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Stability results of a suspension-bridge with nonlinear damping modulated by a time dependent coefficient

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ABSTRACT. The main goal of this work is to investigate the following weakly damped nonlinear suspension-bridge equation

$$u_{tt}(x,y,t)+\Delta^2 u(x,y,t)+\alpha(t)g(u_t)=0,$$

and establish explicit and general decay results for the energy of solutions of the problem. Our decay results depend on the functions α and g and obtained without any restriction growth assumption on g at the origin. The multiplier method, the properties of the convex and the dual of the convex functions, Jensen's inequality and the generalized Young inequality are used to establish the stability results.

1. INTRODUCTION

The importance of bridges is undeniable and their presence in the human daily life goes back in history for a long time. Though their importance, bridges have brought some challenges, such as collapse and instability due to nature hazards such as winds, earthquakes, … etc. To overcome these difficulties, engineers and scientists have made efforts to find the best designs and models possible. Many mathematical models have appeared since the collapse of Tacoma Narrows Bridge. Motivated by the wonderful book of Rocard [17], where it was pointed out that the correct way to model a suspension bridge is through a thin plate, Ferrero-Gazzola [8] introduced the following hyperbolic problem:

$$\begin{cases} u_{tt}(x,y,t)+\eta u_t+\Delta^2 u(x,y,t)+h(x,y,u)=f, \ in\ \Omega\times\mathbb{R}^+, \\ u(0,y,t)=u_{xx}(0,y,t)=u(\pi,y,t)=u_{xx}(\pi,y,t)=0, \ (y,t)\in(-l,l)\times\mathbb{R}^+, \\ u_{yy}(x,\pm l,t)+\sigma u_{xx}(x,\pm l,t)=0, \ (x,t)\in(0,\pi)\times\mathbb{R}^+, \\ u_{yyy}(x,\pm l,t)+(2-\sigma)u_{xxy}(x,\pm l,t)=0, \ (x,t)\in(0,\pi)\times\mathbb{R}^+, \\ u(x,y,0)=u_0(x,y), \ u_t(x,y,0)=u_1(x,y), \ \text{ in } \Omega\times\mathbb{R}^+, \end{cases} \tag{1.1}$$

where $\Omega=(0,\pi)\times(-l,l)$ is a planar rectangular plate, σ is the well-known Poisson ratio, η is the damping coefficient, h is the nonlinear restoring force of the hangers and f is an external force. After the appearance of the above model, many mathematician showed interest in investigating variants of it, using different kinds of damping in the aim to obtain stability of the bridge modelled though the above problem. Wang [18] considered the equation

$$u_{tt}+\Delta^2 u+\nu u_t=|u|^{p-2}u,$$

together with the above initial and boundary conditions. After showing the uniqueness and existence of local solutions, he gave sufficient conditions for global existence and finite-time blow-up of solutions. Messaoudi and Mukiawa [16] studied the above problem, where the linear frictional damping was replaced by a nonlinear frictional damping and established the existence of a global weak solution and proved exponential and polynomial stability results. Recently, Audu et al. [4] considered a plate equation as

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a model for a suspension bridge with a general nonlinear internal feedback and time-varying weight. Under some conditions on the feedback and the coefficient functions, they established a general decay estimate. More results in this direction can be found in [1, 15, 6, 5, 9, 10, 14, 7].

Our aim in this work is to investigate the following plate problem

$$(1.2)\quad \begin{cases} u_{tt}(x,y,t)+\Delta^2 u(x,y,t)+\alpha(t)g(u_t)=0, \ in\ \Omega\times(0,T), \\ u(0,y,t)=u_{xx}(0,y,t)=u(\pi,y,t)=u_{xx}(\pi,y,t)=0, \ (y,t)\in(-d,d)\times(0,T), \\ u_{yy}(x,\pm d,t)+\sigma u_{xx}(x,\pm d,t)=0, \ (x,t)\in(0,\pi)\times(0,T), \\ u_{yyy}(x,\pm d,t)+(2-\sigma)u_{xxy}(x,\pm d,t)=0, \ (x,t)\in(0,\pi)\times(0,T), \\ u(x,y,0)=u_0(x,y), \ u_t(x,y,0)=u_1(x,y), \ \text{ in } \Omega\times(0,T), \end{cases}$$

where $\Omega=(0,\pi)\times(-d,d)$, $d<<\pi$, $g:\mathbb{R}\to\mathbb{R}$ and $\alpha:[0,+\infty)\to(0,+\infty)$ is a nonincreasing differentiable function, u is the vertical displacement of the bridge and σ is the Poisson ratio. For metals its value lies around 0.3, while for concrete it is between 0.1 and 0.2. For this reason, we shall assume that $0<\sigma<\frac{1}{2}$.
This is a weakly damped nonlinear suspension-bridge problem, in which the damping is modulated by a time dependent coefficient $\alpha(t)$. We establish an explicit and general decay result, depending on g and α, for which the optimal exponential and polynomial decay rate estimates are only special cases. The proof is based on the multiplier method and makes use of some properties of convex functions including the use of the general Young inequality and Jensen's inequality. The proof of the current result is easier than the one in [4]. Moreover, this result gives a better rate of decay (see Remark 3.3 below).
The paper is organized as follows. In section 2, we present some notations and material needed for our work. The statement and the proof of our main result will be given in section 3.

2. Preliminaries

In this section, we present some material needed in the proof of our results. First, we introduce the space

$$(2.3)\qquad H^2_*(\Omega)=\{w\in H^2(\Omega): w=0 \text{ on } \{0,\pi\}\times(-d,d)\},$$

together with the inner product

$$(2.4)\qquad (u,v)_{H^2_*}=\int_\Omega \left(\Delta u\Delta v+(1-\sigma)(2u_{xy}v_{xy}-u_{xx}v_{yy}-u_{yy}v_{xx})\right)\,\mathrm{d}x\,\mathrm{d}y.$$

It is well known that $(H^2_*(\Omega),(\cdot,\cdot)_{H^2_*})$ is a Hilbert space, and the norm $\|.\|^2_{H^2_*}$ is equivalent to the usual H^2, see [8]. We also let

$$(2.5)\qquad \mathcal{H}(\Omega):= \text{ The dual space of } H^2_*(\Omega).$$

Throughout this paper, c is used to denote a generic positive constant.

Lemma 2.1. [18] *Let $u\in H^2_*(\Omega)$ and assume that $1\le p<\infty$, then, there exists a positive constant $C_e=C_e(\Omega,p)>0$ such that*

$$\|u\|_p^p\le C_e\|u\|^p_{H^2_*(\Omega)}.$$

Lemma 2.2. (Jensen's inequality) *Let $G:[a,b]\longrightarrow\mathbb{R}$ be a convex function. Assume that the functions $f:(0,L)\longrightarrow[a,b]$ and $r:(0,L)\longrightarrow\mathbb{R}$ are integrable such that $r(x)\ge0$, for any $x\in(0,L)$ and $\int_0^L r(x)dx=k>0$. Then,*

$$G\left(\frac{1}{k}\int_0^L f(x)r(x)dx\right) \le \frac{1}{k}\int_0^L G(f(x))r(x)dx. \tag{2.6}$$

We consider the following hypotheses
(H1) $g : \mathbb{R} \to \mathbb{R}$ is nondecreasing C^1 function such that there exists a C^2 convex and increasing function $G : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $G(0) = 0$ and $G'(0) = 0$ or G is linear on $[0, \varepsilon]$ such that

$$\begin{aligned} &c_1|s| \le |g(s)| \le c_2|s|, \text{ if } |s| \ge \varepsilon,\\ &|s|^2 + g^2(s) \le G^{-1}(sg(s)), \text{ if } |s| \le \varepsilon, \end{aligned} \tag{2.7}$$

where ε, c_1, c_2 are positive constant.
(H2) $\alpha : [0, +\infty) \to (0, +\infty)$ is a nonincreasing differentiable function.

Remark 2.1. Hypothesis (H1) implies that $sg(s) > 0$, for all $s \neq 0$.

Remark 2.2. If g satisfies

$$\begin{aligned} &g_0(|s|) \le |g(s)| \le g_0^{-1}(|s|), \quad |s| \le \varepsilon\\ &c_1|s| \le |g(s)| \le c_2|s|, \quad |s| \ge \varepsilon \end{aligned} \tag{2.8}$$

for some strictly increasing function $g_0 \in C^1([0, +\infty))$, with $g_0(0) = 0$, and positive constants c_1, c_2, ε and the function $G(s) = \sqrt{\frac{s}{2}}g_0\left(\frac{s}{2}\right)$, is strictly convex C^2 function on $(0, \varepsilon]$ when g_0 is nonlinear, then (H1) is staisfied. This kind of hypotheses, where (H1) is weaker, was considered by Liu and Zuazua [13] and Alabau-Boussouira [2].

For completeness we state the following existence and regularity result whose proof can be established similarly to that in [12] and [16].

Theorem 2.1. *Let $(u_0, u_1) \in H^2_*(\Omega) \times L^2(\Omega)$. Assume that (H1) and (H2) hold. Then problem* (1.2) *has a unique weak global solution*

$$u \in C([0, T), H^2_*(\Omega)), \qquad u_t \in C([0, T), L^2(\Omega)).$$

The energy functional associated with problem (1.2) is

$$E(t) = \frac{1}{2}\left(\|u_t\|_2^2 + \|u\|^2_{H^2_*(\Omega)}\right). \tag{2.9}$$

Direct differentiation of (2.9), using (1.2), leads to

$$E'(t) = -\alpha(t)\int_\Omega u_t g(u_t)dx \le 0. \tag{2.10}$$

3. Main Result

In this secction, we state and prove our main result. We first start with the following lemma.

Lemma 3.3. *The functional*

$$L(t) = NE(t) + \int_\Omega u u_t dx$$

satisfies, along the solution of (1.2),

$$L'(t) \le -c_1 E(t) + c_2 \int_\Omega \left(u_t^2 + g^2(u_t)\right) dx \tag{3.11}$$

and

$$L \sim E, \tag{3.12}$$

where c_1 and c_2 are positive constants.

Proof. Differentiate $L(t)$, using (1.2), (2.9) and (2.10), to get

$$L'(t) = NE'(t) + \|u_t\|_2^2 - \|u\|_{H_*^2(\Omega)}^2 - \alpha(t)\int_\Omega ug(u_t)dx. \tag{3.13}$$

The use of Young's inequality and the embedding lemma 2.1 gives

$$L'(t) \le NE'(t) + \|u_t\|_2^2 - (1-c\delta)\|u\|_{H_*^2(\Omega)}^2 + c(\delta)\int_\Omega g^2(u_t)dx.$$

Selecting δ small enough and N large enough, we easily obtain (3.11) and (3.12).

□

We are now ready to state and prove our main result.

Theorem 3.2. *Let $(u_0, u_1) \in H_*^2(\Omega) \times L^2(\Omega)$. Assume that $(A1)$- $(A2)$ hold. Then there exist positive constants ε_0, d_1, d_2, k_1, k_2 and k_3 such that*

$$E(t) \le d_1 e^{-d_2\int_0^t \alpha(s)ds}, \text{ if } G \text{ is linear} \tag{3.14}$$

and

$$E(t) \le k_1 W_1^{-1}(k_2\int_0^t \alpha(s)ds + k_3), \text{ otherwise,} \tag{3.15}$$

where

$$W_1(\tau) = \int_\tau^1 \frac{1}{W_2(s)} \text{ and } W_2(t) = tG'(\varepsilon_0 t).$$

Proof. We multiply (3.11) by $\alpha(t)$ to get

$$\alpha(t)L'(t) \le -c_1\alpha(t)E(t) + c_2\alpha(t)\int_\Omega \left(u_t^2 + g^2(u_t)\right)dx. \tag{3.16}$$

Case 1. G is linear on $[0,\varepsilon]$. Then, using $(A1)$ and (2.10), estimate (3.16) becomes

$$\alpha(t)L'(t) \le -c_1\alpha(t)E(t) + c\alpha(t)\int_\Omega u_t g(u_t)dx = -c_1\alpha(t)E(t) - cE'(t),$$

which gives,

$$(\alpha L + cE)' \le -c_1\alpha(t)E(t).$$

Hence, using the fact that $\alpha L + cE \sim E$, we get (3.14).

Case 2. G is nonlinear on $[0,\varepsilon]$.

Let $0 < \varepsilon_1 \le \varepsilon$ such that

$$sg(s) \le \min\{\varepsilon, G(\varepsilon)\} \text{ for all } |s| \le \varepsilon_1. \tag{3.17}$$

Recalling (H1) and Remark 2.2, we have, for $\varepsilon_1 \le |s| \le \varepsilon$,

$$|g(s)| \le \frac{g_0^{-1}(|s|)}{|s|}|s| \le \frac{g_0^{-1}(|\varepsilon|)}{|\varepsilon_1|}|s| = c_2'|s|$$

and

$$|g(s)| \ge \frac{g_0(|s|)}{|s|}|s| \ge \frac{g_0(|\varepsilon_1|)}{|\varepsilon|}|s| = c_1'|s|.$$

Therefore, we deduce that

$$\begin{cases} s^2 + g^2(s) \le G^{-1}(sg(s)) & \text{for all } |s| \le \varepsilon_1 \\ c_1'|s| \le |g(s)| \le c_2'|s| & \text{for all } |s| \ge \varepsilon_1. \end{cases} \tag{3.18}$$

To estimate the last integral in (3.16), we use the following partition which was first introduced by Komornik [11]:

$$\Omega_1 = \{x \in \Omega : |u_t| \le \varepsilon_1\}, \ \Omega_2 = \{x \in \Omega : |u_t| > \varepsilon_1\}.$$

With

$$J(t) := \frac{1}{|\Omega_1|} \int_{\Omega_1} u_t g(u_t) dx,$$

Jensen's inequality gives (note that G^{-1} is concave)

$$G^{-1}(J(t)) \ge c \int_{\Omega_1} G^{-1}(u_t g(u_t)) dx. \tag{3.19}$$

Thus, combining (2.10), (3.18) and (3.19), we arrive at

$$\begin{aligned} \alpha(t) \int_\Omega \left(u_t^2 + g^2(u_t)\right) dx &\le \alpha(t) \int_{\Omega_1} G^{-1}(u_t g(u_t))\, dx + \alpha(t) \int_{\Omega_2} \left(u_t^2 + g^2(u_t)\right) dx \\ &\le c\alpha(t) G^{-1}(J(t)) + c\alpha(t) \int_{\Omega_2} u_t g(u_t) dx \\ &\le c\alpha(t) G^{-1}(J(t)) - cE'(t). \end{aligned} \tag{3.20}$$

A combination of (3.16) and (3.20) yields

$$L_1'(t) \le -c_1 \alpha(t) E(t) + c\alpha(t) G^{-1}(J(t)), \quad \forall t \in \mathbb{R}^+, \tag{3.21}$$

where $L_1 = \alpha L + cE$, which is clearly equivalent to E. Now, for $\varepsilon_0 < \varepsilon$ and $c_0 > 0$, let

$$L_2(t) := G'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) L_1(t) + c_0 E(t),$$

provided that $E(0) > 0$; otherwise $E(t) = 0$, $\forall t \in \mathbb{R}^+$, hence the theorem is verified. By using (3.21) and exploiting the properties of E and G, we conclude that L_2 satisifies

$$\begin{aligned} L_2'(t) =& \varepsilon_0 \frac{E'(t)}{E(0)} G''\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) L_1(t) + G'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) L_1'(t) + c_0 E'(t) \\ \le& - c_1 \alpha(t) E(t) G'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\alpha(t) G^{-1}(J(t))\, G'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) \\ &+ c_0 E'(t). \end{aligned} \tag{3.22}$$

Let G^* be the convex conjugate of G in the sense of Young (see [3], p.61-64), then, for $s \in (0, G'(\varepsilon)]$,

$$G^*(s) = s(G')^{-1}(s) - G[(G')^{-1}(s)] \le s(G')^{-1}(s). \tag{3.23}$$

Using the general Young inequality

$$AB \le G^*(A) + G(B), \quad \text{if } A \in (0, G'(\varepsilon)], \ B \in (0, \varepsilon],$$

for

$$A = G'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) \quad \text{and} \quad B = G^{-1}(J(t)),$$

we get

$$L_3'(t) \le -c_1\alpha(t)E(t)G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + c\alpha(t)\varepsilon_0\frac{E(t)}{E(0)}G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) - cE'(t)$$
$$+ c_0E'(t)$$
$$= -(c_1E(0) - C\varepsilon_0)\alpha(t)\frac{E(t)}{E(0)}G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) - (c-c_0)E'(t).$$

Consequently, by picking ε_0 small enough so that $c_1E(0) - C\varepsilon_0 > 0$ and c_0 large such that $c - c_0 < 0$, we obtain, for all $t \in \mathbb{R}^+$,

$$\alpha_1L_2(t) \le E(t) \le \alpha_2L_2(t), \tag{3.24}$$

for some $\alpha_1, \alpha_2 > 0$ and for some $k > 0$,

$$L_2'(t) \le -k\alpha(t)\frac{E(t)}{E(0)}G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) = -k\alpha(t)W_2\left(\frac{E(t)}{E(0)}\right), \tag{3.25}$$

where

$$W_2(s) = sG'(\varepsilon_0 s).$$

Thus, with $R(t) = \frac{\alpha_1L_2(t)}{E(0)}$ and using (3.24) and (3.25), we have

$$R(t) \sim E(t) \tag{3.26}$$

and, for some $k_2 > 0$,

$$R'(t) \le -k_2\alpha(t)W_2(R(t)). \tag{3.27}$$

Inequality (3.27) implies that $(W_1(R))' \ge k_2\alpha(t)$, where

$$W_1(\tau) = \int_\tau^1 \frac{1}{W_2(s)}ds \quad \text{for } \tau \in (0,1].$$

So, by integrating over $[0,t]$, we get

$$R(t) \le W_1^{-1}\left(k_2\int_0^t \alpha(s)ds + k_3\right), \quad \forall t \in \mathbb{R}^+. \tag{3.28}$$

Finally, we obtain (3.15) by virtue of (3.26) and (3.28). □

Example 3.1. We give some examples to illustrate the energy decay rates given by Theorem 3.2. Here, we assume that g satisfies (2.8) near the origin with the following various examples for g_0:

1. If $g_0(s) = cs^q$ and $q \ge 1$, then $G(s) = cs^{\frac{q+1}{2}}$ satisfies $(A1)$ and, consequently, Theorem (3.2) yields

$$E(t) \le d_1e^{-d_2\int_0^t \alpha(s)ds}, \quad \text{if } q = 1,$$

$$E(t) \le k_1\left(k_2\int_0^t \alpha(s)ds + k_3\right), \quad \text{if } q > 1.$$

2. If $g_0(s) = e^{-\frac{1}{s}}$, then $(A1)$ is satisfied for $G(s) = \sqrt{\frac{s}{2}}e^{-\sqrt{\frac{2}{s}}}$ near zero. Therefore, we get

$$E(t) \le k_1\left(\ln\left(k_2\int_0^t \alpha(s)ds + k_3\right)\right)^{-2}.$$

Remark 3.3. This work improves the work of Audu et al. [4] in several aspects: first, we asuume less conditions on the nonlinear feedback, this allows a wider class of functions. Second, our decay result is explicit and clearer than the one obotained in [4]. Third, it is easy to compute the decay rate of the well-known nonlinear feedback.

Remark 3.4. Note that the exponential and the polynomial decay results are only special cases.

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