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On evolution of finite energy solutions for a Cosserat thermoelastic body

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ABSTRACT. In our study we approached the mixed problem in the context of linear theory of thermoelasticity for Cosserat bodies. After we define the finite energy solution for this mixed problem, some results with respect to the existence and uniqueness are proven of this kind of solution. To this aim, we generalized some analogous results established by Dafermos, for a mixed problem from linear theory of the classical elasticity. In another important result of our study, we introduce some specific conditions which give the possibility that the evolution of the finite energy solution to be controlled.

1. INTRODUCTION

In our present work we consider a structure of the materials of Cosserat type. This structure introduced by E. and F. Cosserat [6] is a branch of the general microstructure theory. This covers a great number of generalizations of the theory of classical elasticity, like microstretch theory, micromorphic theory, dipolar theory, micropolar structure, etc.
It is considered that the theories dedicated to the microstructure were initiated by Eringen (see [9], [10]).
After that, the concern of specialists regarding these theories increased a lot, fact proven by the large number of results published in an extremely large number of works in this context. Of these we list only a few, like [1], [2], [4], [5], [7], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [26], [27], [28]. Some important applications of this theory are highlighted in geological structures, such as soils and rocks, and also in the manufacture of porous products, such as pressed powders and ceramics. Our results included in the present study are important generalizations of those established by Fichera in [11] and Dafermos [8]. These refer to the existence and uniqueness of solutions for the mixed problem with initial-boundary values, in the context of linear theory of elasticity. Our context, that of Cosserat media, is much more complex both in terms of the differential equation system that governs the evolution of these media as well as the initial and boundary data. Moreover, in the last part of our work, we address the issue of asymptotic stability in the case of a solution having finite energy. It is necessary to specify that in approaching his results, Dafermos was inspired by the works of Visik, see the work [25].
The structure of this work is as follows. After we put down the important differential equations, initial data and, also, the boundary conditions, which characterizes a mixed problem in the theory of the thermoelastic Cosserat media, we define the notion of a solution with finite energy and we explain a way to extend the set of solutions with finite energy. Our first result with respect to the existence of a finite energy solution is deduced by considering that the initial data are null. After that, the results regarding the existence of this kind of solution and its uniqueness are deduced in the general situation of initial

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data and boundary conditions in their non-homogeneous form.
Our last result provide some auxiliary estimates which are used for the controllability of the behavior of the solution having finite energy.

2. Problem formulation

Consider that our Cosserat thermoelastic material occupies at the initial moment $t = 0$ a enough regular region D in the Euclidian space R^3. The border of D is denoted by ∂D and suppose this is a regular surface, which allows the application of the theorem of divergence. By $\bar{D}$ we denoted the closure of the domain D and we have $\bar{D} = D \cup \partial D$. We adopted the usual tensor and vector notation. The time derivative of a function is designated by a superposed dot on respective function. To designate a partial derivative of a function regarding its spatial variables we use a subscript preceded by a comma. The Einstein summation is used in the case of a monomial in which an index is repeated. It is possible that the time argument and/or the spatial variable of a function to be omitted, if there is no risk of confusion.
To characterize the evolution of a Cosserat thermoelastic body there are used the following variables:
- $v_m(x,t)$ - the vector of displacement;
- $\phi_m(x,t)$ - the microrotation vector;
- θ - the variation of the temperature, as difference between the current value and the initial one T_0, that is,

$$\theta = T - T_0.$$

With the help of previous independent variables $v_m(x,t),\ \phi_i(x,t)$, we define the components of the strain tensors, namely e_{mn} and ε_{ij}, as follows:

$$e_{mn} = u_{n,m} + e_{mnk}\phi_k,\ \varepsilon_{mn} = \phi_{n,m}. \tag{2.1}$$

It is considered that that the body has no initial and couple stress and, also, the flux rate is null. Our following considerations are made in the context of a linear theory, as such, we can consider that the internal energy is a form of quadratic type, having the expression:

$$\begin{aligned} \varrho_0 e = \frac{1}{2}A_{klmn}e_{kl}e_{mn} + B_{klmn}e_{kl}\varepsilon_{mn} + \frac{1}{2}C_{klmn}\varepsilon_{kl}\varepsilon_{mn} - \\ -\alpha_{mn}e_{mn}\theta - \beta_{mn}\varepsilon_{mn}\theta - \frac{1}{2}c\theta^2 + \frac{1}{2}K_{mn}\theta_{,m}\theta_{,n}. \end{aligned} \tag{2.2}$$

We can be inspired by the technique used by Eringen in the work [3]. So, considering that

$$\tau_{mn} = \frac{\partial e}{\partial e_{mn}},\ \sigma_{mn} = \frac{\partial e}{\partial \varepsilon_{mn}},\ \eta = -\frac{\partial e}{\partial \theta},\ q_m = \frac{\partial e}{\partial \theta_{,m}},$$

we obtain the correspondence of tensors of stress with strain tensors, namely, the constitutive relations that follow:

$$\begin{aligned} \tau_{mn} &= A_{klmn}e_{kl} + B_{klmn}\varepsilon_{kl} - \alpha_{mn}\theta, \\ \sigma_{mn} &= B_{klmn}e_{kl} + C_{klmn}\varepsilon_{kl} - \beta_{mn}\theta, \\ \varrho\eta &= \alpha_{mn}e_{mn} + \beta_{mn}\varepsilon_{mn} + c\theta, \\ q_m &= K_{mn}\theta_{,n}. \end{aligned} \tag{2.3}$$

Furthermore, using the same procedure we deduce the basic equations (see [10]):
- the motion equations:

$$\begin{aligned} \tau_{mn,n} + \varrho f_m &= \varrho \ddot{v}_m, \\ \sigma_{mn,n} + e_{knm}\tau_{kn} + \varrho g_m &= I_{mn}\ddot{\phi}_n; \end{aligned} \tag{2.4}$$

- the equation of evolution of energy:

$$\varrho T_0 \dot{\eta} = q_{m,m} + \varrho r. \tag{2.5}$$

In the above relations we have used the notations that follow:

ϱ - the mass density;
η - the specific entropy;
T_0 - the temperature in the undeformed state;
e_{mn}, ε_{mn} - the tensors of strain;
τ_{mn}, σ_{mn} - the tensors the stress;
q_m - heat flux vector;
f_m, g_m - body forces;
r - supply of heat;

The coefficients $A_{klmn}, B_{klmn}, C_{klmn}, \alpha_{mn}, \beta_{mn}, c, K_{mn}$ are for the characterization of the elastic properties of the material. For these we have the next symmetry relations:

$$\begin{aligned} &A_{klmn} = A_{mnkl},\ \alpha_{mn} = \alpha_{nm}, \\ &C_{klmn} = C_{mnkl},\ K_{mn} = K_{nm}. \end{aligned} \tag{2.6}$$

We must specify that in our subsequent considerations, we will take into account a body in the most general state, namely it is inhomogeneous and anisotropic.
Considering the inequality of Clausius-Duhem, with other words, the inequality of the production of entropy, we come to the conclusion that the tensor of the thermal conductivity K_{ij} is semi-definite positive, namely:

$$K_{mn}\,\theta_{,\,m}\,\theta_{,\,n} \geq 0. \tag{2.7}$$

It is not difficult to remark that the above equations (2.4) and (2.5) are analogous of those used in the thermoelasticity theory.
If we substitute the constitutive equations (2.3) in the main equations (2.4) and (2.5), we are led to the following system of equations:

$$\begin{aligned} \ddot{v}_m &= \frac{1}{\varrho}\left[(A_{klmn}e_{kl})_{,\,n} + (B_{klmn}\varepsilon_{kl})_{,\,n} - (\alpha_{mn}\theta)_{,\,n}\right] + f_m, \\ \ddot{\phi}_m &= \frac{1}{I_{mn}}\Big[(B_{klmn}e_{kl})_{,\,j} + (C_{klmn}\varepsilon_{kl})_{,\,n} - (\beta_{mn}\theta)_{,\,n} + \\ &\qquad + \epsilon_{mnj}\left(A_{njkl}e_{kl} + B_{njkl}\varepsilon_{kl} - \alpha_{nj}\theta\right) + g_m\Big], \\ \alpha_{mn}\dot{e}_{mn} &+ \beta_{mn}\dot{\varepsilon}_{mn} + c\dot{\theta} = \frac{\varrho}{T_0}\left(K_{mn}\theta_{,\,n}\right)_{,\,m} + \frac{1}{T_0}\,r. \end{aligned} \tag{2.8}$$

The mixed problem in the context of the theory of the Cosserat thermoelastic materials will be complete if we indicate certain initial conditions and certain boundary relations.
At the moment $t = 0$ we have the given initial values:

$$\begin{aligned} &v_m(0) = v_m^0,\ \dot{v}_m(0) = v_m^1,\ \text{in}\ \bar{D}, \\ &\phi_m(0) = \phi_m^0,\ \dot{\phi}_m(0) = \phi_m^1,\ \text{in}\ \bar{D}, \\ &\theta(0) = \theta^0,\ \ \text{in}\ \bar{D}. \end{aligned} \tag{2.9}$$

Further, at the moment $t = 0$ we prescribe the following boundary values:

$$\begin{aligned} &v_m = \bar{v}_m\ on\ \partial D_v,\ t_m = \bar{t}_m\ on\ \partial D_v^c, \\ &\phi_m = \bar{\phi}_m\ on\ \partial D_\phi,\ m_l = \bar{m}_l\ on\ \partial D_\phi^c, \\ &\theta = \bar{\theta}\ on\ \partial D_\theta,\ q = \bar{q}\ on\ \partial D_\theta^c. \end{aligned} \tag{2.10}$$

Here $t_k \equiv \tau_{kl}n_l$, $m_k \equiv \sigma_{kl}n_l$, $q \equiv q_l n_l$ and $n = (n_l)$ is the normal vector to the surface ∂D, outward oriented. Also, we must specify that the time t_0 can take the value ∞. The sets $\partial D_v, \partial D_\phi$, ∂D_θ and respectively $\partial D_v^c, \partial D_\phi^c$, ∂D_θ^c are subsets of the border ∂D, so that:

$$\partial D_v \cup \partial D_v^c = \partial D_\phi \cup \partial D_\phi^c = \partial D_\theta \cup \partial D_\theta^c = \partial D,$$

$$\partial D_v \cap \partial D_v^c = \partial D_\phi \cap \partial D_\phi^c = \partial D_\theta \cap \partial D_\theta^c = \emptyset.$$

Also, $v_m^0,\ v_m^1,\ \phi_m^0,\ \phi_m^1,\ \theta^0,\ \bar{v}_m, \bar{t}_m,\ \bar{\phi}_m,\ \bar{m}_k,\ \bar{\theta}$ and $\bar{q}$ are prescribed and enough smooth functions.

We will denote by $\mathcal{P}$ the mixed problem with initial and boundary values, in the theory of thermoelastic Cosserat material, which consists of (8)-(10).

The ordered array (v_m, ϕ_m, θ) is a solution of the problem $\mathcal{P}$ if, for any $(x, t) \in Q_0 = D \times [0, t_0)$, it satisfies the equations (2.8) and meet the initial values (2.9) and prescribed boundary conditions (2.10).

3. Basic results

Let us suppose that our thermoelastic Cosserat body occupies a bounded domain D from the Euclidean space R^3.

We denote by $C^n\left(\bar{D}\right)$ the set of the scalar functions which are defined on D and admit all derivative up to order n, in any point of the domain D and the derivative of n order is continuous in all points of $\bar{D}$.

The norm of the space $C^n\left(\bar{D}\right)$ can be defined as:

$$\|h\|_{C^n\left(\bar{D}\right)} = \sum_{p=0}^{n} \sum_{i_1, i_2, \ldots,\ i_p} \max\ \left| h_{,\ i_1 i_2 \ldots i_p} \right|,\ for\ any\ h \in C^n\left(\bar{D}\right).$$

The set of vectorial functions composed of seven elements, each element being a function from the space $C^n\left(\bar{D}\right)$, is denoted by $\mathbf{C}^n\left(\bar{D}\right)$. The set $\mathbf{C}^n\left(\bar{D}\right)$ will be equipped by the known norm:

$$\|\mathbf{u}\|_{\mathbf{C}^n\left(\bar{D}\right)} = \sum_{k=1}^{7} \|u_k\|_{C^n\left(\bar{D}\right)}.$$

We can complete the space $C^n\left(\bar{D}\right)$ considering the norm $\|.\|_{W_n(D)}$ until to the space $W_n\left(D\right)$, and it is clearly that the space $W_n\left(D\right)$ is a space of Hilbert type, because that the next scalar product:

$$(f, g)_{W_n(D)} = \sum_{p=0}^{n} \int_D f_{,i_1 i_2 \ldots i_k}\ g_{,i_1 i_2 \ldots k_k} dV,$$

generates the norm $\|.\|_{W_n(D)}$.

By using the same procedure, we can complete the space $\mathbf{C}^n\left(\bar{D}\right)$ by using the norm $\|\cdot\|_{\mathbf{W}_n(D)}$ until the space $\mathbf{W}_n\left(D\right)$, and, also, the space $\mathbf{W}_n\left(D\right)$ is a Hilbert, considering that the norm $\|.\|_{\mathbf{W}_n(D)}$ is generated by the following scalar product:

$$(\mathbf{u}, \mathbf{v})_{\mathbf{W}_n(D)} = \sum_{m=1}^{7} (u_m, v_m)_{W_n(D)}.$$

We base our last statement on the fact that for any product of some normed spaces, its norm is equal with the sum of the norms of all factor spaces.

Let B be a Banach space. We denote by $C^n\left([0, t_0)\,; D\right)$ the set of the functions $f : [0, t_0) \to B$, which have the derivatives up to $n-$th order with respect to t variable, the derivative of order n being a continuous function defined on $[0, t_0)$. We can formulate similar definitions for the spaces $L_1((0,t_0);D)$ and $L_2((0,t_0);D)$, given the clear meaning of notations for the spaces L_1 and L_2.

Definition 3.1. The next notations are useful in the following considerations:

$$\hat{C}^1(D) = \left\{\theta \in C^1(\bar{D}) \ : \ \theta = 0 \ on \ \partial\bar{D}_\theta\right\}$$

$$\hat{\mathbf{C}}^1(D) = \left\{u = (\mathbf{v}, \boldsymbol{\phi}, \theta) \in \mathbf{C}^1(\bar{D}) \ : \ v_m = 0 \ on \ \partial\bar{D}_v, \right.$$
$$\left. \phi_m = 0 \ on \ \partial\bar{D}_\phi, \ \ \theta = 0 \ on \ \partial\bar{D}_\theta\right\}$$

$\hat{W}_1(D) =$ the completion of $\hat{C}^1(D)$ by using the norm $\|.\|_{W_1(D)}$
$\hat{\mathbf{W}}_1(D) =$ the completion of $\hat{\mathbf{C}}^1(D)$ by using the norm $\|.\|_{\mathbf{W}_1(D)}$.

With the help of the pairs of functions $(\mathbf{v}, \boldsymbol{\phi}) \in \hat{\mathbf{C}}^1(D)$, $(\mathbf{w}, \boldsymbol{\psi}) \in \hat{\mathbf{C}}^1(D)$, we define the next functional:

$$F_1((\mathbf{v}, \boldsymbol{\phi}), (\mathbf{w}, \boldsymbol{\psi})) = \frac{1}{2}\int_D \{A_{klmn} e_{mn}(\mathbf{v}, \boldsymbol{\phi}) e_{kl}(\mathbf{w}, \boldsymbol{\psi}) + C_{klmn}\varepsilon_{mn}(\mathbf{v}, \boldsymbol{\phi})\varepsilon_{kl}(\mathbf{w}, \boldsymbol{\psi})$$
$$+B_{klmn}\left[e_{mn}(\mathbf{v}, \boldsymbol{\phi})\varepsilon_{kl}(\mathbf{w}, \boldsymbol{\psi}) + e_{mn}(\mathbf{w}, \boldsymbol{\psi})\varepsilon_{kl}(\mathbf{v}, \boldsymbol{\phi})\right]\} \, dV. \tag{3.11}$$

Similarly, for the functions pair (θ, T) we consider the second functional:

$$F_2(\theta, T) = \int_D K_{mn}\theta_{,m}T_{,n}dV. \tag{3.12}$$

The functional F_1 can be extended by continuity to the hole space $\hat{\mathbf{W}}_1(D)$ and the functional F_2 can be extended by continuity to the hole space $\hat{W}_1(D)$.
The main anticipated results of us will be obtained if we impose the next assumptions with regards to the material properties

(3.13) $i)$ $$\varrho > 0, \ I_{mn}, \ c > 0, \ T_0 > 0;$$

$ii)$ there is a positive constant c_1 so that the functional F_1 satisfies the estimate:

$$F_1((\mathbf{w}, \boldsymbol{\psi}), \ (\mathbf{w}, \boldsymbol{\psi})) \geq$$
$$\geq c_1 \int_D \sum_{m,n=1}^{3} \left[e_{mn}^2((\mathbf{v}, \boldsymbol{\psi}), (\mathbf{v}, \boldsymbol{\psi})) + \varepsilon_{mn}^2((\mathbf{v}, \boldsymbol{\psi}), (\mathbf{v}, \boldsymbol{\psi}))\right] dV, \tag{3.14}$$

for all $((\mathbf{v}, \boldsymbol{\psi}), \ (\mathbf{v}, \boldsymbol{\psi})) \in \hat{\mathbf{C}}^1(D)$;
$iii)$ there is a constant $c_2 > 0$ so that the functional F_2 satisfies the estimate:

$$F_2(\theta, \theta) \geq c_2 \int_D K_{mn}T_{,m}T_{,n}dV, \tag{3.15}$$

for any $T \in \hat{C}^1(D)$.
Of course, the estimate (3.14) can be extended for $(\mathbf{v}, \boldsymbol{\psi}) \in \hat{\mathbf{W}}_1(D)$ and the estimate (3.15) can be extended for $T \in \hat{W}_1(D)$.
By using a suggestion given in [25] and [26] one can determine a constant $k > 0$ so that takes place the next estimate:

$$F_1((\mathbf{v}, \boldsymbol{\phi}), \ (\mathbf{v}, \boldsymbol{\phi})) \geq$$
$$\geq k \int_D \left[v_m v_m + \phi_m\phi_m + v_{m,n}v_{m,n} + \phi_{m,n}\phi_{m,n}\right] dV. \tag{3.16}$$

Considering the inequality (3.16) it can be deduced that the functional $F_1((\mathbf{v},\boldsymbol{\phi}),\ (\mathbf{v},\boldsymbol{\phi}))$ is a coercive quadratic form relative to the space $\hat{\mathbf{W}}_1(D)$.
We now introduce some sets that are useful in what follows.

$$\begin{aligned}
\tilde{C}^0(D) = \Big\{ &(f_m,\ g_m,\ r) : (f_m,\ g_m,) \in \mathbf{C}^0(\bar{D}),\ r \in C^0(\bar{D});\\
&\qquad \text{if meas } (\partial \bar{D}_v) = \text{meas } (\partial \bar{D}_\phi) = 0\\
&\qquad\qquad \Rightarrow \int_D \varrho F dV = 0,\ \int_D \varrho (x \times F) dV = 0;\\
&\text{if meas } (\partial \bar{D}_v) = 0,\ \text{meas } (\partial \bar{D}_\phi) = 0,\ \text{meas } (\partial \bar{D}_\theta) \neq 0,\ \Rightarrow \int_D \varrho F dV = 0;\\
&\text{if meas } (\partial \bar{D}_\theta) = 0,\ \Rightarrow \int_D \varrho r\ dV = 0 \Big\};\\
\tilde{C}^1(D) = \Big\{ &(u = (v_m, \phi_m),\ \theta) :\ (u,\ \theta) \in \hat{\mathbf{C}}^1(D) \times \hat{C}^1(D)\ and\\
&\text{if meas } (\partial \bar{D}_\theta) = 0\ \Rightarrow \int_D \left[\alpha_{mn} e_{mn}(u) + \beta_{mn}\varepsilon_{mn}(u) + c\theta\right] dV = 0 \Big\};\\
\mathcal{D}(Q_0) = C^\infty &\Big([0, t_0)\,;\ \tilde{C}^1(D) \Big);\\
\breve{\mathcal{D}}(Q_0) = &\left\{ (v = (v_m, \psi_m),\ T)\ :\ (v,\ T) \in \mathcal{D}(Q_0)\ and\ v = 0\ on\ D \times \{0\} \right\}.
\end{aligned}$$

For the elements $y = ((v_m, \phi_m), \theta) \in \mathcal{D}(Q_0)$, $w = ((v_m, \psi_m), T) \in \breve{\mathcal{D}}(Q_0)$ and considering the charges $z = (f_m, g_m, r) \in \mathbf{C}^\infty \Big([0, t_0)\,;\ \tilde{C}^0(D)\Big)$, we can introduce the bilinear forms $F_3(y, \omega)$ and $F_4(z, \omega)$ by:

$$\begin{aligned}
F_3(y, w) = \int_0^{t_0} \int_D \Big\{ (t - t_0) \Big[&\varrho \dot{v}_m \ddot{v}_m + I_{mn} \dot{\phi}_m \ddot{\psi}_n - A_{klmn} e_{mn}(y) \dot{e}_{kl}(w) -\\
&- B_{klmn} \left(e_{mn}(y) \dot{\varepsilon}_{kl}(w) + \dot{e}_{mn}(w) \dot{\varepsilon}_{kl}(y) \right) - C_{klmn} \varepsilon_{mn}(y) \dot{\varepsilon}_{kl}(w) -\\
&+ c\, \theta\, \dot{T} + \alpha_{mn}\, \theta\, \dot{e}_{mn}(w) + \beta_{mn}\, \theta\, \dot{\varepsilon}_{mn}(w) \Big] +\\
&+ \varrho\, \dot{v}_m\, \dot{v}_m + I_{mn} \dot{\phi}_m \dot{\psi}_n + c\, \theta\, T +\\
&+ \alpha_{mn}\, T\, e_{mn}(y) + \beta_{mn}\, T\, \varepsilon_{mn}(y) + \frac{1}{T_0} \int_0^t K_{mn}\, \theta_{,\,n}\, T_{,\,m}\, d\tau \Big\} dV dt;\\
F_4(z, w) = \int_0^{t_0} \int_D (t - t_0) \Big[&- \varrho f_m \dot{v}_m - I_{mn} g_m \dot{\psi}_n - \frac{\varrho r}{T_0} T \Big] dV dt.
\end{aligned}$$

Furthermore, considering the initial values $\delta = \left(v_m^0,\ \phi_m^0,\ \theta^0 \right)$ such that $v_m^0,\ \phi_m^0 \in \hat{\mathbf{C}}^1(D)$, $\theta^0 \in \tilde{C}^1(D)$, we define the next functional:

$$\begin{aligned}
F_5(\delta, w) = t_0 \int_D \Big[&\varrho v_m^0 w_m|_{t=0} + I_{mn} \phi_m^0 \psi_n|_{t=0} + c\, T|_{t=0} \theta^0 +\\
&+ \alpha_{mn} T|_{t=0} e_{mn}\left(u^0, \theta^0\right) + \beta_{mn}\, T|_{t=0} \varepsilon_{mn}\left(u^0, \theta^0\right) \Big] dV.
\end{aligned}$$

We substitute y by w in the expression of $F_3(z,w)$, so that we are led to the equality identity:

$$\begin{aligned}F_3(w,w)=&\int_0^{t_0}\int_D\Big\{(t-t_0)\Big[\varrho\dot{w}_m\dot{w}_m+I_{mn}\dot{\psi}_m\dot{\psi}_n+A_{klmn}e_{mn}(w)e_{kl}(w)-\\ &+2B_{klmn}e_{mn}(w)\varepsilon_{kl}(w)+C_{klmn}\varepsilon_{mn}(w)\varepsilon_{kl}(w)+\\ &+cT^2+\frac{1}{T_0}\int_0^t K_{mn}\,T_{,\,n}\,T_{,\,m}\,d\tau\Big]dVdt+\\ &+\frac{t_0}{2}\int_D\Big[\varrho\dot{w}_m\dot{w}_m+I_{mn}\dot{\psi}_m\dot{\psi}_n+cT^2\Big]_{t=0}dV.\end{aligned}\tag{3.17}$$

We intend to determine a solution that has a finite energy for our problem $\mathcal{P}$, which consists of systems the equations (2.8), the initial values (2.9) and of boundary relations (2.10). To reach this desire we will introduce other new spaces.

So, we will complete the space $\mathcal{D}\left(Q_0\right)$, by using the norm $|.|$, until space $V\left(Q_0\right)$, which is a Hilbert space, considering that the norm $|.|$ is determined by the next scalar product:

$$\begin{aligned}<\left(v_m,\phi_m,\theta\right),\left(w_m,\psi_m,T\right)>=&\int_0^{t_0}\int_D\Big[v_m\dot{w}_m+\phi_m\dot{\psi}_m+\\ &+\dot{v}_m\dot{w}_m+\dot{\phi}_m\dot{\psi}_m+v_{m,n}w_{m,n}+\phi_{m,n}\psi_{m,n}+\theta T+\int_0^t\theta_{,m}T_{,m}ds\Big]dVdt.\end{aligned}\tag{3.18}$$

After that, we will complete the space $\breve{\mathcal{D}}\left(Q_0\right)$ until the space $\breve{V}\left(Q_0\right)$, by using the previous norm $|.|$. It can be easily found that the manifold $\breve{V}\left(Q_0\right)$ is a linear and closed set.

We complete now the set $\breve{\mathcal{D}}\left(Q_0\right)$ until the Hilbert space be denoted by $\breve{U}\left(Q_0\right)$, by using the norm determined by the next scalar product:

$$\begin{aligned}&\left[\left(v_m,\phi_m,\theta\right),\left(w_m,\psi_m,T\right)\right]=\\ &\qquad=<(v_m,\phi_m,\theta),(w_m,\psi_m,T)>+<\left(\dot{v}_m,\dot{\phi}_m,\dot{\theta}\right),\left(\dot{w}_m,\dot{\psi}_m,\dot{T}\right)>.\end{aligned}$$

Here we used the scalar product $<(u,\theta),(v,T)>$ that was introduced in (3.18).

At the next step we complete the set $\tilde{C}^0(D)$ until the space $G(D)$, by using the norm $\|.\|_{\mathbf{W}_0(D)\times W_0(D)}$. We will make the last completion for the set:

$$\Big\{(u,\ v,\ \theta):\left(u=\left(v_m,\phi_m\right),\theta\right)\in\tilde{\mathbf{C}}^1(D),\ \left(v=\left(w_m,\psi_m\right),\ \theta\right)\in\hat{\mathbf{C}}^1(D)\Big\},$$

by using the norm:

$$\begin{aligned}|(u,\ v,\ \theta)|_0=&\left\{\frac{1}{2}\int_D\left[\varrho v_m w_m+I_{mn}\phi_m\psi_n+A_{klmn}e_{kl}(u)e_{mn}(v)+\right.\right.\\ &\left.\left.+2B_{klmn}e_{kl}(u)\varepsilon_{mn}(v)+C_{klmn}\varepsilon_{kl}(u)\varepsilon_{mn}(v)+a\theta^2\right]dv\right\}^{1/2},\end{aligned}$$

and the completion space is denoted by $H_0(B)$.

By using the Sobolev's theorem of embedding (see [3]) and considering the Schwarz's inequality, can be extended the functional $F_3(y,w)$, by continuity, to the space of product $\breve{U}\left(Q_0\right)\times V\left(Q_0\right)$ and, similar, can be extended the functional $F_4(z,\delta)$, by continuity, to the space $\breve{U}\left(Q_0\right)\times L_1\left((0,\ t_0)\,;\ G(D)\right)$.

Analogous, can be extended the functional $F_5(\delta,w)$ to be defined for $\delta\in H_0(D)$, $w\in\breve{U}\left(Q_0\right)$.

Considering the previous extensions and completions and taking into account the assumptions (3.13)-(3.16) and the equality (3.18), it can determine a constant $c_3>0$, which depends only on c, T_0 and k, in order to obtain this estimate:

$$|w|^2 \leq c_3 F_3(w, w),\ \forall\ w \in \breve{U}\left(Q_0\right). \tag{3.19}$$

Consider $u = (v_m, \phi_m, \theta)$ and $y \equiv (u, \theta) \in V\left(Q_0\right)$, y being a solution of our mixed problem $\mathcal{P}$.
We now intend to introduce the concept of a finite energy solution, for this mixed problem.

Definition 3.2. It is considered that $y = (u, \theta)$ satisfy the relations:

$$F_3(y, w) = F_4(z, w) + F_5(\delta, w),\ \forall w = \left(\left(w_m, \psi_m, \chi\right), T\right) \in \breve{U}\left(Q_0\right), \tag{3.20}$$

$$\lim_{t \to 0} u(t) = u_0 = \left(v_m^0, \phi_m^0, \theta^0\right),\ in\ \ \mathbf{W}_0(D). \tag{3.21}$$

Then, y will be called a solution with finite energy which verifies the system of equations (2.8) and satisfies the boundary relations (2.10) and are corresponding to the initial values $\delta \equiv \left(v_m^0, \phi_m^0, \theta^0\right) \in \mathbf{H}_0(D)$ and to the charges $z \equiv (\mathbf{f}, \mathbf{g}, r) \in \mathbf{L}_1\left((0, t_0);\ G(D)\right)$.

In the following theorem there is shown a procedure to determine a new solution with finite energy, starting from one already known.

Theorem 3.1. *Let us take y a finite energy solution, $y \in V\left(Q_0\right)$, that satisfies the equations* (2.8)*, and verifies the boundary relations* (2.10)*, which is defined on the domain Q_0 and which is corresponding to the charges $z = (f_m, g_m, r)$ and to the initial values $\delta \equiv \left(v_m^0, \phi_m^0, \theta^0\right)$.*
Consider the function $\breve{y}$ defined by:

$$\breve{y}(t) = \int_0^t y(x, \tau) d\tau.$$

Then $\breve{y}(t)$ is also a finite energy solution of the problem (2.8), (2.10)*, considered in the domain Q_0, that corresponds to the initial values $\breve{\delta} = (0, 0, 0)$ and to the volume charges $\breve{z} = \left(\breve{f}_m, \breve{g}_m, \breve{r}\right)$. Here the following notations are used:*

$$\begin{aligned}
&\breve{f}_m(x, t) = \int_0^t f_m(x, s) ds + w_m^0(x), \\
&\breve{g}_m(x, t) = \int_0^t g_m(x, s) ds + \psi_m^0(x), \\
&\breve{r}(x, t) = \int_0^t r(x, s) ds + \\
&\qquad\qquad + \frac{T_0}{\varrho}\left[\alpha_{mn}(x) e_{mn}\left(v^0\right) + \beta_{mn}(x) \varepsilon_{mn}\left(v^0\right) + c(x) \theta^0\right].
\end{aligned}$$

Proof. The affirmation can be easy proven by applying the same technique used by Dafermos in the paper [8]. □

Further, with the help an algorithm similar to that used by Dafermos in [8], one can obtain the result regarding the uniqueness of the solution of the mixed problem, in relation to a solution which possesses finite energy.

Theorem 3.2. *In the thermoelasticity of Cosserat bodies, the mixed problem $\mathcal{P}$, which consists of system* (2.8) *and the boundary values* (2.10)*, has at most one solution having a finite energy, that is corresponding to certain given initial values and certain prescribed charges.*

To begin with, we will limit ourselves to the simpler case when the initial values are zero and the charges are from space $L_2\left((0,t_0)\,;\;G(D)\right)$. In this situation, we created the possibility to establish two results, both regarding the existence of a solution having a finite energy, and a significant estimation for respective solution. We included both these results in next the theorem.

Theorem 3.3. *For the Cosserat thermoelastic materials, it is considered the problem $\mathcal{P}$ which consists of the above equations* (2.8) *along with boundary relations* (2.10) *which corresponds to zero initial values and to the volume charges*

$$z = \left(\left(f_m, g_m\right), r\right) \in \mathbf{L}_2\left((0,t_0)\,; G(D)\right).$$

Then, the this problem certainly admits a solution with a finite energy y, where $y = (v_m, \phi_m, \theta) \in \breve{V}(Q_0)$. Furthermore, it can be determined a constant $c_4 > 0$, that depends only on c_3 (from (3.18)*), T_0, ϱ and t_0 such that the following estimate is fulfilled:*

$$|y| \le c_4 \|z\|_{\mathbf{L}_2(Q_0)\times L_2(Q_0)}.$$

In the next stage we intend to generalize the existence result regarding a solution having a finite energy, from the simpler case of the zero initial values, formulated in Theorem 3.3, to the general situation non-homogeneous initial values. But to achieve this desired, we need new definitions, exposed in definition that follows.

Definition 3.3. Let us consider a Hilbert space, denoted by $H(D)$, and defined by:

$$H(D) = H_0(D) \cap \left\{\hat{\mathbf{W}}_1(D) \times \hat{\mathbf{W}}_1(D) \times \hat{W}_1(D)\right\}.$$

For the source $z = \left(\left(f_m, g_m\right), r\right) \in G(B)$, the application $\mathcal{S}(z) : H_0(D) \to H(D)$ which leads the array $(w_m, \psi_m, T) \in H_0(D)$ in the array $(v_m, \phi_m, \theta) \in H(D)$, for which $(v_m, \phi_m, \theta) \in \hat{\mathbf{W}}_1(D) \times \hat{W}_1(D)$, is called the solution of the next system of relations:

$$\begin{aligned} F_1(u,\omega) = \int_D &\left[\left(\alpha_{mn} e_{mn}(\omega) + \beta_{mn}\varepsilon_{mn}(\omega) + cT\right)\theta - \right.\\ &\left. -\varrho v_m w_m - I_{mn}\phi_m\psi_n + \varrho f_m w_m + \varrho g_m \psi_m\right], \forall \omega = (w_m, \psi_m, T) \in \hat{\mathbf{W}}_1(D),\\ F_2(T,\theta) = -\int_D &\left\{T_0\left[\alpha_{mn} e_{mn}(u) + \beta_{mn}\varepsilon_{mn}(u) + cT\right] - \varrho r\right\}\theta dV, \forall T \in \hat{W}_1(D). \end{aligned}$$

Now, we intend to describe a behavior of the solution defined with the help the map $\mathcal{S}(z)$, above defined. We will do this in Theorem 3.4, which follows. Also in Theorem 3.4 we will investigate the regularity of this type of solution, by considering certain assumptions on the initial values and on certain characteristic charges.
These two auxiliary results will be useful in proving the general result regarding if there exists a solution in the case of some non-homogeneous initial values.

Theorem 3.4. *The map $\mathcal{S}(z)$ is defined for all $z \in H_0(D)$ and is an one to one application which admits the inverse application $\mathcal{S}^{-1}(z)$ that is well defined in all points z, which belong to the codomain of $\mathcal{S}(z) \subset H(D)$ into $H_0(D)$.*
Furthermore, the solution, constructed with the help of the application $\mathcal{S}(z)$ satisfies the next estimate, regarding the non-homogeneous initial values $\delta = \left(v_m^0, \phi_m^0, \theta^0\right)$ and the sources $z = (f_m, g_m, r)$:

$$\|\mathcal{S}(z)\|_{\mathbf{W}_1(D)\times\mathbf{W}_1(D)\times W_1(D)} \le c_5\left\{|\delta|_0 + \|z\|_{\mathbf{W}_0(D)\times W_0(D)}\right\},$$

where $c_5 > 0$ is a calculable constant.

Proof. Both results can be proven by using the technique described by Fischera in his work [23]. Also, it is important to remember that the functional $F_1(v,v)$ satisfies the estimate

(3.14). Furthermore, we can take into account that the bilinear form F_1 has the property of coercivity, on $\hat{\mathbf{W}}_1(D)$, relative to the norm $\|.\|_{\mathbf{W}_1(D)}$. This is obtained immediately, by using the estimate (3.16). □

We consider now certain arbitrary charges $z_0,\ z_1, ...,\ z_{m-1}$, so that $z_0,\ z_1, ...,\ z_{m-1} \in G(D)$ and introduce the map

$$\mathcal{S}_m\left(z_0,\ z_1, ...,\ z_{m-1}\right) = \mathcal{S}(z_0) \circ \mathcal{S}(z_1) \circ ... \circ \mathcal{S}(z_{m-1}).$$

The codomain of the application

$$\mathcal{S}_m\left(y_0,\ y_1, ...,\ y_{m-1}\right) : H_0(D) \to H(D),$$

is designated by $H_m\left(B;\ y_0,\ y_1, ...,\ y_{m-1}\right)$. If we set $H_m(D) = H_m\left(D;\ y_0,\ y_1, ...,\ y_{m-1}\right)$, for $\delta \in H_m(B)$ we consider a new norm:

$$|\delta|_m = \left|\mathcal{S}_m^{-1}(0,\ 0,\ ...,\ 0)\ \delta\right|_0.$$

The notation $\overset{(m)}{f} = \frac{\partial^m f}{\partial z_i^m}$ will be used to designate the usual m partial derivative for a scalar application $f = f\left(z_1, z_2, ..., z_n\right)$.

Considering the above explanations, we can approach now the issue of existence for a solution with energy finite, in the situation of certain non-zero initial values.

Theorem 3.5. *Consider that the problem $\mathcal{P}$, consists of the system* (3.11), *satisfies in the space $Q(0)$, verifies the boundary relations* (2.10), *has the sources*

$$z = \left(f_m, g_m, r\right) \in C^{m-1}\left([0,\ t_0)\,;\ G(D)\right),\ \overset{(m)}{z} \in L_1\left((0,\ t_0)\,;\ G(D)\right)$$

and satisfies the initial values

$$\delta = \left(v_m^0, \phi_m^0, \theta^0\right) \in H_m\left(D;\ z(0),\ \overset{(1)}{z}(0),\ ...,\ \overset{(m-1)}{z}(0)\right),\ \textit{for}\ \ m = 1, 2, ...$$

Then this problem certainly admits a solution with energy finite y, so that $y \in V\left(Q_0\right)$.

Proof. The demonstration follows step by step the algorithm used by Fischera in his known work [23]. □

Inspired by the procedure introduced by Dafermos in [8] and Fichera in [11], we can prove the next three estimates, which, in fact, characterizes the control property for the solution with finite energy, approached in Theorem 3.5.

Theorem 3.6. *The following estimates are fulfilled by all finite energy solutions which satisfy the problem $\mathcal{P}$:*

i)
$$\left(v_m, \phi_m, \dot{v}_m, \dot{\phi}_m, \theta\right) \in C^m\left([0, t_0)\,;\ H_0(D)\right),$$
$$\left(v_m, \phi_m, \dot{v}_m, \dot{\phi}_m, \theta\right)|_{(0)} = \left(v_m^0, \phi_m^0, \dot{v}_m^1, \dot{\phi}_m^1, \theta^0\right);$$

ii) *for all $t \in [0, t_0)$ and for any $\ k = 0, 1, ..., m,$ we get*

$$\left(\overset{(k)}{v}_m, \overset{(k)}{\phi}_m, \overset{(k+1)}{v_m}, \overset{(k+1)}{\phi_m}, \overset{(k)}{\theta}\right) \in \mathcal{S}_{m-k}\left(D;\ \overset{(k)}{z(t)},\ \overset{(k)+1}{z(t)},\ ...,\ \overset{(m-1)}{z(t)}\right),$$

$$\left(\overset{(k)}{v}_m, \overset{(k)}{\phi}_m, \overset{(k+1)}{v_m}, \overset{(k+1)}{\phi_m}, \overset{(k)}{\theta}\right)(t) = P_{m-k}\left(\overset{(k)}{z(t)},\ \overset{(k)+1}{z(t)},\ ...,\ \overset{(m-1)}{z(t)}\right) =$$

$$= \left(\overset{(m)}{v}_m, \overset{(m)}{\phi}_m, \overset{(m+1)}{v_m}, \overset{(m+1)}{\phi_m}, \overset{(m)}{\theta}\right)(t);$$

iii) we have the estimate

$$\left(\left| \left(\overset{(k)}{v}_m, \overset{(k)}{\phi}_m, \overset{(k+1)}{v_m}, \overset{(k+1)}{\phi_m}, \overset{(k)}{\theta} \right)(t) \right|_0^2 + \frac{1}{T_0} \int_0^t F_2 \left(\overset{(k)}{\theta}, \overset{(k)}{\theta} \right)(s) ds \right)^{1/2} \leq$$

$$\leq \left| \mathcal{S}_k^{-1} \left(z(0), \overset{(1)}{z}(0), ..., \overset{(m-1)}{z}(0) \right) \right|_0 + c_6 \int_0^t \left\| \overset{(k)}{z}(s) \right\|_{\mathbf{W}_0(D) \times W_0(D)} ds, \tag{3.22}$$

where $c_6 > 0$ is a constant that depends only on c, T_0, I_{mn} and ϱ.
Also, the estimate (3.22) is satisfied by all $t \in [0, t_0)$ and any $k = 0, 1, ..., m$.

Remark 3.1. It is easy to observe how in the simple situation of zero sources, $z \equiv 0$, the inequality (3.22) becomes an equality.

4. Conclusions

We considered the motion equations, the energy equations, the initial values and the relations to the limit which are characteristic of a general problem in the context of thermoelastic Cosserat bodies. After that, we have generalized the algorithm used by Dafermos in paper [8] and Fichera in paper [11], which determines the results of existence and uniqueness of the formulated problem. Our first result proven that the solution with energy finite is the unique. Next, we established a result of existence for a solution with energy finite in the special situation of zero initial values. After that, the previous result is generalized to more general situation of non-zero initial values. The last result of our work approaches certain estimates with regards to the property of controllability of the solution with energy finite. It is important to emphasize the consistence of our results with respect to questions of the uniqueness and of the existence of the solution with energy finite, exactly like in the theory of classical elasticity, although we considered in our work a much more complicated mixed problem, by taking into account the thermal effect and the effect of the Cosserat structure.

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