such that

$$\int_\Omega M\left(\frac{|u(x)|}{\lambda}\right) dx < +\infty \quad \text{for some} \quad \lambda > 0.$$

The set $L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_M = \inf\left\{\lambda > 0 : \int_\Omega M\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1\right\},$$

We now turn to the Orlicz-Sobolev space, $W^1L_M(\Omega)$ is the space of all functions $u$ such that $u$ and its distributional derivatives up to order 1 lie in $L_M(\Omega)$. It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_M.$$

Let $W^{-1}L_{\overline{M}}(\Omega)$ denote the space of distributions on $\Omega$ which can be written as sums of derivatives of order $\leq 1$ of functions in $L_{\overline{M}}(\Omega)$. It is a Banach space under the usual quotient norm (for more details see [3]).
The inhomogeneous Orlicz-Sobolev spaces are defined as follows

$$W^{1,x}L_M(Q_T) = \left\{u \in L_M(Q_T) : \nabla_x^\alpha u \in L_M(Q_T), \forall \alpha \in \mathbb{N}^N, |\alpha| \leq 1\right\},$$

where $\nabla_x^\alpha$ the distributional derivative on $Q_T$ of order $\alpha$ with respect to the variable $x \in \mathbb{R}^N$.
The $W^{1,x}L_M(Q_T)$ is a Banach space under the norm

$$\|u\| = \sum_{|\alpha|\leq 1} \|\nabla_x^\alpha u\|_{M,Q_T}.$$

**Proposition 2.2.** ([3, 26]) *Under (2.2) and (2.3), $L_M(\Omega)$, $W^1L_M(\Omega)$ and $W^{1,x}L_M(\Omega)$ are separable and reflexive Banach spaces.*

Let define the modular $\varrho(u) = \int_\Omega (M(|u|) + M(|\nabla u|))dx$ for any $u \in W^{1,x}L_M(\Omega)$. Then

**Proposition 2.3.** ([33, 32]) *For any $u_n, u \in W^1L_M(\Omega)$, we have*

*(1)* $\|u\|_{1,M}^{p^*} \leq \varrho(u) \leq \|u\|_{1,M}^{p_*}$, *if* $\|u\|_{1,M} < 1$,
*(2)* $\|u\|_{1,M}^{p_*} \leq \varrho(u) \leq \|u\|_{1,M}^{p^*}$, *if* $\|u\|_{1,M} > 1$,
*(3)* $\|u_n - u\|_{1,M} \to 0 \Leftrightarrow \varrho(u_n - u) \to 0$,
*(4)* $\|u_n - u\|_{1,M} \to \infty \Leftrightarrow \varrho(u_n - u) \to \infty$.

**Lemma 2.1.** ([25]) *Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ with the segment property. Then*

$$\left\{u \in W_0^{1,x}L_M(Q_T) \mid \frac{\partial u}{\partial t} \in W^{-1,x}L_{\bar{M}}(Q_T) + L^1(Q_T)\right\} \subset C\left([0,T], L^1(\Omega)\right).$$

**Lemma 2.2.** ([30]) *For all $u \in W_0^1 L_M(Q_T)$ with meas$(\Omega) < +\infty$, one has*

$$\int_{Q_T} M\left(\frac{|u|}{\lambda}\right) dxdt \leq \int_{Q_T} M(|\nabla u|)dxdt. \tag{2.6}$$

*where $\lambda = diam(Q_T)$, is the diameter of $Q_T$.*

**Proposition 2.4.** ([3]) *Let $M_1$ and $M_2$ be two N-functions. $L_{M_1}(Q_T) \subset L_{M_2}(Q_T)$ if and only if it exists $s_0 > 0$ and $\alpha_0 > 0$ such that*

$$M_2(s) \leq \alpha_0 M_1(s), \forall s \geq s_0.$$

**Lemma 2.3.** *Let $u \in L_M(Q_T)$ such that $\int_{Q_T} M(u)dxdt > 1$. Then for any $p > 1$, we have*

$$\|u\|_M^p \leq \int_{Q_T} M(u)dxdt. \tag{2.7}$$

*Proof.* We set $\sigma = \int_{Q_T} M(u)dxdt > 1$ and since $\psi(s) = s^p M(s)$ is increasing, we have

$$\left(\frac{u}{\sigma^{\frac{1}{p}}}\right)^p M\left(\frac{u}{\sigma^{\frac{1}{p}}}\right) \leq u^p M(u),$$

and thus $M\left(\frac{u}{\sigma^{\frac{1}{p}}}\right) \leq \sigma M(u)$. This yields that

$$\int_{Q_T} M\left(\frac{u}{\sigma^{\frac{1}{p}}}\right) dxdt \leq \sigma \int_{Q_T} M(u)dxdt = 1,$$

so that $\|u\|_M \leq \sigma^{\frac{1}{p}}$ and then we obtain the lemma 2.3. □

Now we present our functional framework for the periodic solutions to the problem, we set

$$\Lambda = \{u \mid u(x,0) = u(x,T), u \in L^2(0,T; H^1(\Omega)\},$$
$$\Lambda_0 = \{u \mid u(x,0) = u(x,T), u \in L^2(0,T; H_0^1(\Omega)\},$$
$$L_M^T(\Omega) = \{u | u(x,0) = u(x,T), u \in L_M(Q_T)\},$$
$$W_0^{1,x}L_M^T(\Omega) = \{u \mid u(x,0) = u(x,T), u \in W_0^{1,x}L_M(Q_T)\}.$$

We consider the Banach space $\mathbf{W}_T$ given as follows

$$\mathbf{W}_T = \left\{u \in W_0^{1,x}L_M^T(Q_T) / \frac{\partial u}{\partial t} \in W^{-1,x}L_{\bar{M}}^T(Q_T)\right\}$$

provided with its standard norm

$$\|u\|_{\mathbf{W_T}} = \|u\|_{W^{1,x}L_M^T(Q_T)} + \left\|\frac{\partial u}{\partial t}\right\|_{W^{-1,x}L_{\bar{M}}^T(Q_T)}.$$

**Theorem 2.1.** ([15, 18]) *If $A$ is a monotone, hemicontinuous mapping from $\Lambda_0$ to $\Lambda^*$ such that $A$ is coercive, then $Range(A) = \Lambda^*$.*

**Theorem 2.2.** ([15, 18, 31]) *Let $\mathcal{L}$ be a linear closed, densely defined operator from the reflexive Banach space $W_0^{1,x}L_M^T(Q_T)$ to $\left(W_0^{1,x}L_M^T(Q_T)\right)^*$, $\mathcal{L}$ maximal monotone and let $\mathcal{B}$ a bounded hemicontinuous, monotone mapping from $W_0^{1,x}L_M^T(Q_T)$ to $\left(W_0^{1,x}L_M^T(Q_T)\right)^*$. Then $\mathcal{L}+\mathcal{B}$ is maximal monotone in $W_0^{1,x}L_M^T(Q_T)\times\left(W_0^{1,x}L_M^T(Q_T)\right)^*$. Moreover, if $\mathcal{L}+\mathcal{B}$ is coercive then $Range(\mathcal{L}+\mathcal{B})=\left(W_0^{1,x}L_M^T(Q_T)\right)^*$.*

## 3. Assumptions

Let us now introduce the hypothesis which we assume throughout this section. We consider that for functions defined in $Q_T$, we are automatically imposing the time periodicity, and $M$ and $P$ be two $N$-functions such that $P \ll M$.
The second-order partial differential operator

$$A : D(A) \subset W_0^{1,x}L_M^T(Q_T) \mapsto W^{-1,x}L_{\bar{M}}^T(Q_T)$$

in divergence form $A(u) = -\operatorname{div} a(x,t,u,\nabla u)$, where $a : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying, for almost every $(x,t) \in Q_T$ and for all $s, s_1, s_2 \in \mathbb{R}, \xi, \xi^* \in \mathbb{R}^N$,

(3.8) $$|a(x,t,s,\xi)| \leq c(x,t) + \bar{M}^{-1}(P(s)) + \bar{M}^{-1}(M(|\xi|)),$$

(3.9) $$(a(x,t,s,\xi) - a(x,t,s,\xi^*))(\xi-\xi^*) \geq \alpha M(|\xi-\xi^*|),$$

(3.10) $$a(x,t,s,0) = 0,$$

where $c(.,.) \in E_{\bar{M}}(Q_T)$ and $\alpha > 0$.

(3.11) $\rho \in C(\mathbb{R})$ and there exist $\rho_*, \rho^* \in \mathbb{R}$ such that $0 < \rho_* \leq \rho(s) \leq \rho^*,$ for all $s \in \mathbb{R}$.

(3.12) $\varphi_0$ is a T-periodic and bounded function on $\Sigma,$ with an extension to $\Omega$ denoted by $\widetilde{\varphi_0} \in L^\infty(0,T;W^{1,\infty}(\Omega))$.

(3.13) $$u_0 \in L^2(\Omega).$$

We assume the following continuous inclusions hold:

(3.14) $$L_M(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L_{\overline{M}}(\Omega).$$

**Remark 3.2.** According to the proposition 2.4 and (3.14), there exist two positive constants $u_0$, $\gamma_0$ such that

(3.15) $$|u|^2 \leq \gamma_0 M(u), \quad \text{for all} \quad u \geq u_0.$$

we deduce also, that

(3.16) $$W_0^1 L_M(\Omega) \hookrightarrow H_0^1(\Omega),$$

(3.17) $$H^{-1}(\Omega) \hookrightarrow W^{-1}L_{\overline{M}}(\Omega),$$

(3.18) $$L^2\left(0,T;H^{-1}(\Omega)\right) \hookrightarrow W^{-1,x}L_{\overline{P}}(Q_T) \hookrightarrow W^{-1,x}L_{\overline{M}}(Q_T).$$

**Example 3.1.**
We provide some examples of N-functions $M$ that check for the previous assumptions as follows,

(1) $M(t) = \log(1+|t|^\alpha)|t|^{p-2}t$ with $p, \alpha > 1$, $t \in \mathbb{R}$.
(2) $M(t) = |t|^p + |t|^q$, with $1 < p < q$, $t \in \mathbb{R}$.

(3) $M(t)=\int_0^{|t|} m(s)ds$ such that $t \leq m(t)$, $t\in\mathbb{R}$.

**Remark 3.3.** Note that the Sobolev spaces $H_0^2(\Omega)$ are special cases of the Orlicz spaces defined by $M(t)=t^2$. Thus $\overline{M}(t)=t^2$ and $\overline{M}^{-1}M(t)=t$. Then in this case, we can cite two examples of the operator $A$: $a(x,t,u,\nabla u)=-\nabla u$ in [6], or $a(x,t,u,\nabla u)=a(u)\nabla u$ with $0<a_0\leq a(s)\leq a_1$ in [9], and one can easily verify that the two previous operators satisfy the conditions (3.8)-(3.10)

## 4. Main result

Our main result is composed of two theorems, namely the existence theorem (Theorem 4.3) and the uniqueness theorem (Theorem 4.5).

### 4.1. **Existence results.**

**Theorem 4.3.** *Assume that the assumptions (3.8)-(3.14) hold. Then there exists a weak solution* $(u,\varphi)$ *to system* (1.1)*, that is,*

$$\begin{cases} u\in W_0^1L_M^T(Q_T),\ a(x,t,u,\nabla u)\in L_{\overline{M}}^T(Q_T)^N, \varphi-\varphi_0\in\Lambda_0\cap L^\infty(Q_T),\\ \displaystyle\int_{Q_T}\frac{\partial u}{\partial t}\xi dxdt+\int_{Q_T} a(x,u,\nabla u)\nabla\xi dxdt=-\int_{Q_T}\rho(u)\varphi\nabla\varphi\nabla\xi dxdt, \quad \textit{for all}\quad \xi\in W_0^1L_M^T(Q_T),\\ \displaystyle\int_{Q_T}\rho(u)\nabla\varphi\nabla\xi dxdt=0, \quad \textit{for all}\quad \xi\in\Lambda_0,\\ u=0,\ \varphi=\varphi_0,\ \textit{on}\quad \Sigma. \end{cases}$$

*Proof.*
The proof is divided into 4 steps.
In steps 1 and 2, we will show certain results by using the monotone operator method.
**Step 1: The electrical potential problem**
In this step, we prove the existence of periodic solutions for the elliptic equation in (1.1).
Fixed $\omega\in L_M^T(Q_T)$, we resolve the following problem, in the weak sens

$$\begin{cases} \operatorname{div}\left(\rho(\omega)\left(\nabla v+\nabla\varphi_0\right)\right)=0 & \text{in } Q_T,\\ v(x,t)=0 & \text{on } \Sigma, \end{cases} \tag{4.19}$$

where $v:=\varphi-\varphi_0$.

**Definition 4.1.** A function $v\in\Lambda_0$ is a weak periodic solution to (4.19) if

$$\int_{Q_T}\rho(\omega)\left(\nabla v+\nabla\varphi_0\right)\nabla\zeta dxdt=0,\ \text{for every } \zeta\in\Lambda_0. \tag{4.20}$$

Let define the mapping $A:\Lambda_0\to\Lambda^*$, by setting

$$\langle A(v),\zeta\rangle:=\int_{Q_T}\rho(\omega)\left(\nabla v+\nabla\varphi_0\right)\nabla\zeta dxdt,\ \text{for every } \zeta\in\Lambda_0.$$

Thus, the problem (4.20) can be rewritten as

$$A(v)=0. \tag{4.21}$$

**Proposition 4.5.** *If* (3.11)-(3.12) *hold, there exists a unique weak periodic solution to* (4.21).

*Proof.* The proposition 4.5 is the direct application of The theorem 2.1. For that we will show that the mapping $A$ checks the following properties,

**Proposition 4.6.** *If assumptions* (3.11), (3.12) *are fulfilled, the mapping $A$ is hemicontinuous, monotone and coercive.*

Indeed, let starte by proving that $A$ is hemicontinuous; For that applying the Hölder inequality, one has

$$|\langle A(v), \zeta\rangle| \leq \rho^* \left(\int_{Q_T} |\nabla v + \nabla\varphi_0|^2 \, dxdt\right)^{1/2} \|\zeta\|_\Lambda,$$

hence

$$\|A(v)\|_{\Lambda^*} \leq \sqrt{2}\rho^* \left(\|v\|_\Lambda + \|\nabla\varphi_0\|_{L^2(Q_T)}\right),$$

and the hemicontinuity results of [[28], Theorems 2.1 and 2.3].
Now it's easy to show that $A$ is monotone, since we have,

$$\langle A(v_1) - A(v_2), v_1 - v_2\rangle = \int_{Q_T} \rho(w) |\nabla(v_1 - v_2)|^2 \, dxdt \geq 0.$$

For the coercivity, one has

$$\begin{aligned}\langle A(v), v\rangle &= \int_{Q_T} \rho(\omega)\left(\nabla v + \nabla\varphi_0\right)\nabla v dxdt \\ &\geq \rho_*\|\nabla v\|^2_{L^2(Q_T)} - \rho^* \|\nabla\varphi_0\|_{L^2(\Omega)} \|\nabla v\|_{L^2(Q_T)}.\end{aligned}$$

Using the Poincaré inequality, there exists a constant $c_p$ such that

$$\langle A(v), v\rangle \geq c_p\rho_*\|v\|^2_\Lambda - \rho^* \|\nabla\varphi_0\|_{L^2(\Omega)} \|v\|_\Lambda,$$

thus

$$\frac{\langle A(v), v\rangle}{\|v\|_\Lambda} \geq c_p\rho_*\|v\|_\Lambda - \rho^* \|\nabla\varphi_0\|_{L^2(Q_T)} \to +\infty, \text{ as } \|v\|_\Lambda \to +\infty.$$

Finally, from Proposition 4.6 and Theorem 2.1, results the existence of weak periodic solution, while the strict monotonicity implies the uniqueness of the solution. □

Thus, for any $\omega \in L^T_M(Q_T)$, the following problem admets a unique weak periodic solution

$$\left\{\begin{array}{ll} \operatorname{div}(\rho(\omega)\nabla\varphi) = 0 & \text{in } Q_T, \\ \varphi(x,t) = \varphi_0(x,t) & \text{on } \Sigma, \\ \varphi(x,0) = \varphi(x,T) & \text{in } Q_T. \end{array}\right. \tag{4.22}$$

Keeping in mind that $\widetilde{\varphi}_0 \in L^\infty(0,T;W^{1,\infty}(\Omega))$, the weak maximum principle gives

$$\|\varphi\|_{L^\infty(Q_T)} \leq \operatorname*{esssup}_{Q_T}|\widetilde{\varphi}_0(x,t)|. \tag{4.23}$$

Therefore, $\varphi \in L^2(0,T;H^1(\Omega)) \cap L^\infty(\Omega)$.
The energy estimate is obtained using $\varphi - \varphi_0$ as a test function in (4.22). Indeed,

$$\begin{aligned}\rho_* \int_{Q_T} |\nabla\varphi|^2 dxdt &\leq \int_{Q_T} \rho(\omega)|\nabla\varphi|^2 dxdt \\ &= \int_{Q_T} \rho(\omega)\nabla\varphi\nabla\varphi_0 dxdt \\ &\leq \rho^*\|\nabla\varphi\|_{L^2(Q_T)}\|\nabla\varphi_0\|_{L^2(Q_T)},\end{aligned}$$

and, then

$$\|\nabla\varphi\|_{L^2(Q_T)} \leq \frac{\rho^*}{\rho_*}\|\nabla\varphi_0\|_{L^2(Q_T)}, \tag{4.24}$$

as a result from (4.23) and (4.24), we obtain

$$\|\varphi\|^2_{L^2(Q_T)} + \|\nabla\varphi\|^2_{L^2(Q_T)} \leq C. \tag{4.25}$$

**Step 2: Temperature problem**

The purpose of this step is to exploit the maximal monotone theory and to show the periodicity of solutions for the nonlinear heat equation (4.26). Hence, let's consider the following variational formulation problem.

**Definition 4.2.** A function $u \in W_0^{1,x}L_M^T(Q_T)$ is called a weak periodic solution to (1.1) corresponding to $\omega \in L_M^T(Q_T)$, if $u$ satisfies

(4.26)
$$\int_{Q_T} \frac{\partial u}{\partial t}\xi dxdt+\int_{\Omega} a(x,t,\omega,\nabla u)\nabla\xi dxdt = -\int_{Q_T} \rho(\omega)\varphi\nabla\varphi\nabla\xi dxdt, \text{ for any } \xi \in W_0^{1,x}L_M^T(Q_T).$$

Let $\mathcal{L} : \mathbf{W}_T \to \left(W_0^{1,x}L_M^T(Q_T)\right)^*$ be the mapping defined by

$$\langle \mathcal{L}(u), \xi\rangle := \int_{Q_T} \frac{\partial u}{\partial t}\xi dxdt, \quad \forall \xi \in W_0^{1,x}L_M^T(Q_T),$$

on the dense set $\mathbf{W}_T$, because $C_0^\infty(Q_T) \subset \mathbf{W}_T$ is dense in $W_0^{1,x}L_M^T(Q_T)$. The linear operator $\mathcal{L}$ is closed, skew-adjoint i.e. $\mathcal{L} = -\mathcal{L}^*$ (integrating by parts and using the periodicity) and maximal monotone (see [[31], Lemma l.1, p. 313]).

Instead, the mapping $\mathcal{B} : W_0^{1,x}L_M^T(Q_T) \to \left(W_0^{1,x}L_M^T(Q_T)\right)^*$ is defined as follows

$$\langle \mathcal{B}(u), \xi\rangle := \int_{Q_T} a(x,t,\omega,\nabla u).\nabla\xi dxdt, \quad \forall \xi \in W_0^{1,x}L_M^T(Q_T).$$

We observe that $\mathcal{B}$ satisfies the above conditions $(i)-(iii)$ of Proposition 4.6. Indeed,

i) $B$ is hemicontinuous: Choosing $\xi \in W_0^{1,x}L_M(Q_T)$ such that $\|\nabla\xi\|_{M,Q_T} \le 1$, then

$$|\langle \mathcal{B}(u), \xi\rangle| \le \int_{Q_T} \left[c(x,t) + \bar{M}^{-1}(P(\omega)) + \bar{M}^{-1}(M(|\nabla u|))\right] \nabla\xi dxdt.$$

Using Holder's inequality and $P \ll M$, we get

$$|\langle \mathcal{B}(u), \xi\rangle| \le \Big(\|c(.,.)\|_{\overline{M}} + \|\omega\|_M + \|\nabla u\|_M + C_1\Big)\||\nabla\xi|\|_{M,Q_T},$$

so that

$$\|\mathcal{B}(u)\|_* \le C_2.$$

ii) $B$ is monotone: According to (3.9)

$$\begin{aligned}\langle \mathcal{B}(u_1) - \mathcal{B}(u_2), u_1 - u_2\rangle &= \int_{Q_T} (a(x,t,\omega,\nabla u_1) - a(x,t,\omega,\nabla u_2))\nabla(u_1-u_2)\, dxdt \\ &\ge \alpha\int_{Q_T} M(|\nabla u_1 - \nabla u_2|)\, dxdt \ge 0.\end{aligned}$$

iii) $B$ is coercive: For $\|u\|_{M,Q_T}$ large enough, using (3.10), lemma 2.3 with $p > 2$ and the Poincaré inequality, we get

$$\begin{aligned}\langle \mathcal{B}(u), u\rangle &= \int_{Q_T} a(x,t,\omega,\nabla u)\nabla u dxdt \\ &\ge \alpha\int_{Q_T} M(|\nabla u|)dxdt \\ &\ge \alpha\|\nabla u\|_{M,Q_T}^p \\ &\ge \alpha C\|u\|_{M,Q_T}^p,\end{aligned}$$

hence,

$$\frac{\langle \mathcal{B}(u), u\rangle}{\|u\|_{M,Q_T}} \ge \alpha\|u\|_{M,Q_T}^{p-1} \to +\infty, \text{ as } \|u\|_{M,Q_T} \to +\infty.$$

Now, let denote $\mathcal{M} \in \left(W_0^{1,x}L_M^T(Q_T)\right)^*$, the linear functional defined by setting

$$\langle \mathcal{M}, \xi \rangle := -\int_{Q_T} \rho(\omega)\varphi\nabla\varphi\nabla\xi dxdt, \quad \forall \xi \in W_0^{1,x}L_M^T(Q_T),$$

then, problem (4.26) assumes the equivalent form

$$\mathcal{L}(u) + \mathcal{B}(u) = \mathcal{M}. \tag{4.27}$$

**Theorem 4.4.** *If assumptions* (3.8)-(3.14) *are fulfilled,* (4.27) *has a unique weak periodic solution.*

*Proof.* From Theorem 2.2, we deduce easily the existence of weak periodic solutions, whereas the uniqueness is due to classical results. □

**Step 3: The approximating problem and apriori estimates**
Let $\omega_n \in L_M^T(Q_T)$ be a sequence such that $\omega_n \to \omega$ in $L_M^T(Q_T)$ and $\rho(\omega_n) \to \rho(\omega)$ strongly in $L^2(Q_T)$. We consider $(u_n, \varphi_n)$ the weak periodic solution of

$$\begin{cases} \frac{\partial u_n}{\partial t} - \operatorname{div}(a(x,t,\omega_n,\nabla u_n)) = \rho(\omega_n)|\nabla\varphi_n|^2 & \text{in } Q_T \\ u_n(x,t) = 0 & \text{on } \Sigma, \\ u_n(x,0) = u_n(x,T) & \text{in } Q_T \end{cases} \tag{4.28}$$

and

$$\begin{cases} \operatorname{div}(\rho(\omega_n)\nabla\varphi_n) = 0 & \text{in } Q_T, \\ \varphi_n(x,t) = \varphi_0(x,t) & \text{on } \Sigma, \\ \varphi_n(x,0) = \varphi_n(x,T) & \text{in } Q_T. \end{cases} \tag{4.29}$$

Taking $\varphi_n - \varphi_0$ as a test function in (4.29), we obtain

$$\|\nabla\varphi_n\|_{L^2(Q_T)} \leq \frac{\rho^*}{\rho_*}\|\nabla\varphi_0\|_{L^2(Q_T)}, \tag{4.30}$$

and by the maximum principle

$$\|\varphi_n\|_{L^\infty(Q_T)} \leq \operatorname*{esssup}_{Q_T}|\widetilde{\varphi}_0(x,t)|. \tag{4.31}$$

Combining (4.30) and (4.44), we deduce the energy estimate

$$\int_{Q_T}|\varphi_n|^2dxdt + \int_{Q_T}|\nabla\varphi_n|^2dxdt \leq C, \tag{4.32}$$

Here and below, $C$ is always, a positive constant, independent of $n$ and generally different from place to place.
By (4.32), we assure that $\varphi_n$ is an uniformly bounded sequence in the norm $\Lambda$. Accordingly, there exists a subsequence such that

$$\varphi_n \to \varphi, \text{ in } L^2(0,T;H^1(\Omega)) \text{ and a.e. in } Q_T. \tag{4.33}$$

From (4.30) and (4.31), we conclude that there exists a subsequence such that

$$\varphi_n \to \varphi \text{ weakly-* in } L^\infty(0,T;H^1(\Omega)), \tag{4.34}$$

and

$$\varphi_n \to \varphi \text{ weakly-* in } L^\infty(Q_T). \tag{4.35}$$

Moreover,

**Lemma 4.4.** *The sequence $\nabla\varphi_n$ converges strongly to $\nabla\varphi$ in $\left(L^2(Q_T)\right)^N$.*

*Proof.* Taking $\varphi_n - \varphi$ as a test function in (4.29), one has

$$\int_{Q_T} \rho(\omega_n)|\nabla(\varphi_n-\varphi)|^2 dxdt = -\int_{Q_T} \rho(\omega_n)\nabla\varphi\nabla(\varphi_n-\varphi)dxdt,$$

thus,

$$\rho_* \int_{Q_T} |\nabla(\varphi_n-\varphi)|^2 dxdt \le -\int_{Q_T} \rho(\omega_n)\nabla\varphi\nabla(\varphi_n-\varphi)dxdt.$$

The strong convergence $\rho(\omega_n) \to \rho(w)$ in $L^2(Q_T)$ implies that $\rho(\omega_n)\nabla\varphi \to \rho(\omega)\nabla\varphi$ in $(L^2(Q_T))^N$ and using (4.35), make obvious the weak convergence of $\rho(\omega_n)\nabla(\varphi-\varphi_n) \to 0$ weakly-* in $\left(L^2(Q_T)\right)^N$. This completes the proof. □

**Lemma 4.5.** *We have*

*(1) $u_n \to u$ strongly in $L^T_M(Q_T)$ and a.e. in $Q_T$,*

*(2) The sequence $\nabla u_n \to \nabla u$ strongly in $\left(L^T_M(Q_T)\right)^N$.*

*Proof.* (1) Choosing $u_n$ as a test function in (4.28) and integrating over $Q_T$, one has

$$\int_{Q_T} \frac{\partial u_n}{\partial t} u_n dxdt + \int_{Q_T} a(x,t,\omega_n,\nabla u_n)\nabla u_n dxdt = -\int_{Q_T} \rho(\omega_n)\varphi_n \nabla\varphi_n \nabla u_n dxdt. \tag{4.36}$$

Because of the periodicity of $u_n$, we get

$$\int_{Q_T} \frac{\partial u_n}{\partial t} u_n dxdt = 0.$$

Assumptions (3.8), (3.12) and the Young inequality give us

$$\begin{aligned} \alpha \int_{Q_T} M(|\nabla u_n|)\,dxdt &\le \int_{Q_T} a(x,t,\omega_n,\nabla u_n)\nabla u_n dxdt \\ &\le \frac{\gamma_0}{2\alpha}\left(\rho^* \operatorname*{esssup}_{Q_T} |\widetilde{\varphi}_0(x,t)|\right)^2 \int_{Q_T} |\nabla\varphi_n|^2\,dxdt + \frac{\alpha}{2\gamma_0}\int_{Q_T} |\nabla u_n|^2\,dxdt, \end{aligned} \tag{4.37}$$

with (3.15) and (4.32), we get

$$\int_{Q_T} M(|\nabla u_n|)\,dxdt \le C, \tag{4.38}$$

hence

$$\|u_n\|_{W^{1,x}L^T_M(Q_T)} \le C. \tag{4.39}$$

Also from (4.37), (4.32) and (4.38), there exists a positive constant such that

$$\int_{Q_T} a(x,t,\omega_n,\nabla u_n)\nabla u_n dxdt \le C. \tag{4.40}$$

Also from (3.8), (3.11), (4.23), (4.25) and (4.40 ) one obtains that $\frac{\partial u_n}{\partial t}$ is bounded with respect to the norm of $W^{-1,x}L^T_{\overline{M}}(Q_T)$ and this ensures that $u_n$ belongs to a bounded set of $\mathbf{W}_T$ i.e.

$$\|u_n\|_{\mathbf{W}_T} \le C,$$

Thus, we can choose a subsequence, still denoted by $u_n$, such that

$$u_n \rightharpoonup u \text{ in } \mathbf{W}_T,$$

that allows us to have also

$$\nabla u_n \rightharpoonup \nabla u \text{ weakly in } L^T_M(Q_T), \tag{4.41}$$

And since the embedding $W_0^{1,x}L_M^T(Q_T) \hookrightarrow L_M^T(Q_T)$ is compact, we have

$$u_n \to u \text{ strongly in } L_M^T(Q_T) \text{ and a.e. in } Q_T \text{ as } n \to +\infty.$$

(2) Taking into account (4.38), (3.8) and the strong convergence of $\omega_n$ to $\omega$ in $L_M^T(Q_T)$, implies that $a(x,t,\omega_n,\nabla u_n)$ is bounded in $(L_{\bar{M}}^T(Q_T))^N$. In fact, for any $\psi \in W_0^{1,x}L_M^T(Q_T)$ with $\|\nabla\psi\|_{L_M^T(Q_T)} \le 1$, we have

$$\begin{aligned}
\int_{Q_T} a(x,t,\omega_n,\nabla u_n)\nabla\psi dxdt &\le \int_{Q_T} a(x,t,\omega_n,\nabla u_n)\nabla u_n dxdt \\
&\quad - \int_{Q_T} a(x,t,\omega_n,\nabla\psi)(\nabla u_n - \nabla\psi)dxdt \\
&\le C + \int_{Q_T} |a(x,t,\omega_n,\nabla\psi)||\nabla u_n|dxdt \\
&\quad + \int_{Q_T} a(x,t,\omega_n,\nabla\psi)\nabla\psi)dxdt \\
&\le C + 3(\int_{Q_T} \overline{M}\Big(\frac{|a(x,t,\omega_n,\nabla\psi)|}{3}\Big)dxdt \\
&\quad + \int_{Q_T} M(|\nabla u_n|)dxdt) \\
&\quad + 3(\int_{Q_T} \overline{M}\Big(\frac{|a(x,t,\omega_n,\nabla\psi)|}{3}\Big)dxdt + \int_{Q_T} M(|\nabla\psi|)dxdt)
\end{aligned}$$

and thus, using (3.8), $P \ll M$ and Young's inequality,

$$\overline{M}\Big(\frac{|a(x,t,\omega_n,\nabla\psi)|}{3}\Big) \le \frac{1}{3}(\overline{M}(c(x,t)) + M(|\omega_n|) + M(|\nabla\psi|) + C$$

$$\begin{aligned}
\int_{Q_T} a(x,t,\omega_n,\nabla u_n)\nabla\psi dxdt &\le 2\int_{Q_T} (\overline{M}(c(x,t)) + M(|\omega_n|) + M(|\nabla\psi|) \\
&\quad + 6\int_{Q_T} M(|\nabla u_n|)dxdt + C
\end{aligned}$$

Since $\omega_n \to \omega$ in $L_M^T(Q_T)$ and considering Remark 2.1, which implies that $\{\omega_n\}_n$ is bounded, thus with (4.38), we conclude that there exist a positive constant $C$ and $L \in (L_{\bar{M}}^T(Q_T))^N$ such that

$$\|a(x,t,\omega_n,\nabla u_n)\|_{(L_{\bar{M}}^T(Q_T))^N} \le C, \tag{4.42}$$

and

$$a\,(x,t,\omega_n,\nabla u_n) \rightharpoonup L \text{ in } (L_{\bar{M}}^T(Q_T))^N.$$

Now, letting $n \to +\infty$ in (4.36), yields

$$\begin{aligned}
\lim_n \int_{Q_T} a\,(x,t,\omega_n,\nabla u_n)\,\nabla u_n dxdt &= \int_{Q_T} L.\nabla u dxdt \\
&= -\int_{Q_T} \rho(\omega)\varphi\nabla\varphi\nabla u dxdt.
\end{aligned}$$

On the other hand, since $\omega_n \to w$ in $L_M^T(Q_T)$, $a(x,t,\omega_n,\nabla u)$ is Carathéodory function and verifies (3.8), it is then sufficient to apply the Dominated convergence

theorem to have $a(x,t,\omega_n,\nabla u)\to a(x,t,\omega,\nabla u)$ strongly in $(L^T_{\bar{M}}(Q_T))^N$.
Since,

$$\begin{aligned}\alpha\int_{Q_T}M(|\nabla(u_n-u)|)dxdt &\leq \textstyle\int_{Q_T}(a(x,t,\omega_n,\nabla u_n)-a(x,t,\omega_n,\nabla u)\nabla(u_n-u))dxdt\\ &=\int_{Q_T}a(x,t,\omega_n,\nabla u_n)\nabla u_n dxdt\\ &-\int_{Q_T}a(x,t,\omega_n,\nabla u_n)\nabla u dxdt\\ &-\int_{Q_T}a(x,t,\omega_n,\nabla u)\nabla(u_n-u)dxdt.\end{aligned}$$

Passing to the limit with all the above, and (4.41), we conclude that

$$\lim_n\int_{Q_T}M\left(\left|\nabla\left(u_n-u\right)\right|\right)dxdt\leq 0,$$

that is

$$\nabla u_n\to\nabla u \text{ a.e on } Q_T. \tag{4.43}$$

□

The equation (4.43) implies that

$$L=a(x,t,\omega,\nabla u) \text{ a.e. on } Q_T,$$

so that

$$a(x,t,\omega_n,\nabla u_n)\rightharpoonup a(x,t,\omega,\nabla u) \text{ in } (L^T_{\bar{M}}(Q_T))^N.$$

**Step 4: Fixed points**
The existence of weak periodic solutions for system (1.1), depends on the research of fixed points for an operator equation.
Let $\Phi : L^T_M(Q_T)\to L^T_M(Q_T)$ be the nonlinear mapping defined by $\Phi(\omega)=u$, where $u$ is the unique weak periodic solution of (4.26). $\Phi$ is well defined and its continuity is based on a strong convergence of $\nabla\varphi_n$ in $L^2(Q_T)$ and the weak convergence of $a(x,t,\omega_n,\nabla u_n)$ in $(L^T_{\bar{M}}(Q_T))^N$.

**Lemma 4.6.** *The operator $\Phi$ is continuous and bounded in $L^T_M(Q_T)$.*

*Proof.* All convergences archived,

$$\begin{cases}\omega_n\to\omega \text{ in } L^T_M(Q_T); \rho(\omega_n)\to\rho(\omega) \text{ in } L^2(Q_T), \text{ and a.e in } Q_T\\ u_n\to u \text{ in } L^T_M(Q_T) \text{ and a.e in } Q_T; \nabla u_n\to\nabla u \text{ a.e in } Q_T,\\ \nabla u_n\to\nabla u \text{ in } (L^T_M(Q_T))^N \text{ and a.e in } Q_T,\\ \nabla\varphi_n\rightharpoonup\nabla\varphi \text{ in } (L^2(Q_T))^N \text{ and a.e in } Q_T,\\ a(x,t,\omega_n,\nabla u_n)\rightharpoonup a(x,t,\omega,\nabla u) \text{ in } (L^T_{\bar{M}}(Q_T))^N.\end{cases}$$

allows us to conclude that $\Phi$ is continuous and $\Phi(\omega_n)=u_n$ converges strongly to $\Phi(\omega)=u$ in $L^T_M(Q_T)$.
Besides, from (4.39), passing to the limit as $n\to+\infty$, there exists a constant $R>0$ such that

$$\|\Phi(\omega)\|_{L^T_M(Q_T)}\leq R, \text{ for every } \omega\in L^T_M(Q_T).$$

Now, since $\Phi(L^T_M(Q_T))\subset L^T_M(Q_T)$ and the embedding $\mathbf{W}_T\hookrightarrow L^T_M(Q_T)$ is compact, $\Phi$ is a compact operator from $L^T_M(Q_T)$ to itself. □

Finally, to complete the proof of Theorem 4.3, remark that Lemmas 4.6, implies that the mapping $\Phi$ is both continuous and compact. Hence, by the Schauder fixed point theorem, it is possible to affirm the existence of at least one fixed point for $\Phi$, which corresponds to a weak periodic solution to systems (1.1). □

4.2. **Uniqueness.**

First of all be $\varphi \in L^2\left(0,T;H^1(\Omega)\right) \cap L^\infty(Q_T)$ a solution of (4.28) and taking $\psi\varphi$ ($\psi \in \mathcal{D}(Q_T)$) as a test function in, we have

$$\int_{Q_T} \rho(u)\nabla\varphi\nabla(\psi\varphi)dx = 0,$$

then

$$\int_{Q_T} \rho(u)|\nabla\varphi|^2\psi dx = -\int_{Q_T} \rho(u)\varphi\nabla\varphi\nabla\psi dx = \langle \operatorname{div}(\rho(u)\varphi\nabla\varphi), \psi\rangle_{\mathcal{D}'(Q_T),\mathcal{D}(Q_T)}.$$

Thus

$$\rho(u)|\nabla\varphi|^2 = \operatorname{div}(\rho(u)\varphi\nabla\varphi) \quad \text{in } \mathcal{D}^{'}(Q_T). \tag{4.44}$$

Now to prove the uniqueness of the solution, we need to impose some assumptions on the term $a$ and $\rho$ as follows,

**Theorem 4.5.** *Assume that assumptions (3.1)-(3.6) hold true, there exist $A \in L^\infty(Q_T), B \in L^\infty(\mathbb{R})$ and a constant $C_0$ such that $\forall s, \bar{s} \in \mathbb{R}$,*

$$\varphi \in L^\infty(0,T,W^{1,\infty}(\Omega)) \tag{4.45}$$

$$|\rho(s) - \rho(\bar{s})| \le C_0|s-\bar{s}|, \tag{4.46}$$

*and*

$$|a(x,t,s,\xi) - a(x,t,\bar{s},\xi)| \le (A(x,t) + B(|\xi|))|s-\bar{s}| \tag{4.47}$$

*for almost every $(x,t) \in Q_T$ and for every $\xi \in \mathbb{R}^N$. Then the problem* (1.1) *has a unique weak solution.*

*Proof.* Consider two weak solutions $(u_1,\varphi_1)$ and $(u_2,\varphi_2)$ of (1.1).
From elliptic equation of (1.1), we have

$$\begin{aligned}\nabla\cdot\left(\rho\left(u_1\right)\nabla\varphi_1\right) &= \nabla\cdot\left(\rho\left(u_2\right)\nabla\varphi_2\right)\\ \nabla\cdot\left(\rho\left(u_1\right)\nabla\left(\varphi_1-\varphi_2\right)\right) &= \nabla\cdot\left(\left(\rho\left(u_2\right)-\rho\left(u_1\right)\right)\cdot\nabla\varphi_2\right).\end{aligned}$$

So, multiplying by $\varphi_1 - \varphi_2$ and integrating over $\Omega$ a.e. in $t$, we get

$$\int_\Omega \rho\left(u_1\right)\left|\nabla\left(\varphi_1-\varphi_2\right)\right|^2 \mathrm{d}x \le \int_\Omega \left(\rho\left(u_2\right)-\rho\left(u_1\right)\right)\nabla\varphi_2\nabla\left(\varphi_1-\varphi_2\right)\mathrm{d}x.$$

Using (3.11), (4.45) and (4.46) we easily obtain

$$\begin{aligned}\rho_*\int_\Omega \left|\nabla\left(\varphi_1-\varphi_2\right)\right|^2 \mathrm{d}x &\le \int_\Omega \rho\left(u_1\right)\left|\nabla\left(\varphi_1-\varphi_2\right)\right|^2 \mathrm{d}x\\ &\le C_1\int_\Omega \left|u_1-u_2\right|\left|\nabla\left(\varphi_1-\varphi_2\right)\right|\mathrm{d}x.\end{aligned}$$

So, by the Cauchy-Schwarz inequality we obtain

$$\int_\Omega \left|\nabla\left(\varphi_1-\varphi_2\right)\right|^2 \mathrm{d}x \le C_2\int_\Omega \left|u_1-u_2\right|^2 \mathrm{d}x. \tag{4.48}$$

Now considering (4.44), we can write (1.1) as

$$u_t - \operatorname{div}(a(x,t,u,\nabla u)) = \nabla\cdot(\rho(u)\varphi\nabla\varphi). \tag{4.49}$$

Multiplying by $u_1 - u_2$ the difference of equation (1.1) for $u_1$ and $u_2$ and integrating over $\Omega$, we obtain a.e. in $t$

$$\frac{d}{dt}\Big(\frac{1}{2}\int_\Omega |u_1-u_2|^2\,dx\Big)+\int_\Omega (a(x,t,u_1,\nabla u_1)-a(x,t,u_2,\nabla u_2))\nabla(u_1-u_2)dx$$
$$= -\int_\Omega (\rho\,(u_1)\,\varphi_1\nabla\varphi_1-\rho\,(u_2)\,\varphi_2\nabla\varphi_2)\nabla(u_1-u_2)dx$$

$$\frac{d}{dt}\Big(\frac{1}{2}\int_\Omega |u_1-u_2|^2\,dx\Big)+I=J. \tag{4.50}$$

The second integral in the left-side hand, can written as

$$\begin{aligned} I &= \int_\Omega (a(x,t,u_1,\nabla u_1)-a(x,t,u_2,\nabla u_2))\nabla(u_1-u_2)dx \\ &= \int_\Omega (a(x,t,u_1,\nabla u_1)-a(x,t,u_1,\nabla u_2))\nabla(u_1-u_2)dx \\ &+ \int_\Omega (a(x,t,u_1,\nabla u_2)-a(x,t,u_2,\nabla u_2))\nabla(u_1-u_2))dx \\ &= I_1+I_2. \end{aligned}$$

Using (3.9), (3.15) and (3.10), there exists a constant $C_3$ such that

$$I_1 \geq \alpha\int_\Omega M(\nabla(u_1-u_2))dx \geq C_3\int_\Omega |\nabla(u_1-u_2)|^2dx.$$

On the other hand by (4.47), there exists a constant $C_4$ such that

$$\begin{aligned} |I_2| &\leq \int_\Omega |u_1-u_2|\,(A(x,t)+B(|v|))\,|\nabla(u_1-u_2)|dx \\ &\leq C_4\int_\Omega |u_1-u_2|^2dx + C_4\int_\Omega |\nabla(u_1-u_2)|^2dx \end{aligned}$$

For the integral in right-side hand in (4.50), we have

$$\begin{aligned} J &= -\int_\Omega (\rho\,(u_1)\,\varphi_1\nabla\varphi_1-\rho\,(u_2)\,\varphi_2\nabla\varphi_2)\nabla(u_1-u_2)dx \\ &= -\int_\Omega (\rho\,(u_1)-\rho\,(u_2))\,\varphi_1\nabla\varphi_1\cdot\nabla\,(u_1-u_2)\,dx \\ &\quad -\int_\Omega \rho\,(u_2)\,(\varphi_1\nabla\varphi_1-\varphi_2\nabla\varphi_2)\cdot\nabla\,(u_1-u_2)\,dx \\ &= -\int_\Omega (\rho\,(u_1)-\rho\,(u_2))\,\varphi_1\nabla\varphi_1\cdot\nabla\,(u_1-u_2)\,dx \\ &\quad -\int_\Omega \rho\,(u_2)\,(\varphi_1-\varphi_2)\,\nabla\varphi_1\cdot\nabla\,(u_1-u_2)\,dx \\ &\quad -\int_\Omega \rho\,(u_2)\,\varphi_2\nabla\,(\varphi_2-\varphi_1)\cdot\nabla\,(u_1-u_2)\,dx \\ &= J_1+J_2+J_3. \end{aligned}$$

Using (4.45) and (4.46) and the Young inequality, we obtain

$$\begin{aligned} |J_1| &\leq C_5\int_\Omega |u_1-u_2|\,|\nabla\,(u_1-u_2)|\,dx \\ &\leq C_5\alpha\int_\Omega |\nabla(u_1-u_2)|^2dx+\frac{C_5}{\alpha}\int_\Omega |u_1-u_2|^2dx, \end{aligned}$$

where $\alpha$ is a small parameter to be specified later.
Similarly, by (3.11) and (4.45) we get

$$\begin{aligned} |J_2| &\le C_6 \int_\Omega |\varphi_1 - \varphi_2| \, |\nabla (u_1 - u_2)| \, dx \\ &\le C_6 \alpha \int_\Omega |\nabla (u_1 - u_2)|^2 \, dx + \frac{C_6}{\alpha} \int_\Omega |\varphi_1 - \varphi_2|^2 \, dx \end{aligned}$$

and

$$\begin{aligned} |J_3| &\le C_7 \int_\Omega |\nabla (\varphi_1 - \varphi_2)| \, |\nabla (u_1 - u_2)| \, dx \\ &\le C_7 \alpha \int_\Omega |\nabla (u_1 - u_2)|^2 \, dx + \frac{C_7}{\alpha} \int_\Omega |\nabla (\varphi_1 - \varphi_2)|^2 \, dx. \end{aligned}$$

Now by the Poincaré inequality one has for some constant $C_8$,

$$\int_\Omega |\varphi_1 - \varphi_2|^2 \, dx \le C_8 \int_\Omega |\nabla (\varphi_1 - \varphi_2)|^2 \, dx, \tag{4.51}$$

so that

$$\begin{aligned} J_2 &\le C_6 \alpha \int_\Omega |\nabla (u_1 - u_2)|^2 \, dx + \frac{C_6 C_8}{\alpha} \int_\Omega | \nabla (\varphi_1 - \varphi_2)^2 \, dx \\ &\le C_6 \alpha \int_\Omega |\nabla (u_1 - u_2)|^2 \, dx + \frac{C_2 C_6 C_8}{\alpha} \int_\Omega |u_1 - u_2|^2 \, dx \quad \text{(by (4.48))}. \end{aligned}$$

And also

$$J_3 \le C_7 \alpha \int_\Omega |\nabla(u_1 - u_2)|^2 dx + \frac{C_2 C_7}{\alpha} \int_\Omega |u_1 - u_2|^2 \, dx.$$

Combining the result above, we obtain

$$J \le (C_5 + C_6 + C_7)\alpha \int_\Omega |\nabla (u_1 - u_2)|^2 \, dx + (\frac{C_5 + C_2 C_7 + C_2 C_6 C_8}{\alpha}) \int_\Omega |u_1 - u_2|^2 \, dx.$$

Return to equation (4.50) and choosing $\alpha$ such that $(C_5 + C_6 + C_7)\alpha = C_3$, we deduce

$$\frac{d}{dt} \Big( \frac{1}{2} \int_\Omega |u_1 - u_2|^2 \, dx \Big) \le C_9 \int_\Omega |u_1 - u_2|^2 \, dx,$$

where $C_9 = (\frac{C_5 + C_2 C_7 + C_2 C_6 C_8}{\alpha})$.
Finally, by the Gronwall lemma, we obtain

$$\int_\Omega |u_1 - u_2|^2 \, dx \le C_{10} \int_\Omega |u_1(x,0) - u_2(x,0)|^2 \, dx,$$

and the initial condition allows us to have $u_1 = u_2$ and by (4.48) and (4.51) , we have also $\varphi_1 = \varphi_2$. So we have the uniqueness of the weak solution of (1.1).

**Remark 4.4.** The term $K = \dfrac{d}{dt} \Big( \dfrac{1}{2} \displaystyle\int_\Omega |u_1 - u_2|^2 \, dx \Big)$ has played an important role in proving the uniqueness of the solution. Then, if we have integrated the equation (4.49) on $Q_T$, the term $K$ would be equal to zero, because of the periodicity of $u_i, i = 1, 2$. So we only have to integrate on $\Omega$ a.e. in $t$.

□

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