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Faedo-Galerkin Approximations for nonlinear Heat equation on Hilbert Manifold

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ABSTRACT. In this work, we aim to study the well-posedness of a deterministic problem consisting of the non-linear heat equation of gradient type. The evolution equation emerges by projecting the Laplace operator with Dirichlet boundary conditions and polynomial nonlinearity of degree $2n-1$, onto the tangent space of the sphere $\mathcal{M}$ in a Hilbert space $\mathcal{H}$. We are going to deal with the question of existence and uniqueness based on the Faedo-Galerkin compactness method.

1. INTRODUCTION

In this article, we are concerned with the problem Faedo-Galerkin approximations for the non-linear heat flow equation projected on the manifold (Hilbert) M. Firstly, Rybka [20] and Caffarelli Lin [11] studies the heat equation in $\mathcal{L}^2(\mathcal{D})$ projected on the manifold $\mathcal{M}$, where,

$$\mathcal{M} = \left\{ u \in \mathcal{L}^2(\mathcal{D}) \cap C(\mathcal{D}) : \int_D u^k(x)\,dx = C_k, k = 1, 2, \cdots, N \right\}, \tag{1.1}$$

and $\mathcal{D}$ be the bounded, connected region in $\mathbb{R}^2$. Rybka proved the global existence and uniqueness of the solution to the following projected heat equation,

$$\begin{cases} \dfrac{du}{dt} = \Delta u - \displaystyle\sum_{k=1}^{N} \lambda_k u^{k-1} \quad \text{in} \quad \mathcal{D} \subset \mathbb{R}^2 \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\mathcal{D}, \quad u(0,x) = u_0. \end{cases} \tag{1.2}$$

where $\lambda_k = \lambda_k(u)$ are such that u_t is orthogonal to Span $\{u^{k-1}\}$ for more regular initial data.

Secondly, in [11] Cafarelli and Lin established the global well-posedness of the energy-conserving solution to the heat equation. They were then able to extend these results to more general family of singularly perturbed systems of nonlocal parabolic equation. Their main result was to prove the strong convergence of the solutions of these perturbed systems to some weak solutions of the limiting constrained non-local heat flows of maps into singular space.

Recently, in [4] Brzeźniak, Dhariwal, and Mariani studied 2D Navier–Stokes equations with a constraint forcing the conservation of the energy of the solution. They proved the existence and uniqueness of the global solution for the constrained Navier–Stokes equation on $\mathbb{R}^2$ and $\mathbb{T}^2$, by fixed point argument. They also show that the solution of the constrained equation converges to the solution of the Euler equation as the viscosity ν vanishes.

In this paper, we consider a problem that links the aforementioned works. Some of the

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classical and modern references on constrained partial differential equations on manifolds are [2], [3], [5], [8], [9], [10], [12], [13]. Finally, we note that the approach we have used in this paper can be applied to a number of problems, including the projected deterministic and stochastic wave equation, Schrodinger equation, Navier-stokes equations, beam equation with polynomial nonlinearity, and spatially homogeneous noise, on bounded domains. We also believe that what we have done for the sphere in Hilbert space, a similar set of studies can be done for projections on a variety of closed Hilbert manifolds (i.e. compact manifolds without boundaries).
Suppose that $\mathcal{H}$ is Hilbert space and $\mathcal{M}$ is its unit sphere. Let f be a vector field on $\mathcal{H}$ (possibly only densely defined) such that the initial value problem,

$$\begin{aligned} \frac{du}{dt} &= f\left(u(t)\right), \quad t \geq 0 \\ u(0) &= x, \end{aligned} \tag{1.3}$$

has the unique global solution for every $x \in \mathcal{H}$. The semi-flow generated by above initial value problem, denoted by $(\varphi(t,x))_{t\geq 0}$, in general does not stay on $\mathcal{M}$ even though $x \in \mathcal{M}$. The reason for this is because in general, the vector field f is not tangent to $\mathcal{M}$ i.e. It does not satisfy the following,

$$f\left(x\right) \in T_x\mathcal{M}, \quad x \in \mathcal{V} \cap \mathcal{M}, \tag{1.4}$$

where $\mathcal{V} := \mathcal{D}(f)$. However, it is easy to modify f to the new vector field $\widetilde{f}$ such that the property 1.4 is satisfied. This can be achieved by using a map $\pi : \mathcal{V} \to \mathcal{L}(\mathcal{H},\mathcal{H})$, defined by

$$\pi(x) = \{\mathcal{H} \ni y \mapsto y - \langle x, y\rangle\, x \in \mathcal{H}\} \in \mathcal{L}(\mathcal{H},\mathcal{H}), \quad for \;\; every \;\; x \in \mathcal{V}.$$

The remarkable property of π is that when $x \in \mathcal{M}$, the linear map $\pi(x) : \mathcal{V} \to T_x\mathcal{M}$ is the orthogonal projection so that vector field $\widetilde{f}$ defined by,

$$\widetilde{f} : \mathcal{D}(f) \ni x \mapsto \pi(x)[f(x)] \in \mathcal{H}. \tag{1.5}$$

Indeed, for $x \in \mathcal{V} \cap \mathcal{M}$ we have,

$$\left\langle \widetilde{f}(x), x \right\rangle = \langle f(x) - \langle x, f(x)\rangle\, x, x\rangle = \langle f(x), x\rangle - \langle f(x), x\rangle\, |u|^2_{\mathcal{H}} = 0.$$

Hence $\widetilde{f}$ satisfies the property described in (1.4).
If f is globally defined (i.e. $\mathcal{D}(f) = \mathcal{H}$) and locally Lipschitz map then $\widetilde{f}$ is also globally defined and is also locally Lipschitz map. Moreover, the modified equation,

$$\begin{aligned} \frac{du}{dt} &= \widetilde{f}\left(u(t)\right), \quad t \geq 0, \\ u(0) &= x, \end{aligned} \tag{1.6}$$

has local solution for every $x \in \mathcal{V}$. This solution stays on $\mathcal{M}$ whenever $x \in \mathcal{M}$.

The situation is not so clear when f is only densely defined. We will consider the following two special cases for densely defined f:
Let $\mathcal{O} \subset \mathbb{R}^d$, where $d \in \mathbb{N}$, be the bounded domain with sufficiently smooth boundary, A denotes the negative Laplace operator with Dirichlet boundary conditions and $f(u) = -Au$. In the first case, we will see that

$$\widetilde{f}(u) = -Au + |\nabla u|^2_{\mathcal{H}} u.$$

The second case is when $f(u) = -u^{2n-1}$, where the range for n is described in the Assumption 2.1. In this case, one can find that,

$$\widetilde{f}(u) = -u^{2n-1} + |u|^{2n}_{\mathcal{L}^{2n}(\mathcal{O})} u. \tag{1.7}$$

Roughly speaking, the aim of this paper is to give complete treatment to both the examples simultaneously. To be more precise, we prove the existence and uniqueness of the solution initial value problem (1.6). See Theorem 2.2 for a precise description.

2. Assumption, Functional settings, estimates, and key results

In this section, we will give all important assumptions, spaces and estimates that we require throughout the paper. We will also state the main result of the paper.

2.1. **Functional Setting.** Let us assume that $\mathcal{O} \subset \mathbb{R}^d$ be a bounded and smooth domain. We have precisely described the assumptions on $d \in \mathbb{N}$, later in Assumption 2.1. For $p \in [0,\infty)$ let $\mathcal{L}^p(\mathcal{O})$ denotes the Banach space of [equivalence classes] Lebesgue measurable $\mathbb{R}$-valued p-th power integrable functions on the set $\mathcal{O}$. The norm on $\mathcal{L}^p(\mathcal{O})$ is given by,

$$|u|_{\mathcal{L}^p(\mathcal{O})} := \Big(\int_{\mathcal{O}} |u(x)|^p dx\Big)^{\frac{1}{p}}, \quad u \in \mathcal{L}^p(\mathcal{O}).$$

For $p = 2$, the space $\mathcal{L}^2(\mathcal{O})$ is Hilbert space with the standard inner product denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$.

$$\langle u, v \rangle_{\mathcal{H}} := \int_{\mathcal{O}} u(x) \cdot v(x) dx, \quad u, v \in \mathcal{L}^2(\mathcal{O}).$$

For $k \in \mathbb{N}$ and $p \in [0,\infty)$, by $W^{k,p}(\mathcal{O})$ we denote the Sobolev space of all $u \in \mathcal{L}^p(\mathcal{O})$ for which the weak derivative $D^\alpha u \in \mathcal{L}^p(\mathcal{O})$, $|\alpha| \leq k$. In particular for $p = 2$, we denote $\mathcal{H}^k := W^{k,2}(\mathcal{O})$ and its norm by $\|\cdot\|_{\mathcal{H}^k(\mathcal{O})}$. In particular $\mathcal{H}^1(\mathcal{O})$ is a Hilbert space with the following inner product,

$$\langle u, v \rangle_{\mathcal{H}^1} := \langle u, \mathrm{v} \rangle_{\mathcal{H}} + \langle \nabla u, \nabla v \rangle_{\mathcal{H}}, \quad u, v \in \mathcal{H}^1(\mathcal{O}).$$

We also denote by $\mathcal{H}^{1,2}_0(\mathcal{O})$ the closure in $\mathcal{H}^{1,2}(\mathcal{O})$ of the space $C^\infty_0(\mathcal{O})$ equipped with the norm

$$\|u\|^2 := \langle \nabla u, \nabla u \rangle_{\mathcal{H}}, \quad u \in \mathcal{H}^1_0(\mathcal{O}),$$

which in view of the Poincaré inequality, is equivalent to the norm induced by the $\mathcal{H}^1(\mathcal{O})$-norm.

Remark 2.1. Let A be the Laplace operator with Dirichlet boundary conditions, i.e. a linear operator defined by

$$\mathcal{D}(A) = \mathcal{H}^{1,2}_0(\mathcal{O}) \cap \mathcal{H}^{2,2}(\mathcal{O}), \quad Au = -\Delta u, \ u \in \mathcal{D}(A). \tag{2.1}$$

It is well known that, cf. [23], Theorem 4.1.2, page 79, that A is a self-adjoint positive operator in $\mathcal{H}$, $\mathcal{D}(A^{1/2}) = \mathcal{H}^{1,2}_0(\mathcal{O})$ and

$$\|u\|^2 = \left|A^{1/2}u\right|^2_{\mathcal{L}^2(\mathcal{O})} = |\nabla u|^2_{\mathcal{L}^2(\mathcal{O})}, \ u \in \mathcal{H}^{1,2}_0(\mathcal{O}).$$

To simplify the presentation let us introduce the notation $(\mathcal{E}, |\cdot|_{\mathcal{E}})$, $(\mathcal{V}, \|\cdot\|)$, $(\mathcal{H}, |\cdot|_{\mathcal{H}})$ for following spaces,

$$\mathcal{H} = \mathcal{L}^2(\mathcal{O}), \quad \mathcal{V} = \mathcal{H}^{1,2}_0(\mathcal{O}) \text{ and } \mathcal{E} = \mathcal{D}(A). \tag{2.2}$$

Moreover,

$$\mathcal{E} \subset \mathcal{V} \subset \mathcal{H} \subset \mathcal{V}' =: \mathcal{H}^{-1}(\mathcal{O}),$$

the inclusion is continuous and dense.

By $\mathcal{L}(X,Y)$ we mean the space of all bounded linear operators from Banach X to the Banach space Y. For any $b > a \geq 0$, and a separable Banach space X, let us denote by $\mathcal{L}^2\big(a,b;X\big)$ the space of [equivalence classes] of all Borel measurable functions $u : \big[a,b\big] \to X$, such that

$$|u|_{\mathcal{L}^2(a,b;X)} := \Big(\int_a^b |u(t)|_X^2 \; dt\Big)^{1/2} < \infty.$$

For $b > a \geq 0$ we define a Banach space $X_{a,b}$ by

$$\begin{aligned} X_{a,b} &:= \mathcal{L}^2\big(a,b;\mathcal{E}\big) \cap C\big(\big[a,b\big];\mathcal{V}\big), \\ |u|^2_{X_{a,b}} &= \sup_{t\in[a,b]} \|u(t)\|^2 + \int_a^b |u(t)|^2_{\mathcal{E}}\, dt. \end{aligned}$$

For $a = 0$ and $b = T > 0$ we are going to write $X_T := X_{0,T}$. Note that if $u \in X_T$ the map $[0,T] \ni t \mapsto |u|_{X_t}$ is increasing function.

2.2. **Assumptions on Domain.** Let us begin by deriving the relation between n (involved in equation (1.7)) and dimension d of the domain $\mathcal{O}$, required to have a useful embedding. For this, let us recall the following well-known Gagliardo-Nirenberg-Sobolev inequality.

Lemma 2.1. [21] *For any $u \in W^{m,r}(\Omega)\bigcap \mathcal{L}^q(\Omega)$ where Ω is bounded domain with smooth boundary, there are two positive constants C_1, C_2 such that the following inequality holds:*

$$\big|D^j u\big|_{p,\Omega} \leq C_1 \left|D^m u\right|^a_{r,\Omega} |u|^{1-a}_{q,\Omega} + C_2|u|_{q,\Omega}. \tag{2.3}$$

Here

$$\frac{1}{p} = \frac{j}{d} + a\left(\frac{1}{r} - \frac{m}{d}\right) + (1-a)\frac{1}{q}, \tag{2.4}$$

for all $a \in \big[\frac{j}{m}, 1\big]$. If $m - j - \frac{d}{r}$ is non-negative integer, then the equality (2.3) *holds only for $a \in \big[\frac{j}{m}, 1\big)$.*
In particular, for any $u \in W^{m,r}_0(\Omega) \cap \mathcal{L}^q(\Omega)$, the constant C_2 in (2.3) *can be taken as zero.*

Observe that the expression in equation (1.7) involves the $\mathcal{L}^{2n}$ norm. Therefore, at several instances throughout the paper, we will use following particular case of Gagliardo-Nirenberg-Sobolev inequality.
Let us put $m = 1, j = 0, r = 2, q = 2$.
For $\boldsymbol{d = 2}$, we have $m - j - \frac{d}{r} = 0$ and the condition (2.4) becomes,

$$\frac{1}{p} = \frac{0}{d} + a\left(\frac{1}{2} - \frac{1}{2}\right) + \frac{(1-a)}{2} = \frac{(1-a)}{2} \geq 0,$$

In particular, $p = 2n - 1$, where $n \in [1, \infty)$.
For $\boldsymbol{d \geq 3}$, we have

$$\frac{1}{p} = -a\left(\frac{1}{d}\right) + \frac{1}{2}.$$

When $a \leq 1$ we get

$$\frac{1}{p} \geq -\frac{1}{d} + \frac{1}{2}.$$

Plugging values of $r = q = 2, j = 0, m = 1$ and $p = 2n$ in the inequality (2.3) we get,

$$|u|_{\mathcal{L}^{2n}(\mathcal{O})} \leq C \left|\nabla u\right|^a_{\mathcal{H}} |u|^{1-a}_{\mathcal{H}}, \;\; u \in \mathcal{V}.$$

i.e.

$$|u|_{\mathcal{L}^{2n}(\mathcal{O})} \leq c\,\|u\|^{a}\,|u|_{\mathcal{H}}^{1-a}\,. \tag{2.5}$$

Since the embedding $\mathcal{V} \hookrightarrow \mathcal{H}$ is continuous so there exists $C > 0$ such that $|u|_{\mathcal{H}} \leq C\,\|u\|$ for all $u \in \mathcal{V}$. We infer that,

$$|u|_{\mathcal{L}^{2n}(\mathcal{O})} \leq C\,\|u\|\,. \tag{2.6}$$

Now we are able to summarize the above discussion in the form of the following assumption, which is going to be the key for the rest of the paper.

Assumption 2.1. *We assume that $\mathcal{O} \subset \mathbb{R}^d$ is a smooth domain and $n \in [0,\infty)$ such that*

$$\mathcal{H}^{1,2}(\mathcal{O}) \subset \mathcal{L}^{2n}(\mathcal{O}).$$

Thus, for $d = 2$, we assume that $n \geq 1$, for $d = 3$ we take $n = 2$, and in general,

$$\frac{1}{d} \geq \frac{1}{2} - \frac{1}{2n}. \tag{2.7}$$

2.3. **Manifold and Projection.** In this paper, we will deal with the following sub-manifold $\mathcal{M}$ of Hilbert space $\mathcal{H}$, with the inner product denoted by $\langle\cdot,\cdot\rangle$,

$$\mathcal{M} = \left\{u \in \mathcal{H} : |u|_{\mathcal{H}}^2 = 1\right\}.$$

It is well known that $\mathcal{M}$ is Hilbert manifold and sub-manifold of $\mathcal{H}$. Moreover, the tangent space at the point $a \in \mathcal{H}$, can be identified with the following subspace of $\mathcal{H}$:

$$T_a\mathcal{M} = \{v : \langle a, v\rangle = 0\}\,.$$

for $a \in \mathcal{M}$, and let

$$\pi_a : \mathcal{V} \to T_a\mathcal{M}$$

be the orthogonal projection of $\mathcal{H}$ onto the tangent space $T_a\mathcal{M}$. We have the following simple lemma.

Lemma 2.2. *If $a \in \mathcal{M}$, then*

$$\pi_a(v) = v - \langle a, v\rangle\, a, \quad v \in \mathcal{H}.$$

Corollary 2.1. *In framework of Remark 2.1 and Assumption 2.1, for any $u \in \mathcal{E} \cap \mathcal{M}$, we have,*

$$\pi_u\left(\Delta u - u^{2n-1}\right) = \Delta u - u^{2n-1} + \left(\|u\|^2 + |u|_{\mathcal{L}^{2n}}^{2n}\right) u. \tag{2.8}$$

Proof. Fix $u \in \mathcal{E} \cap \mathcal{M}$. Then by definition of A and the Sobolev embedding, Δu and $|u|^{2n}u \in \mathcal{H}$. Using Lemma 2.2 and integration by parts formula, cf. [1, Corollary 8.10, p.82], we have

$$\begin{aligned}
\pi_u\left(\Delta u - u^{2n-1}\right) &= \Delta u - u^{2n-1} - \left\langle \Delta u - u^{2n-1}, u\right\rangle u \\
&= \Delta u - u^{2n-1} + \langle \nabla u, \nabla u\rangle u + \left\langle u^{2n-1}, u\right\rangle u \\
&= \Delta u - u^{2n-1} + \left(\|u\|^2 + |u|_{\mathcal{L}^{2n}}^{2n}\right) u.
\end{aligned}$$

□

We now state the main result of the paper.

Theorem 2.2. *Assume we are in the framework of Remark 2.1 and Assumption 2.1. Then for every $u_0 \in \mathcal{H}_0^{1,2}(\mathcal{O}) \cap \mathcal{M}$, where $\mathcal{M} = \{u \in \mathcal{H} : |u|_{\mathcal{H}} = 1\}$, there exists unique function $u : [0,\infty) \to \mathcal{V}$ such that for every $T > 0$, $u \in \mathcal{X}_T$, solves the following problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - u^{2n-1} + \left(\|u\|^2 + |u|_{\mathcal{L}^{2n}}^{2n}\right) u \\ u(0) = u_0. \end{cases} \tag{2.9}$$

Moreover, the function u stays on $\mathcal{M}$, i.e. $u(t) \in \mathcal{M}$, for all $t \geq 0$.

2.4. **Important Estimates.** In this subsection, we are going to treat $\mathcal{E}, \mathcal{V}$ and $\mathcal{H}$ as in Remark 2.1. The aim of this subsection is to show that the nonlinear part of our projected heat flow problem (2.9) i.e. the function $F : \mathcal{V} \to \mathcal{H}$ defined by:

$$F(u) := \|u\|^2 u - u^{2n-1} + u\,|u|^{2n}_{\mathcal{L}^{2n}}\,, \tag{2.10}$$

is locally Lipschitz, and it satisfies some suitable estimates.
Following elementary inequality is a straightforward consequence of Results 2.20 (page 62) or 2.91 (page 77) in [18].

Lemma 2.3. *If $a, b \geq 0$ and $p \geq 1$ then there exists a constant $c_p > 0$ such that*

$$|a^p - b^p| \leq c_p\,|a-b|\left(a^{p-1} + b^{p-1}\right). \tag{2.11}$$

In particular for $1 \leq p \leq 2$ the constant $c_p = 1$.

Lemma 2.4. [15] *Assume we are in the framework of Remark 2.1 and Assumption 2.1. Consider the map $F : \mathcal{V} \to \mathcal{H}$ is as defined by $F(u) = \|u\|^2 u - u^{2n-1} + u\,|u|^{2n}_{\mathcal{L}^{2n}}$. Then F is locally Lipschitz, i.e. there exists a positive constant C such that, for all $u, v \in \mathcal{V}$*

$$|F(u) - F(v)|_{\mathcal{H}} \leq C \begin{bmatrix} \left(\|u\|^2 + \|v\|^2\right) + (\|u\| + \|v\|)^2 \\ + \left(\|u\|^{2n-1} + \|v\|^{2n-1}\right)(\|u\| + \|v\|) \\ + \left(\|u\|^{2n} + \|v\|^{2n}\right) + \left(1 + \|u\|^2 + \|v\|^2\right)^{1/3} \end{bmatrix} \|u - v\|\,. \tag{2.12}$$

Proof. Set $F(u) = \|u\|^2 u - |u|^{2n-2}u + u\,|u|^{2n}_{\mathcal{L}^{2n}} =: F_1(u) - F_2(u) + F_3(u)$. We will now find the estimate for each F_1, F_2 and F_3.

Fix $u, v \in \mathcal{V}$. Let us begin by deriving the estimate for F_1. Then using triangle inequality

$$\begin{aligned} |F_1(u) - F_1(v)|_{\mathcal{H}} &= \left|\|u\|^2 u - \|v\|^2 v\right|_{\mathcal{H}} = \left|\|u\|^2 u - \|u\|^2 v + \|u\|^2 v - \|v\|^2 v\right|_{\mathcal{H}} \\ &\leq \left(\|u\|^2 + \|v\|^2\right)|u - v|_{\mathcal{H}} + (\|u\| + \|v\|)\,\|u - v\|\,(|u|_{\mathcal{H}} + |v|_{\mathcal{H}})\,. \end{aligned}$$

Since embedding $\mathcal{V} \hookrightarrow \mathcal{H}$ is continuous, so there exists $c_1 > 0$ such that the following holds

$$|F_1(u) - F_1(v)|_{\mathcal{H}} \leq c_1\left[\left(\|u\|^2 + \|v\|^2\right) + (\|u\| + \|v\|)^2\right]\|u - v\|\,. \tag{2.13}$$

Next, consider F_3. Using the elementary inequality (2.11) for $p = 2n - 1$, it follows that,

$$\begin{aligned} |F_3(u) - F_3(v)|_{\mathcal{H}} &= \left|u\,|u|^{2n}_{\mathcal{L}^{2n}} - v\,|v|^{2n}_{\mathcal{L}^{2n}}\right|_{\mathcal{H}} = \left|u\,|u|^{2n}_{\mathcal{L}^{2n}} - u\,|v|^{2n}_{\mathcal{L}^{2n}} + u\,|v|^{2n}_{\mathcal{L}^{2n}} - v\,|v|^{2n}_{\mathcal{L}^{2n}}\right|_{\mathcal{H}} \\ &\leq |u|_{\mathcal{H}}\left||u|^{2n}_{\mathcal{L}^{2n}} - |v|^{2n}_{\mathcal{L}^{2n}}\right|_{\mathcal{H}} + |u - v|_{\mathcal{H}}\,|v|^{2n}_{\mathcal{L}^{2n}}\,, \\ &\leq c_n\left(|u|^{2n-1}_{\mathcal{L}^{2n}} + |v|^{2n-1}_{\mathcal{L}^{2n}}\right)|u|_{\mathcal{H}}\,|u - v|_{\mathcal{L}^{2n}} + C^{2n}\,|u - v|_{\mathcal{H}}\,\|v\|^{2n} \\ &\leq c_2\left(\|u\|^{2n-1} + \|v\|^{2n-1}\right)(\|u\| + \|v\|) + \left(\|u\|^{2n} + \|v\|^{2n}\right)\|u - v\|\,, \end{aligned}$$

where $c_2(n) := c_n C^{2n+1}$.
To prove an estimate for F_2 we require the following inequality,

$$\left||u|^{2n-2}\,u - |v|^{2n-2}v\,\right| \leq c\left(|u|^{2n-2} + |v|^{2n-2}\right)\left|\,u - v\,\right|\,. \tag{2.14}$$

Now, we proceed with a proof of (2.14). Let us suppose initially, that $|u|, |v| \leq 1$, then differentiability at zero of the following function $x \mapsto |x|^p$ for $p > 1$ or one-sided differentiability at zero when $p = 1$ yields that

$$\sup_{u\neq v, |u|,|v|\leq 1} \frac{\left| |u|^{2n-2} u - |v|^{2n-2} v \right|}{|u-v|} =: C_0 < \infty \tag{2.15}$$

When $|u| > 1$ or $|v| > 1$, then we proceed as follows. We take $M_1 = |u|, M_2 = |v|$. Then, we have

$$\frac{\left| |u|^{2n-2} u - |v|^{2n-2} v \right|}{|u-v|} = (M_1 + M_2)^{2n-2} \frac{\left| |u_1|^{2n-2} u_1 - |v_1|^{2n-2} v_1 \right|}{|u_1 - v_1|},$$

where $u_1 = \frac{u}{M_1+M_2}, v_1 = \frac{v}{M_1+M_2}$. Hence, (2.15) yields,

$$\begin{aligned}\frac{\left| |u|^{2n-2} u - |v|^{2n-2} v \right|}{|u-v|} &\leq C_0 \left(M_1 + M_2\right)^{2n-2} = C_0(|u| + |v|)^{2n-2} \\ &\leq C_n \left(|u|^{2n-2} + |v|^{2n-2}\right).\end{aligned}$$

and our claim (2.14) follows.

Now, let us estimate $|F_2(u) - F_2(v)|_{\mathcal{H}}$. Indeed, using (2.14) yields,

$$\begin{aligned}|F_2(u) - F_2(v)|^2_{\mathcal{H}} &\leq C_n^2 \int_{\mathcal{O}} \left(|u(x)|^{2n-2} + |v(x)|^{2n-2}\right)^2 |u(x) - v(x)|^2 dx \\ &\leq C_n^2 \left(\int_{\mathcal{O}} \left(|u(x)|^{2n-2} + |v(x)|^{2n-2}\right)^3 dx\right)^{2/3} \left(\int_{\mathcal{O}} |u(x) - v(x)|^6 dx\right)^{1/3}, \quad (2.16)\end{aligned}$$

where we used the Hölder's inequality with exponents $3/2$ and 3 in the last line. Hence, with the help of Minkowski inequality it follows that

$$|F_2(u) - F_2(v)|_{\mathcal{H}} \leq C_n \left(\left(\int_{\mathcal{O}} |u(x)|^{6n-6} dx\right)^{1/3} + \left(\int_{\mathcal{O}} |v(x)|^{6n-6} dx\right)^{1/3}\right) |u-v|_{\mathcal{L}^6(\mathcal{O})} \tag{2.17}$$

We notice that

$$|u(x)|^{6n-6} \leq \max\{1, |u(x)|\}^{6n-6}. \tag{2.18}$$

Hence,

$$\left(\int_{\mathcal{O}} |u(x)|^{6n-6} dx\right)^{1/3} \leq \begin{cases} \left(|\mathcal{O}| + |u|^2_{\mathcal{L}^{6n}(\mathcal{O})}\right)^{1/3}, & d = 2 \\ \left(|\mathcal{O}| + |u|^2_{\mathcal{L}^{6}(\mathcal{O})}\right)^{1/3}, & d = 3. \end{cases} \tag{2.19}$$

Finally, for $d = 2$ using continuity of embedding $\mathcal{V} \hookrightarrow \mathcal{L}^{6n}(\mathcal{O})$, and for $d = 3$ using embedding $\mathcal{V} \hookrightarrow \mathcal{L}^6(\mathcal{O})$, it follows that there exists $c_n > 0$ such that

$$|F_2(u) - F_2(v)|_{\mathcal{H}} \leq c_n \left(1 + \|u\|^2 + \|v\|^2\right)^{1/3} \|u - v\|. \tag{2.20}$$

Setting $C := \max\{c_1, c_2, c_n\}$ and combining inequalities (2.13), (2.14) and (2.20) we get the desired inequality. $\square$

Moreover, for the convenience of the reader, we present the following [22] (Lemma 1.2, Chapter 3).

Lemma 2.5 (Lemma III 1.2, [22]). *Let $\mathcal{V}, \mathcal{H}$ and $\mathcal{V}'$ be three Hilbert spaces with $\mathcal{V}'$ being the dual space of $\mathcal{V}$ and each included and dense in the following one*

$$\mathcal{V} \hookrightarrow \mathcal{H} \cong \mathcal{H}' \hookrightarrow \mathcal{V}'.$$

If u belongs to $\mathcal{L}^2(0,T;\mathcal{V})$ *and its weak derivative* $\frac{\partial u}{\partial t}$ *belongs to* $\mathcal{L}^2(0,T;\mathcal{V}')$ *then there exists* $\widetilde{u} \in \mathcal{L}^2(0,T;\mathcal{V}) \cap C([0,T];\mathcal{V})$ *such that* $\widetilde{u} = u$ *a.e. and we have the following energy equality:*

$$|u(t)|^2 = |u_0|^2 + 2\int_0^t \langle u'(s), u(s)\rangle\, ds, \text{ for all } t \in [0,T].$$

Remark 2.2. If u satisfies the assumptions of Lemma 2.5 then we will identify u with $\widetilde{u}$. Note that weak derivative of $\widetilde{u}$ exists and is equal to $\frac{\partial u}{\partial t}$.

Moreover, we will make use of the following result in the subsequent sections.

3. A PRIORI ESTIMATES

Let $\{e_k\}$ be the orthonormal basis of, $\mathcal{H}$ consisting of eigenvectors of A. Let $\{\lambda_k\}$ be the set of eigenvalues of A. Let us denote by $\mathcal{H}_m$ the subspace generated by $\{e_k\}_{k=1}^m$. Clearly $\mathcal{H}_m \subset \mathcal{H}_j$ for all $j \geq m$. Consider the linear operator defined in the following manner,

$$\pi_m u := \sum_{k=1}^m \langle u, e_k\rangle e_k.$$

We are going to consider the following finite-dimensional approximate problem.

$$\begin{aligned} \frac{du_m}{dt} + A_m u_m &= F_m(u_m) \\ u_m(0) &= \frac{\pi_m u_0}{|\pi_m u_0|_{\mathcal{H}}}. \end{aligned} \tag{3.1}$$

Here $A_m(\cdot) = \pi_m A(\cdot)$ and $F_m(\cdot) = \pi_m F(\cdot)$.

In this section, we aim to obtain some a priori estimates for the approximated solution $u_m = \sum_{k=1}^m g_{km} e_k$.
Hence, we see

$$\frac{d}{dt}\left(\frac{|u_m(t)|^2_{\mathcal{H}} - 1}{2}\right) = \left\langle u_m(t), \frac{du_m}{dt}(t)\right\rangle.$$

Using equation (3.1) it follows that

$$\begin{aligned} \frac{d}{dt}\left(\frac{|u_m(t)|^2_{\mathcal{H}} - 1}{2}\right) &= \langle u_m(t), -A_m u_m + F_m(u_m)\rangle \\ &= -\langle u_m(t), A_m u_m\rangle + \langle u_m(t), F_m(u_m)\rangle \\ &= -\left\langle A_m^{\frac{1}{2}} u_m(t), A_m^{\frac{1}{2}} u_m\right\rangle \\ &\quad + \left\langle u_m(t), \left(\|u_m(t)\|^2 + |u_m(t)|^{2n}_{\mathcal{L}^{2n}}\right) u_m(t) - u_m(t)^{2n-1}\right\rangle \\ \frac{d}{dt}\left(|u_m(t)|^2_{\mathcal{H}} - 1\right) &= 2\left(\|u_m(t)\|^2 + |u_m(t)|^{2n}_{\mathcal{L}^{2n}}\right)\left(|u_m(t)|^2_{\mathcal{H}} - 1\right). \end{aligned} \tag{3.2}$$

On solving the differential equation (3.2) for $|u_m(t)|^2_{\mathcal{H}} - 1$, we get

$$|u_m(t)|^2_{\mathcal{H}} - 1 = \left(|u_m(0)|^2_{\mathcal{H}} - 1\right) \exp\left[2\int_0^t \left(\|u_m(s)\|^2 + |u_m(s)|^{2n}_{\mathcal{L}^{2n}}\right) ds\right].$$

Since $|u_m(0)|^2_{\mathcal{H}} = \left|\frac{\pi_m u_0}{|\pi_m u_0|_{\mathcal{H}}}\right|^2 = \frac{|\pi_m u_0|^2_{\mathcal{H}}}{|\pi_m u_0|^2_{\mathcal{H}}} = 1$, hence last equation reduces to

$$|u_m(t)|^2_{\mathcal{H}} - 1 = 0, \text{ for all } t \in [0,T].$$

Next, consider the same for the energy norm

$$\frac{d}{dt}\left(\frac{\|u_m(t)\|^2}{2}\right)=\frac{1}{2}\frac{d}{dt}\left\langle A^{\frac{1}{2}}u_m(t),A^{\frac{1}{2}}u_m(t)\right\rangle=\left\langle Au_m(t),\frac{du_m}{dt}(t)\right\rangle. \tag{3.3}$$

Using equation (3.1) it follows that

$$\frac{d}{dt}\left(\frac{\|u_m(t)\|^2}{2}\right) = \left\langle -\frac{du_m}{dt}(t)+F_m\left(u_m\right),\frac{du_m}{dt}(t)\right\rangle \tag{3.4}$$

$$\begin{aligned} &= -\left|\frac{du_m}{dt}(t)\right|_{\mathcal{H}}^2-\left\langle u_m(t)^{2n-1},\frac{du_m}{dt}(t)\right\rangle \\ &\quad+\left(\|u_m(t)\|^2+|u_m(t)|_{\mathcal{L}^{2n}}^{2n}\right)\left\langle u_m(t),\frac{du_m}{dt}(t)\right\rangle. \end{aligned} \tag{3.5}$$

Since $u_m(t)\in M$ and $\left\langle u_m(t),\frac{du_m}{dt}(t)\right\rangle=0$, for all $t\in[0,T]$. Therefore, equation (3.5) reduces to the following

$$\frac{d}{dt}\left(\Psi\left(u_m(t)\right)\right)=-\left|\frac{du_m}{dt}(t)\right|_{\mathcal{H}}^2. \tag{3.6}$$

Here $\Psi\left(u_m\right):=\frac{\|u_m(t)\|^2}{2}+\frac{|u_m(t)|_{\mathcal{L}^{2n}}^{2n}}{2n}$, is indeed a non-increasing function. Integrating both sides gives,

$$\begin{aligned} \Psi\left(u_m(t)\right)-\Psi\left(u_m(0)\right) &= -\int_0^t\left|\frac{du_m}{dt}(s)\right|_{\mathcal{H}}^2ds\le 0 \\ \sup_{t\in[0,T]}\|u_m(t)\|^2 &\le 2\Psi\left(u_m(0)\right)<\infty. \end{aligned} \tag{3.7}$$

Thus the sequence (u_m) remains in the bounded set of $\mathcal{L}^\infty(0,T;\mathcal{V})$.
Consider again

$$\begin{aligned} \frac{d}{dt}\left(\frac{\|u_m(t)\|^2}{2}\right) &= \left\langle A_mu_m(t),\frac{du_m}{dt}(t)\right\rangle \\ &= \left\langle A_mu_m(t),-Au_m(t)+F_m\left(u_m(t)\right)\right\rangle \\ &= -\left|A_mu_m(t)\right|_{\mathcal{H}}^2+\left(\|u_m(t)\|^2+|u_m(t)|_{\mathcal{L}^{2n}_{2n}}\right)\|u_m(t)\|^2 \\ &\quad-\left\langle A_mu_m(t),u_m(t)^{2n-1}\right\rangle. \end{aligned} \tag{3.8}$$

Using Cauchy-Schwartz inequality and Young's inequality, we infer that

$$\begin{aligned} \frac{d}{dt}\left(\frac{\|u_m(t)\|^2}{2}\right) &\le -\left|A_mu_m(t)\right|_{\mathcal{H}}^2+\left(\|u_m(t)\|^2+|u_m(t)|_{\mathcal{L}^{2n}}^{2n}\right)\|u_m(t)\|^2 \\ &\quad+\left|A_mu_m(t)\right|_{\mathcal{H}}\left|u_m(t)^{2n-1}\right|_{\mathcal{H}} \\ &\le -\left|A_mu_m(t)\right|_{\mathcal{H}}^2+\left(\|u_m(t)\|^2+|u_m(t)|_{\mathcal{L}^{2n}}^{2n}\right)\|u_m(t)\|^2+\frac{1}{2}\left|A_mu_m(t)\right|_{\mathcal{H}}^2 \\ &\quad+\left|u_m(t)^{2n-1}\right|_{\mathcal{H}}^2 \\ &= -\frac{1}{2}\left|A_mu_m(t)\right|_{\mathcal{H}}^2+\left(\|u_m(t)\|^2+|u_m(t)|_{\mathcal{L}^{2n}}^{2n}\right)\|u_m(t)\|^2+|u_m(t)|_{\mathcal{L}^{4n-2}}^{4n-2}. \end{aligned} \tag{3.9}$$

For the n as described in the Assumption 2.1, using inequality (2.6) it follows that there exist constants c_1 and c_2 such that,

$$\begin{aligned}\frac{d}{dt}\left(\frac{\|u_m(t)\|^2}{2}\right) &\le -\frac{1}{2}|A_m u_m(t)|^2_{\mathcal{H}} + \left(\|u_m(t)\|^2 + c_1^{2n}\|u_m(t)\|^{2n}\right)\|u_m(t)\|^2 \\ &\quad + c_2^{4n-2}\|u_m(t)\|^{4n-2},\end{aligned}$$

for all $t \in [0,T]$. By using the estimate (3.7), we infer that there exists a constant $C > 0$ such that

$$\frac{d}{dt}\left(\frac{\|u_m(t)\|^2}{2}\right) \le -\frac{1}{2}|A_m u_m(t)|^2_{\mathcal{H}} + C, \tag{3.10}$$

where $C := \left(C^2 + c_1^{2n}C^{2n}\right)C^2 + c_2^{4n-2}C^{4n-2}$. Integrating both sides between 0 to T, we get

$$\frac{1}{2}\int_0^T |A_m u_m(t)|^2_{\mathcal{H}}\,dt \le \frac{\|u_m(T)\|^2}{2} + \frac{1}{2}\int_0^T |A_m u_m(t)|^2_{\mathcal{H}}\,dt \le \frac{\|u_m(0)\|^2}{2} + CT. \tag{3.11}$$

Since $u_m(0) \in \mathcal{V}$ hence the sequence (u_m) remains in bounded set of $\mathcal{L}^2(0,T;\mathcal{E})$. Thus, we infer from estimates (3.7) and (3.11) that the sequence (u_m) remains in the bounded subset of X_T.

Finally, we want to show that the sequence $(u_m'(t))$ is uniformly bounded in $\mathcal{L}^2(0,T;\mathcal{H})$. Let us begin by observing following,

$$\begin{aligned}\int_0^T \left|\frac{du_m}{dt}\right|^2_{\mathcal{H}} dt &= \int_0^T |A_m u_m(t)|^2_{\mathcal{H}}\,dt + \int_0^T |F_m(u_m(t))|^2_{\mathcal{H}}\,dt \\ &\quad + 2\int_0^T \langle -A_m u_m(t), F_m(u_m(t))\rangle\,dt.\end{aligned} \tag{3.12}$$

Using Cauchy-Schwartz inequality followed by Young's inequality and inequality (2.12) we get

$$\begin{aligned}\int_0^T \left|\frac{du_m}{dt}(s)\right|^2_{\mathcal{H}} ds &\le \int_0^T |A_m u_m(t)|^2_{\mathcal{H}}\,dt + \int_0^T |F_m(u_m(t))|^2_{\mathcal{H}}\,dt \\ &\quad + 2\int_0^T |A_m u_m(t)|_{\mathcal{H}}|F_m(u_m(t))|_{\mathcal{H}}\,dt \\ &= 2\int_0^T |A_m u_m(t)|^2_{\mathcal{H}}\,ds + 3\int_0^T |F_m(u_m(t))|^2_{\mathcal{H}}\,dt.\end{aligned} \tag{3.13}$$

Thus using estimates (3.11) and (3.7) the above inequality simplifies to following

$$\int_0^T \left|\frac{du_m}{dt}(s)\right|^2_{\mathcal{H}} ds \le 2K + 3\int_0^T C^{2n}dt = 2K + 3C^{2n}T < \infty. \tag{3.14}$$

Thus (u_m') is uniformly bounded in $\mathcal{L}^2(0,T;\mathcal{H})$. Using compactness theorem III. 2.1 (Aubin-Lion Lemma) of [21] in the light of estimates (3.7), (3.11) and (3.14), we infer that

$$u_m \to u \quad \text{strongly in} \quad \mathcal{L}^2(0,T;\mathcal{V}). \tag{3.15}$$

4. Existence and Uniqueness

This section is dedicated to proving the main result of this paper, i.e. Theorem 2.2. In particular, we will show that the solution of the problem (2.9) exists, and it is unique. We also show that trajectories of the solution are continuous in $\mathcal{V}$-norm on $[0,T]$.

Proof. From the apriori estimates (3.7), (3.11) and (3.15) we know that there exists an element u such that $m \to \infty$

$$\begin{aligned} u_m &\to u \quad \text{in} \quad \mathcal{L}^\infty(0,T;\mathcal{V}) \ \text{weak* sense} \\ u_m &\to u \quad \text{in} \quad \mathcal{L}^2(0,T;\mathcal{E}) \ \text{weakly} \\ u_m &\to u \quad \text{in} \quad \mathcal{L}^2(0,T;\mathcal{V}) \ \text{strongly.} \end{aligned}$$

Let us begin by proving that this limiting function u is indeed the solution to the main Problem (2.9).

Let us fix C^1 -class function $\psi : [0,T] \to \mathbb{R}$ such that $\psi(T) = 0$, and let $\phi \in \mathcal{H}_m$, for some $m \in \mathbb{N}$. Multiply with equation (3.1) with $\psi(\cdot)\phi$ and integrating with respect to the space variable, we get

$$\langle u_m'(t), \psi(t)\phi\rangle = \langle -A_m u_m(t), \psi(t)\phi\rangle + \langle F_m\left(u_m(t)\right), \psi(t)\phi\rangle .$$

Integrating both sides with respect to time on interval 0 to T

$$\int_0^T \langle u_m'(t), \psi(t)\phi\rangle\, dt = \int_0^T \langle -A_m u_m(t), \psi(t)\phi\rangle\, dt + \int_0^T \langle F_m\left(u_m(t)\right), \psi(t)\phi\rangle\, dt.$$

Integrating by parts on left-hand sides, we infer that

$$\begin{aligned} \int_0^T \langle u_m(t), \psi'(t)\phi\rangle\, dt &= \int_0^T \langle A_m u_m(t), \psi(t)\phi\rangle\, dt - \int_0^T \langle F_m\left(u_m(t)\right), \psi(t)\phi\rangle\, dt \\ &\quad - \langle u_0, \psi(0)\phi\rangle . \end{aligned} \tag{4.1}$$

Our aim is to pass limit $m \to \infty$ in above equation.

We will study each term in equation (4.1) on both sides. Recall the well-known fact that, cf. [1], for a bounded domain $\mathcal{O}$ in $\mathbb{R}^d$ then Dirichlet Laplacian operator $A := -\Delta : \mathcal{H}_0^{1,2}(\mathcal{O}) \cap \mathcal{H}^{2,2}(\mathcal{O}) \to \mathcal{H}$ then $A^{-1} := (-\Delta)^{-1} : L^2(\mathcal{O}) \to \mathcal{H}^{2,2}(\mathcal{O})$ is continuous and Hilbert-Schmidt. Moreover, the inner product on $\mathcal{L}^2(0,T;\mathcal{D}(A))$ can be given as

$$\begin{aligned} \langle u, v\rangle_{\mathcal{L}^2(0,T;\mathcal{D}(A))} &= \int_0^T \langle Au(t), Av(t)\rangle_{\mathcal{H}}\, dt \qquad \text{or} \\ \left\langle A^{-1}u, A^{-1}v\right\rangle_{\mathcal{L}^2(0,T;\mathcal{D}(A))} &= \int_0^T \langle u(t), v(t)\rangle_{\mathcal{H}}\, dt. \end{aligned} \tag{4.2}$$

Let us begin with the first term on the right-hand side. For $m \in \mathbb{N}$, we have $\pi_m(\phi) = \phi$, for all $\phi \in \mathcal{H}_m$.

$$\begin{aligned} \int_0^T \langle A_m u_m(t), \psi(t)\phi\rangle\, dt &= \int_0^T \langle \pi_m A u_m(t), \psi(t)\phi\rangle\, dt = \int_0^T \langle A u_m(t), \psi(t)\pi_m\phi\rangle\, dt \\ &= \int_0^T \langle A u_m(t), \psi(t)\phi\rangle\, dt = \int_0^T \left\langle u_m(t), A^{-1}\psi(t)\phi\right\rangle_{\mathcal{D}(A)} dt. \end{aligned} \tag{4.3}$$

Using $A^{-1}\psi(\cdot)\phi \in \mathcal{L}^2(0,T;\mathcal{D}(A))$, equation (4.3) and the fact that $u_m \to u$ in $\mathcal{L}^2(0,T;\mathcal{D}(A))$ weakly, we infer that

$$\begin{aligned} \int_0^T \langle A_m u_m(t), \psi(t)\phi\rangle\, dt - \int_0^T \left\langle u(t), A^{-1}\psi(t)\phi\right\rangle_{\mathcal{D}(A)} dt &= \int_0^T \left\langle u_m(t) - u(t), A^{-1}\psi(t)\phi\right\rangle_{\mathcal{D}(A)} \\ &\to 0 \ \text{as} \ m \to \infty. \end{aligned} \tag{4.4}$$

Let us move towards the second term on the right-hand side of the equation (4.1). Consider

$$\begin{aligned}\int_0^T \langle F_m(u_m(t)), \psi(t)\phi\rangle\, dt &= \int_0^T \langle \pi_m F(u_m(t)), \psi(t)\phi\rangle\, dt = \int_0^T \langle F(u_m(t)), \psi(t)\pi_m\phi\rangle\, dt \\ &= \int_0^T \langle F(u_m(t)), \psi(t)\phi\rangle\, dt. \end{aligned} \tag{4.5}$$

Therefore

$$\begin{aligned}&\int_0^T \langle F_m\left(u_m(t)\right), \psi(t)\phi\rangle\, dt - \int_0^T \langle F\left(u(t)\right), \psi(t)\phi\rangle\, dt \\ &= \int_0^T \langle F\left(u_m(t) - F(u(t)\right), \psi(t)\phi\rangle\, dt \to 0.\end{aligned} \tag{4.6}$$

Using the Cauchy-Schwartz inequality it follows that

$$\left|\int_0^T \langle F\left(u_m(t) - F(u(t)\right), \psi(t)\phi\rangle\, dt\right| \leq \left(\int_0^T |F\left(u_m(t) - F(u(t)\right)|_{\mathcal{H}}^2\, dt\right)^{\frac{1}{2}} \left(\int_0^T |\psi(t)\phi|_{\mathcal{H}}^2\, dt\right)^{\frac{1}{2}}. \tag{4.7}$$

To show that the right-hand side in the inequality above goes to zero as m goes to infinity, it is enough to show that $\int_0^T |F\left(u_m(t) - F(u(t)\right)|_{\mathcal{H}}^2\, dt$ goes to zero as m goes to infinity. In the following, we write $u_m(t)$ as u_m and $u(t)$ as u. Recall following elementary Cauchy-Schwartz inequality

$$\left(\sum_{i=1}^m x_i\right)^2 = \left(\sum_{i=1}^m 1\cdot x_i\right)^2 \leq m\cdot\sum_{i=1}^m x_i^2. \tag{4.8}$$

Using the estimate (2.4) and inequality (4.8). Consider the following sequence of inequalities

$$\begin{aligned}&\int_0^T |F\left(u_m(t) - F(u(t)\right)|_{\mathcal{H}}^2\, dt \\ &\leq c\int_0^T \left[\begin{array}{c}\left(\|u_m\|^2 + \|u\|^2\right) + \left(\|u_m\| + \|u\|\right)^2 \\ + \left(\|u_m\|^{2n-1} + \|u\|^{2n-1}\right)\left(\|u_m\| + \|u\|\right) \\ + \left(\|u_m\|^{2n} + \|u\|^{2n}\right) + \left(1 + \|u_m\|^2 + \|u\|^2\right)^{1/3}\end{array}\right]^2 \|u_m - u\|^2\, dt \\ &\leq 5c\int_0^T \left[\begin{array}{c}\left(\|u_m\|^2 + \|u\|^2\right)^2 + \left(\|u_m\| + \|u\|\right)^4 \\ + \left(\|u_m\|^{2n-1} + \|u\|^{2n-1}\right)^2\left(\|u_m\| + \|u\|\right)^2 \\ + \left(\|u_m\|^{2n} + \|u\|^{2n}\right)^2 + \left(1 + \|u_m\|^2 + \|u\|^2\right)^{2/3}\end{array}\right] \|u_m - u\|^2\, dt \\ &\leq 5c\int_0^T \left[\begin{array}{c}\left(K^2 + L^2\right)^2 + (K+L)^4 \\ + \left(K^{2n-1} + L^{2n-1}\right)^2 (K+L)^2 \\ + \left(K^{2n} + L^{2n}\right)^2 + \left(1 + K^2 + L^2\right)^{2/3}\end{array}\right] \|u_m - u\|^2\, dt \\ &\leq C\,\|u_m - u\|_{\mathcal{L}^2([0,T,\mathcal{V})}^2 \to 0 \text{ as } m \to \infty,\end{aligned} \tag{4.9}$$

where $K := \sup_{t\in[0,T]} \|u_m(t)\|^2$, $L := \sup_{t\in[0,T]} \|u(t)\|^2$ are both finite because $u_m, u \in L^\infty(0,T;\mathcal{V})$. Also

$$C \;:=\; 5c\left[\begin{array}{c}\left(K^2+L^2\right)^2+(K+L)^4+\left(K^{2n-1}+L^{2n-1}\right)^2(K+L)^2\\ +\left(K^{2n}+L^{2n}\right)^2+\left(1+K^2+L^2\right)^{2/3}\end{array}\right]<\infty.$$

Finally, consider the left-hand side of equation (4.1)

$$\begin{aligned}\int_0^T \langle u_m(t)-u(t),\psi'(t)\phi\rangle\,dt &\leq \int_0^T \|u_m(t)-u(t)\|\,|\psi'(t)\phi|_{\mathcal{V}'}\,dt\\ &\leq \left[\int_0^T \|u_m(t)-u(t)\|^2\,dt\right]^{\frac12}\left[\int_0^T |\psi'(t)\phi|^2_{\mathcal{V}'}\,dt\right]^{\frac12}\\ &\to 0 \text{ as } m\to\infty. \end{aligned}\tag{4.10}$$

Let us pass the limit to the last equation (4.1)

$$\begin{aligned}-\int_0^T \langle u(t),\psi'(t)\phi\rangle\,dt &= -\int_0^T \langle Au(t),\psi(t)\phi\rangle dt+\int_0^T\langle F(u(t)),\psi(t)\phi\rangle dt\\ &\quad+\langle u_0,\psi(0)\phi\rangle,\end{aligned}\tag{4.11}$$

for all $\phi\in\bigcup_{m=1}\mathcal{H}_m$. Since $\bigcup_{m=1}\mathcal{H}_m$ it is dense in $\mathcal{V}$, by a standard continuity argument, equation (4.11) holds for any $\phi\in\mathcal{V}$ and $\psi\in C0^1([0,T])$. Thus, u satisfies the evolution equation of problem (2.9) in $\mathcal{L}^2([0,T];\mathcal{H})$.

Next, let us show that the initial condition for the problem (2.9) is satisfied.

Consider an arbitrary $\phi\in\mathcal{V}$ and $\psi\in C_0^1([0,T])$ such that $\psi(0)=1$. Multiplying equation (2.1) by $\psi(t)\phi$ and using integration by parts, we get

$$\begin{aligned}-\int_0^T \langle u(t),\psi'(t)\phi\rangle\,dt &= -\int_0^T \langle Au(t),\psi(t)\phi\rangle dt+\int_0^T\langle F(u(t)),\psi(t)\phi\rangle dt\\ &\quad+\langle u(0),\psi(0)\phi\rangle.\end{aligned}\tag{4.12}$$

Comparing equation (4.11) and (4.12) we get

$$\langle u(0)-u_0,\psi(0)\phi\rangle=0.$$

Since $\psi(0)=1$ so it follows that,

$$\langle u(0)-u_0,\phi\rangle=0,\ \text{ for all }\ \phi\in\mathcal{V}.$$

Hence, the density of $\mathcal{V}$ in $\mathcal{H}$ implies that $u(0)-u_0=0$, i.e. $u(0)=u_0$. Thus, u satisfies the initial value problem (2.9).

It remains now to show that the solution $u\in\mathcal{X}_T=\mathcal{L}^2(0,T;\mathcal{E})\cap C([0,T];\mathcal{V})$ for all $T\geq 0$, and that u is unique. Given the a priori estimates, to show that $u\in\mathcal{X}_T$, it is sufficient to demonstrate that $u\in C([0,T];\mathcal{V})$.

From Equation (3.15), we know that $u\in\mathcal{L}^2(0,T;\mathcal{V})$. Thus, in order to satisfy the assumptions of Lemma 2.5, we only need to show that $\frac{\partial u}{\partial t}\in\mathcal{L}^2(0,T;\mathcal{H})$.

Since u satisfies problem (2.9), i.e.

$$\frac{\partial u}{\partial t}=\Delta u+F(u)\tag{4.13}$$

Taking $\mathcal{H}$-inner product with $\frac{\partial u}{\partial t}$ on both sides, integrating from 0 to T and using the Minkowski's inequality for $p=2$, on the right-hand side it follows that

$$\begin{aligned}\int_0^T \left|\frac{\partial u}{\partial t}\right|_{\mathcal{H}}^2 dt &\leq \left(\int_0^T |Au(t)|_{\mathcal{H}}^2\, dt\right)^{\frac{1}{2}} + \left(\int_0^T |F(u(t))|_{\mathcal{H}}^2\, dt\right)^{\frac{1}{2}} \\ &\leq |u|_{\mathcal{L}^2(0,T;\mathcal{E})} + \left(\int_0^T |F(u(t))|_{\mathcal{H}}^2\, dt\right)^{\frac{1}{2}}. \end{aligned} \tag{4.14}$$

From Equation (3.11), we know that $u \in \mathcal{L}^2(0,T;\mathcal{E})$, therefore, the first term in Equation (4.14) is finite.

For the second term on the right-hand side of Equation (4.14), arguing along the same line of reasoning used to derive inequality (4.9), it follows that,

$$\begin{aligned}\int_0^T |F(u(t)|_{\mathcal{H}}^2\, dt &\leq 5c\int_0^T \left(2\|u(t)\|^4 + 2\|u(t)\|^{4n} + \left(1+\|u(t)\|^2\right)^{2/3}\right)\|u(t)\|^2\, dt \\ &\leq 5c\int_0^T \left(2K^4 + 2K^{4n} + \left(1+K^2\right)^{2/3}\right)K^2 dt \leq CT < \infty, \end{aligned} \tag{4.15}$$

where, $K := \sup_{t\in[0,T]} \|u(t)\|^2$ and $C := 5c\left(2K^4 + 2K^{4n} + \left(1+K^2\right)^{2/3}\right)K^2$, are finite because $u \in \mathcal{L}^\infty(0,T;\mathcal{V})$. From (4.14), it follows that $\frac{\partial u}{\partial t} \in \mathcal{L}^2(0,T;\mathcal{H})$. Invoking Lemma 2.5 gives $u \in C([0,T];\mathcal{V})$. Thus $u \in X_T$.

Uniqueness of Solution: Assume that u and v be two solutions of problem (2.9). Set $z = u - v \in X_T = \mathcal{L}^2(0,T;\mathcal{E}) \cap C([0,T];\mathcal{V})$. As $u', v' \in \mathcal{L}^2(0,T;\mathcal{H})$ so $z' = u' - v' \in \mathcal{L}^2(0,T;\mathcal{H})$. Moreover, z solves

$$\begin{aligned}\frac{dz}{dt} + Az &= F(u) - F(v), \\ z(0) &= 0.\end{aligned}$$

Clearly, z is regular enough to be identified as v in Lemma 2.5 and so the following holds in $\mathcal{V}$-norm

$$\begin{aligned}\frac{1}{2}\|z(t)\|^2 - \frac{1}{2}\|z(0)\|^2 &= \int_0^t \left\langle \nabla z(s), \nabla\frac{dz}{dt}(s)\right\rangle ds = \int_0^t \left\langle -\Delta z(s), \frac{dz}{dt}(s)\right\rangle \\ &= -\int_0^t \left\langle \frac{dz}{dt}(s), \frac{dz}{dt}(s)\right\rangle ds + \int_0^t \left\langle \frac{dz}{dt}(s) - \Delta z(s), \frac{dz}{dt}(s)\right\rangle ds\end{aligned}$$

As $z(0)=0$, and using Cauchy-Schwartz inequality, elementary inequality $ab \le \left(\frac{a^2+b^2}{2}\right)$ and then Lemma 2.4, in the last equality, it follows that for all $t \in [0,T]$

$$\begin{aligned} \frac{1}{2}\|z(t)\|^2 &\le -\int_0^t \left|\frac{dz}{dt}(s)\right|_{\mathcal{H}}^2 ds + \frac{1}{2}\int_0^t \left|\frac{dz}{dt}(s)\right|_{\mathcal{H}}^2 ds \\ &\quad + \frac{1}{2}\int_0^t \left|\frac{dz}{dt}(s) - \Delta z(s)\right|_{\mathcal{H}}^2 ds \\ \|z(t)\|^2 + \int_0^t \left|\frac{dz}{dt}(s)\right|_{\mathcal{H}}^2 ds &\le \int_0^t \left|\frac{dz}{dt}(s) - \Delta z(s)\right|_{\mathcal{H}}^2 ds \\ &= \int_0^t |F(u(s)) - F(v(s))|_{\mathcal{H}}^2\, ds, \end{aligned}$$

$$\|z(t)\|^2 \le \int_0^t \psi(s)\,\|z(s)\|^2\, ds \tag{4.16}$$

where,

$$\psi(s) := C \begin{bmatrix} (\|u(s)\| + \|v(s)\|)^2 + \left(\frac{2n-1}{2}\right)\left(\|u(s)\|^{2n-1} + \|v(s)\|^{2n-1}\right)(\|u(s)\| + \|v(s)\|) \\ \left(\|u(s)\|^2 + \|v(s)\|^2\right) + \left(\|u(s)\|^{2n} + \|v(s)\|^{2n}\right) + \left(\|u(s)\|^{2n-3} + \|v(s)\|^{2n-3}\right) \end{bmatrix}.$$

Using Gronwall inequality it follows that

$$\begin{aligned} \|z(t)\|^2 &\le \int_0^t \psi(s)\,\|z(t)\|^2\, ds \\ \|z(t)\|^2 &\le 0 \cdot \exp\psi(t) = 0. \end{aligned}$$

Hence $z(t)=0$, for all $t \in [0,T]$ and the uniqueness follows.

Invariance of Manifold: Let us take $u_0 \in \mathcal{M}$. For $t \in [0,T)$, using the Lemma 2.5, consider the following chain of equations

$$\begin{aligned} \frac{1}{2}\left(|u(t)|_{\mathcal{H}}^2 - 1\right) &= \frac{1}{2}\left(|u_0|_{\mathcal{H}}^2 - 1\right) + \int_0^t \langle u'(s), u(s)\rangle_{\mathcal{H}}\, ds = \int_0^t \langle u'(s), u(s)\rangle_{\mathcal{H}}\, ds \\ &= \int_0^t \left\langle -Au(s) + \|u(s)\|^2 u(s) - u(s)^{2n-1} + u(s)\,|u(s)|_{\mathcal{L}^{2n}}^{2n}, u(s)\right\rangle_{\mathcal{H}} ds \\ &= \int_0^t \|u(s)\|^2\left(|u(s)|_{\mathcal{H}}^2 - 1\right) ds + \int_0^t |u(s)|_{\mathcal{L}^{2n}}^{2n}\left(|u(s)|_{\mathcal{H}}^2 - 1\right) ds \\ \phi(t) &= 2\int_0^t \left(\|u(s)\|^2 + |u(s)|_{\mathcal{L}^{2n}}^{2n}\right)\phi(s)\, ds, \end{aligned}$$

where $\phi(t) := |u(t)|_{\mathcal{H}}^2 - 1$, $t \in [0,T)$. Using the fact that $u \in \mathcal{L}^2([0,T];\mathcal{V})$ and the continuity of the embedding of $\mathcal{V} \hookrightarrow \mathcal{L}^{2n}(\mathcal{O})$, it implies that $\int_0^T \left(|\nabla u(t)|_{\mathcal{H}}^2 + |u(t)|_{\mathcal{L}^{2n}}^{2n}\right) dt < \infty$. Therefore, from Gronwall's Lemma, it follows that

$$\phi(t) = \phi(0)e^{2\left(|\nabla u(t)|_{\mathcal{H}}^2 + |u(t)|_{\mathcal{L}^{2n}}^{2n}\right)} = 0, \qquad t \in [0,T).$$

Thus $|u(t)|_{\mathcal{H}}^2 = 1$, for all $t \in [0,\tau)$. This completes the proof. □

5. Conclusions

The focus of this work was to study the well-posedness deterministic problem consisting of a non-linear heat equation of gradient type. Using the Faedo-Galerkin compactness method we showed that there exists a unique solution to the proposed constrained problem, and also demonstrated that for all time t, the solution stays on the manifold.

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