CARPATHIAN J. MATH.
Volume **40** (2024), No. 1,
Pages 121 - 137

Online version at https://www.carpathian.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2024.01.09

Subadditive and Superadditive Inequalities for Convex and Superquadratic Functions

HAMID REZA MORADI, NICUŞOR MINCULETE, SHIGERU FURUICHI and MOHAMMAD SABABHEH

ABSTRACT. Convex functions and their analogues have been powerful tools in almost all mathematical fields, including optimization, fractional calculus, mathematical analysis, functional analysis, operator theory, and mathematical physics. It is well established in the literature that a convex function $f:[0,\infty)\to[0,\infty)$ with $f(0)=0$ is necessarily superadditive, while a concave function $f:[0,\infty)\to[0,\infty)$ is subadditive. The converses of these two assertions are not valid in general. The main target of this article is to study the subadditivity and superadditivity of convex and superquadratic functions. In particular, we obtain several results extending, refining, and reversing some known inequalities in this direction. Further discussion of superquadratic functions in this line will be given.

1. INTRODUCTION AND PRELIMINARIES

Convex functions are widely used in pure mathematics, functional analysis, optimization theory, and mathematical economics. Giant attention has been given to analyzing convex functions and their properties. See, for example, [22], which contains an extensive bibliography.

The theory of convex functions is closely connected to the theory of mathematical inequalities. Indeed, any convex function, ($J\subset\mathbb{R}$) fulfills the inequality

$$f\left(\left(1-\nu\right)a+\nu b\right)\le\left(1-\nu\right)f\left(a\right)+\nu f\left(b\right);\ 0\le\nu\le1 \tag{1.1}$$

for all $a,b\in J$. This says a function is convex if its graph lies below its secants. A function f is called mid-convex if, for every set of two elements $a,b\in J$, we have

$$f\left(\frac{a+b}{2}\right)\le\frac{f\left(a\right)+f\left(b\right)}{2}. \tag{1.2}$$

Inequalities for convex functions delivered significant interest in providing many elegant results with applications. Several authors, including Marshall and Proschan [14], Wright [27], and Mercer [15], presented a large number of significant results for convex functions and related inequalities.

The famous Hermite-Hadamard inequality furnishes estimates of the mean value of a continuous convex function (see, e.g., [20, p. 50]). This inequality asserts that every convex function f on an interval $[a,b]$ (with $a<b$) enjoys the inequality

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f\left(t\right)dt\le\frac{f\left(a\right)+f\left(b\right)}{2}.$$

Received: 29.01.2023. In revised form: 21.09.2023. Accepted: 28.09.2023
2010 *Mathematics Subject Classification.* Primary 26A51, 26B25, Secondary 26D15, 39B62, 52A40, 26D15.
Key words and phrases. *Convex function, superquadratic, inequality, subadditive, Hermite-Hadamard, mid-convex function.*
Corresponding author: Nicuşor Minculete; minculete.nicusor@unitbv.ro

Notice that every continuous function is convex if and only if it satisfies the Hermite-Hadamard inequality. We encourage interested readers to explore the literature on convex functions and their inequalities, especially [16, 18, 19, 23, 24, 25] and the references therein.

A class of functions strongly related to convex functions is the class of superquadratic functions. A function $f:[0,\infty)\to\mathbb{R}$ is said to be superquadratic provided that for all $s\geq 0$, there exists a constant $C_s\in\mathbb{R}$ such that

$$f(s)+C_s(t-s)+f(|t-s|)\leq f(t) \tag{1.3}$$

for all $t\geq 0$. We say that f is subquadratic if $-f$ is superquadratic.

For instance, the function $f(x)=x^p$ $(x\geq 0)$ is superquadratic for $p\geq 2$, and subquadratic for $0<p\leq 2$. Notice that in this situation C_s in (1.3) is equal to px^{p-1} and when $p=2$, (1.3) is an equality.

Primary definitions and properties of superqudratic functions are presented by Abramovich, Jameson, and Sinnamon in [1] and [2].

Condition (1.3) appears more powerful than the convexity condition. In [2], it was proved that this statement is valid for nonnegative f. More exactly, any superquadratic function fulfills the following conditions:

(P_1) $f(0)\leq 0$.
(P_2) If $f(0)=f'(0)=0$ and f is differentiable at s, then $C_s=f'(s)$.
(P_3) If $f\geq 0$, then f is convex and $f(0)=f'(0)=0$.

From (1.3) we easily get that if $f:[0,\infty)\to\mathbb{R}$ is a superquadratic function, then for any $\nu_1,\nu_2\geq 0$ with $\nu_1+\nu_2=1$ (see e.g., [1, (1.4)]),

$$f(\nu_1a+\nu_2b)\leq \nu_1f(a)+\nu_2f(b)-\nu_2f(\nu_1|a-b|)-\nu_1f(\nu_2|a-b|) \tag{1.4}$$

for all $a,b\geq 0$.

Remark 1.1. Notice that if f is superquadratic and $\nu_1,\nu_2>0$ with $\nu_1+\nu_2\leq 1$, then

$$\begin{aligned}
&f(\nu_1a+\nu_2b)\\
&=f\left((\nu_1+\nu_2)\left(\frac{\nu_1}{\nu_1+\nu_2}a+\frac{\nu_2}{\nu_1+\nu_2}b\right)+(1-(\nu_1+\nu_2))\cdot 0\right)\\
&\leq(\nu_1+\nu_2)f\left(\frac{\nu_1}{\nu_1+\nu_2}a+\frac{\nu_2}{\nu_1+\nu_2}b\right)+(1-(\nu_1+\nu_2))f(0)\\
&\quad-(1-(\nu_1+\nu_2))f(\nu_1a+\nu_2b)-(\nu_1+\nu_2)f\left(\left(\frac{1}{\nu_1+\nu_2}-1\right)(\nu_1a+\nu_2b)\right)\quad\text{(by (1.4))}\\
&\leq(\nu_1+\nu_2)f\left(\frac{\nu_1}{\nu_1+\nu_2}a+\frac{\nu_2}{\nu_1+\nu_2}b\right)\\
&\quad-(1-(\nu_1+\nu_2))f(\nu_1a+\nu_2b)-(\nu_1+\nu_2)f\left(\left(\frac{1}{\nu_1+\nu_2}-1\right)(\nu_1a+\nu_2b)\right)\quad\text{(by }(P_1))\\
&\leq\nu_1f(a)+\nu_2f(b)-\nu_1f\left(\frac{\nu_2|a-b|}{\nu_1+\nu_2}\right)-\nu_2f\left(\frac{\nu_1|a-b|}{\nu_1+\nu_2}\right)\\
&\quad-(1-(\nu_1+\nu_2))f(\nu_1a+\nu_2b)-(\nu_1+\nu_2)f\left(\left(\frac{1}{\nu_1+\nu_2}-1\right)(\nu_1a+\nu_2b)\right)\quad\text{(by (1.4))}
\end{aligned}$$

i.e.,

$$\begin{aligned}
f(\nu_1a+\nu_2b)&\leq\nu_1f(a)+\nu_2f(b)-\nu_1f\left(\frac{\nu_2|a-b|}{\nu_1+\nu_2}\right)-\nu_2f\left(\frac{\nu_1|a-b|}{\nu_1+\nu_2}\right)\\
&\quad-(1-(\nu_1+\nu_2))f(\nu_1a+\nu_2b)-(\nu_1+\nu_2)f\left(\left(\frac{1}{\nu_1+\nu_2}-1\right)(\nu_1a+\nu_2b)\right).
\end{aligned}$$

We refer the interested reader to [3, 4, 5, 21] to see some inequalities and related discussions on this topic.

A function $f\colon [0,\infty) \to \mathbb{R}$ is superadditive provided

$$f(a) + f(b) \le f(a+b)\,;\ a, b \ge 0. \tag{1.5}$$

If the reversed inequality holds, then f is said to be subadditive.

These functions play a significant role in the theory of differential equations, semigroups, and the theory of convex bodies. The fundamental properties of sub and superadditive functions can be found in [6].

If a function f is convex on $[0,\infty)$, and $f(0) \le 0$, then f is superadditive. However, superadditivity does not imply convexity. Counterexamples can be seen in [7, 8]. Meanwhile, if a function f is concave, and $f(0) \ge 0$, then f is subadditive on $[0,\infty)$. This topic attracted the attention of many mathematicians over the decades, and they generalize, improve and extend these inequalities in several forms; see [13, 17].

This paper presents superadditive and subadditive inequalities for convex and superquadratic functions. Also, we discuss convex inequalities when $\nu_1, \nu_2 > 0$ are such that $\nu_1 + \nu_2 \le 1$, rather than $\nu_1 + \nu_2 = 1$.

It is worth noting that another class of superquadratic functions is defined in [26] and was treated, for example, in [12]. In some cases, the two classes overlap but only sometimes. The differences and similarities between these classes are clarified in [1] and in [11].

Among many results, we show that if $f : [0,\infty) \to \mathbb{R}$ is a convex function with $f(0) = 0$, then for any $a, b > 0$,

$$f\left(\frac{ab}{a+b}\right) \le \frac{ab}{(a+b)^2}\left(f(a) + f(b)\right),$$

as a new type of convex inequalities. The $n-$tuple version of this result will be shown too. Further, we show that such a convex function satisfies the two inequalities

$$tf(x) \le f(tx) - (t - |2-t|)\left(\frac{f(tx)}{2} - f\left(\frac{tx}{2}\right)\right),$$

and

$$tf(x) \ge f(tx) + (t + |2-t|)\left(\frac{f(tx)}{2} - f\left(\frac{tx}{2}\right)\right),$$

where $x \ge 0$ and $t \ge 1$.

On the other hand, if f is superquadratic, we show that

$$f(a) + f(b) \le f(a+b) - \frac{2a}{a+b} f(b) - \frac{2b}{a+b} f(a),$$

as a superadditive-type inequality for superquadratic functions. Many other related results will be discussed and proven.

2. Main results

2.1. **Inequalities for Convex Functions.**

We begin with a simple result containing a comparison of $f(b-a)$ and $f(b) - f(a)$ when f is a convex function.

Lemma 2.1. *Let $f\colon [0,\infty) \to \mathbb{R}$ be a convex function with $f(0) \le 0$. Then for any $0 \le a \le b$,*

$$f(b-a) \le f(b) - f(a).$$

Proof. This follows immediately noting that f is superadditive and that $b = (b-a)+a$. □

We should remark that in Theorem 2.4 below, we present a refinement and a reverse for the inequality in the above lemma.

Remark 2.2. We emphasize that Lemma 2.1 can be obtained directly from the Jensen-Mercer inequality [15], though we put its proof for the reader's convenience.

Utilizing the above lemma, we can state the following result.

Theorem 2.1. *Let* $f\colon [0,\infty) \to \mathbb{R}$ *be a convex function with* $f(0) \le 0$. *If* $0 < a, b < c$, *then*

$$f\left(c - \frac{a+b}{2}\right) \le f(a+b+c) - \frac{f(2a+b) + f(2b+a)}{2}.$$

Proof. Exploiting Lemma 2.1, we have

$$f(c-a) = f(a+b+c-(2a+b)) \le f(a+b+c) - f(2a+b) \tag{2.6}$$

and

$$f(c-b) = f(a+b+c-(a+2b)) \le f(a+b+c) - f(2b+a). \tag{2.7}$$

By adding these two inequalities together, we get

$$\frac{f(c-a) + f(c-b)}{2} \le f(a+b+c) - \frac{f(2a+b) + f(2b+a)}{2}. \tag{2.8}$$

On the other hand, by the convexity of f, we have

$$f\left(c - \frac{a+b}{2}\right) \le \frac{f(c-a) + f(c-b)}{2}. \tag{2.9}$$

Therefore, by incorporating (2.8) and (2.9), we reach

$$f\left(c - \frac{a+b}{2}\right) \le f(a+b+c) - \frac{f(2a+b) + f(2b+a)}{2},$$

as desired. □

Now we list some remarks concerning Theorem 2.1.

Remark 2.3.

(i) If we drop the condition $f(0) \le 0$, in Theorem 2.1, then we obtain

$$f\left(c - \frac{a+b}{2}\right) \le f(a+b+c) + f(0) - \frac{f(2a+b) + f(2b+a)}{2}. \tag{2.10}$$

(ii) Let $r = \min\{\nu, 1-v\}$, $R = \max\{\nu, 1-v\}$, and $0 < \nu < 1$ with $\nu \ne 1/2$. Setting $a = r/2$, $b = R/2$, and $c = 1/2$, in the inequality (2.10), we deduce:

$$\frac{1}{2}\left(f\left(\frac{1+r}{2}\right) + f\left(\frac{1+R}{2}\right)\right) \le f(0) + f(1) - f\left(\frac{1}{4}\right). \tag{2.11}$$

(iii) From (2.11), we can write

$$\begin{aligned} f\left(\frac{1}{2}\right) &\le \frac{1}{2}\left(f\left(\frac{3}{4}\right) + f\left(\frac{1}{4}\right)\right) \\ &\le \frac{1}{2}\left(\frac{1}{2}\left(f\left(\frac{1+r}{2}\right) + f\left(\frac{1+R}{2}\right)\right) + f\left(\frac{1}{4}\right)\right) \\ &\le \frac{f(0) + f(1)}{2} \end{aligned}$$

where the first and the second inequality are obtained from (1.2). Notice that this provides a considerable improvement of the convex inequality $f\left(\frac{1}{2}\right) \le \frac{f(0)+f(1)}{2}$.

The following is a difference-type inequality for convex functions, with an integral representing a mean value as part of the upper bound.

Theorem 2.2. *Let $f\colon [0,\infty) \to \mathbb{R}$ be a convex function with $f(0) \le 0$. If $0 \le b < a$, then*

$$f(a-b) \le \frac{1}{a-b}\int_{2a+b}^{3a} f(t)\,dt - \frac{f(2a+b)+f(2b+a)}{2}.$$

Proof. As we proved in Theorem 2.1,

$$f\left(c-\frac{a+b}{2}\right) \le f(a+b+c) - \frac{f(2a+b)+f(2b+a)}{2}.$$

For $\nu \ge 0$, we take $c = (1+\nu)a - \nu b$, which implies that $c \ge a \ge b$. So, we obtain

$$f\left((\nu+\frac{1}{2})(a-b)\right) \le f((2+\nu)a+(1-\nu)b) - \frac{f(2a+b)+f(2b+a)}{2}.$$

By integrating over ν on $[0,1]$, we have

$$\int_0^1 f\left((\nu+\frac{1}{2})(a-b)\right)d\nu \le \int_0^1 f((2+\nu)a+(1-\nu)b)\,d\nu - \int_0^1 \frac{f(2a+b)+f(2b+a)}{2}d\nu,$$

so, we deduce

$$\frac{1}{a-b}\int_{\frac{a-b}{2}}^{\frac{3(a-b)}{2}} f(t)\,dt \le \frac{1}{a-b}\int_{2a+b}^{3a} f(t)\,dt - \frac{f(2a+b)+f(2b+a)}{2}.$$

Finally, using the Hermite-Hadamard inequality, we have

$$f(a-b) \le \frac{1}{a-b}\int_{\frac{a-b}{2}}^{\frac{3(a-b)}{2}} f(t)\,dt.$$

Therefore, the inequality of the theorem follows. □

Remark 2.4. If we take $2a+b = x$ and $3a = y$, with $y \ge x \ge \frac{2y}{3}$, in the inequality from the above theorem, we obtain

$$f(y-x) \le \frac{1}{y-x}\int_x^y f(t)\,dt - \frac{f(x)+f(2x-y)}{2},$$

which implies inequality

$$f(y-x) + \frac{f(x)+f(2x-y)}{2} \le \frac{1}{y-x}\int_x^y f(t)\,dt.$$

Theorem 2.3. *Let $f\colon [0,\infty) \to \mathbb{R}$ be a convex function with $f(0) \le 0$. If $0 < a, b$ and $a+b \le c$ then,*

$$f\left(c-\frac{a+b}{2}\right) \le \frac{1}{a+b}\int_{c-a-b}^{c} f(t)\,dt \le f(a+b+c) - \frac{f(2a+b)+f(2b+a)}{2}.$$

Proof. Using inequalities (2.6), (2.7) and multiplying by ν and $1-\nu$, where $\nu \in [0,1]$, we have

$$\nu f(c-a) \le \nu f(a+b+c) - \nu f(2a+b) \tag{2.12}$$

and

$$(1-\nu) f(c-b) \le (1-\nu) f(a+b+c) - (1-\nu) f(2b+a). \tag{2.13}$$

By adding the two inequalities (2.12) and (2.13), we get

$$\nu f(c-a) + (1-\nu) f(c-b) \le f(a+b+c) - \nu f(2a+b) - (1-\nu) f(2b+a). \tag{2.14}$$

On the other hand, by the convexity of f, we have

$$f(c-\nu(a+b)) \le \nu f(c-a) + (1-\nu) f(c-b). \tag{2.15}$$

Taking into account the inequalities (2.14) and (2.15), we obtain

$$f\left(c-\nu(a+b)\right)\le f\left(a+b+c\right)-\nu f\left(2a+b\right)-(1-\nu)f\left(2b+a\right).$$

Integrating over ν from 0 to 1, we deduce

$$\int_0^1 f\left(c-\nu(a+b)\right)d\nu\le f\left(a+b+c\right)-\frac{f\left(2a+b\right)+f\left(2b+a\right)}{2}.$$

But, we have

$$\int_0^1 f\left(c-\nu(a+b)\right)d\nu=\frac{1}{a+b}\int_{c-a-b}^{c} f\left(t\right)dt$$

and using the Hermite-Hadamard inequality, we find

$$\frac{1}{a+b}\int_{c-a-b}^{c} f\left(t\right)dt\ge f\left(c-\frac{a+b}{2}\right).$$

Consequently, the statement is true. □

Remark 2.5. If we take $c=a+b$ in the inequality from the above theorem, we find

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{a+b}\int_0^{a+b} f\left(t\right)dt\le f\left(2(a+b)\right)-\frac{f\left(2a+b\right)+f\left(2b+a\right)}{2},$$

which implies inequality

$$f\left(\frac{a+b}{2}\right)+\frac{f\left(2a+b\right)+f\left(2b+a\right)}{2}\le f\left(2(a+b)\right).$$

But, using the convexity of f, we deduce

$$2f(a+b)\le f\left(\frac{a+b}{2}\right)+\frac{f\left(2a+b\right)+f\left(2b+a\right)}{2}\le f\left(2(a+b)\right). \tag{2.16}$$

Notice that this provides a refinement of the inequality $2f(a+b)\le f(2(a+b))$, which follows immediately from the super-additivity of f.

The following result provides an improvement and a counterpart for the inequality in Lemma 2.1.

Theorem 2.4. *Let $f\colon[0,\infty)\to\mathbb{R}$ be a convex function. Then for any $0\le a<b$,*

$$\begin{aligned}&f\left(b-a\right)+\left(f\left(0\right)+f\left(b\right)-2f\left(\frac{b}{2}\right)\right)\left(1-\frac{|b-2a|}{b}\right)\\&\le f\left(0\right)+f\left(b\right)-f\left(a\right)\\&\le f\left(b-a\right)+\left(f\left(0\right)+f\left(b\right)-2f\left(\frac{b}{2}\right)\right)\left(1+\frac{|b-2a|}{b}\right).\end{aligned}$$

Proof. It has been established in [10] that

$$\begin{aligned}2r\left(\frac{f\left(a\right)+f\left(b\right)}{2}-f\left(\frac{a+b}{2}\right)\right)&\le\left(1-\nu\right)f\left(a\right)+\nu f\left(b\right)-f\left(\left(1-\nu\right)a+\nu b\right)\\&\le 2R\left(\frac{f\left(a\right)+f\left(b\right)}{2}-f\left(\frac{a+b}{2}\right)\right)\end{aligned} \tag{2.17}$$

where $r=\min\left\{\nu,1-\nu\right\}$, $r=\max\left\{\nu,1-\nu\right\}$, and $0\le\nu\le1$. Since for any $s,t\ge0$,

$$\max\left\{s,t\right\}=\frac{s+t+|s-t|}{2}\text{ and }\min\left\{s,t\right\}=\frac{s+t-|s-t|}{2},$$

we can write (2.17), in the following format

$$(1-|1-2\nu|)\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)\le(1-\nu)f(a)+\nu f(b)-f((1-\nu)a+\nu b)$$
$$\le(1+|1-2\nu|)\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right).$$

Now, applying the above inequality, we have

$$\begin{aligned}
&\left(1-\frac{|x_1+x_2-2x|}{x_2-x_1}\right)\left(\frac{f(x_1)+f(x_2)}{2}-f\left(\frac{x_1+x_2}{2}\right)\right)\\
&\le\frac{x_2-x}{x_2-x_1}f(x_1)+\frac{x-x_1}{x_2-x_1}f(x_2)-f(x)\\
&\le\left(1+\frac{|x_1+x_2-2x|}{x_2-x_1}\right)\left(\frac{f(x_1)+f(x_2)}{2}-f\left(\frac{x_1+x_2}{2}\right)\right)
\end{aligned}\tag{2.18}$$

when $x_1<x<x_2$. Besides,

$$\begin{aligned}
&\left(1-\frac{|x_1+x_2-2x|}{x_2-x_1}\right)\left(\frac{f(x_1)+f(x_2)}{2}-f\left(\frac{x_1+x_2}{2}\right)\right)\\
&\le\frac{x-x_1}{x_2-x_1}f(x_1)+\frac{x_2-x}{x_2-x_1}f(x_2)-f(x_1+x_2-x)\\
&\le\left(1+\frac{|x_1+x_2-2x|}{x_2-x_1}\right)\left(\frac{f(x_1)+f(x_2)}{2}-f\left(\frac{x_1+x_2}{2}\right)\right).
\end{aligned}\tag{2.19}$$

The inequalities (2.18) and (2.19) give, respectively,

$$\left(1-\frac{|x_2-2x|}{x_2}\right)\left(\frac{f(0)+f(x_2)}{2}-f\left(\frac{x_2}{2}\right)\right)\le\frac{x_2-x}{x_2}f(0)+\frac{x}{x_2}f(x_2)-f(x)$$
$$\le\left(1+\frac{|x_2-2x|}{x_2}\right)\left(\frac{f(0)+f(x_2)}{2}-f\left(\frac{x_2}{2}\right)\right),$$

and

$$\left(1-\frac{|x_2-2x|}{x_2}\right)\left(\frac{f(0)+f(x_2)}{2}-f\left(\frac{x_2}{2}\right)\right)\le\frac{x}{x_2}f(0)+\frac{x_2-x}{x_2}f(x_2)-f(x_2-x)$$
$$\le\left(1+\frac{|x_2-2x|}{x_2}\right)\left(\frac{f(0)+f(x_2)}{2}-f\left(\frac{x_2}{2}\right)\right).$$

Thus, by adding the last two inequalities, we conclude that

$$2\left(1-\frac{|x_2-2x|}{x_2}\right)\left(\frac{f(0)+f(x_2)}{2}-f\left(\frac{x_2}{2}\right)\right)\le f(0)+f(x_2)-f(x)-f(x_2-x)$$
$$\le2\left(1+\frac{|x_2-2x|}{x_2}\right)\left(\frac{f(0)+f(x_2)}{2}-f\left(\frac{x_2}{2}\right)\right)$$

i.e.,

$$2\left(1-\frac{|b-2a|}{b}\right)\left(\frac{f(0)+f(b)}{2}-f\left(\frac{b}{2}\right)\right)\le f(0)+f(b)-f(a)-f(b-a)$$
$$\le2\left(1+\frac{|b-2a|}{b}\right)\left(\frac{f(0)+f(b)}{2}-f\left(\frac{b}{2}\right)\right)$$

as desired. □

Remark 2.6. The case $f(0)=0$, in Theorem 2.4, reduces to the following inequality:

$$2\left(1-\frac{|b-2a|}{b}\right)\left(\frac{f(b)}{2}-f\left(\frac{b}{2}\right)\right)\le f(b)-f(a)-f(b-a)$$
$$\le 2\left(1+\frac{|b-2a|}{b}\right)\left(\frac{f(b)}{2}-f\left(\frac{b}{2}\right)\right).$$

The following Hermite-Hadamard-type inequality can be stated as well. The significance of this result is the way it refines and reverses the Hermite-Hadamard inequality $\frac{2}{b}\int_0^b f(t)dt\le f(0)+f(b)$.

Corollary 2.1. *Let $f\colon[0,\infty)\to\mathbb{R}$ be a convex function and $0<b$. Then*

$$\frac{2}{b}\int_0^b f(t)\,dt+\frac{1}{2}\left(f(0)+f(b)-2f\left(\frac{b}{2}\right)\right)$$
$$\le f(0)+f(b)$$
$$\le \frac{2}{b}\int_0^b f(t)\,dt+\frac{3}{2}\left(f(0)+f(b)-2f\left(\frac{b}{2}\right)\right).$$

Proof. If we take integral over $t\in[0,b]$, in Theorem 2.4, we obtain

$$\frac{1}{b}\int_0^b f(b-t)\,dt+\left(f(0)+f(b)-2f\left(\frac{b}{2}\right)\right)\left(1-\frac{1}{b^2}\int_0^b|b-2t|\,dt\right)$$
$$\le f(0)+f(b)-\frac{1}{b}\int_0^b f(t)\,dt$$
$$\le \frac{1}{b}\int_0^b f(b-t)\,dt+\left(f(0)+f(b)-2f\left(\frac{b}{2}\right)\right)\left(1+\frac{1}{b^2}\int_0^b|b-2t|\,dt\right).$$

The result follows by seeing that $\frac{1}{b}\int_0^b f(b-t)\,dt=\frac{1}{b}\int_0^b f(t)\,dt$ and $\int_0^b|b-2t|\,dt=\frac{b^2}{2}$. □

Remark 2.7. The first inequality from the above corollary becomes

$$\frac{2}{b}\int_0^b f(t)\,dt\le\frac{f(0)+f(b)}{2}+f\left(\frac{b}{2}\right),$$

where $f:[0,\infty)\to\mathbb{R}$ is a convex function and $b>0$. This inequality represents the inequality of Bullen [9]

$$\frac{2}{b-a}\int_a^b f(t)\,dt\le\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right),$$

when $a=0$.

Inequality (2.17) can be written equivalently in the following form

$$\begin{aligned}2r\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right) &\le \nu_1 f(a)+\nu_2 f(b)-f(\nu_1 a+\nu_2 b)\\ &\le 2R\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)\end{aligned} \tag{2.20}$$

where $\nu_1,\nu_2 \ge 0$ with $\nu_1+\nu_2=1$. We continue this section by slightly improving the condition $\nu_1+\nu_2=1$ in (2.20).

Proposition 2.1. *Let $f: J\to\mathbb{R}$ be a convex function and let $a,b\in J$. If $\upsilon_1,\nu_2\ge 0$ satisfy $\nu_1+\nu_2\le 1$, then*

$$\begin{aligned}&f(\nu_1 a+\nu_2 b)\\ &\le \nu_1 f(a)+\nu_2 f(b)+(1-(\nu_1+\nu_2))f(0)\\ &\quad -2\left(r\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)-r'\left(\frac{f((\nu_1 a+\nu_2 b)/(\nu_1+\nu_2))+f(0)}{2}-f\left(\frac{\nu_1 a+\nu_2 b}{2(\nu_1+\nu_2)}\right)\right)\right)\end{aligned}$$

where $r=\min\{\nu_1,\nu_2\}$ and $r'=\min\{\nu_1+\nu_2,1-(\nu_1+\nu_2)\}$. In the opposite direction,

$$\begin{aligned}&f(\nu_1 a+\nu_2 b)\\ &\quad +2\left(R\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)-R'\left(\frac{f((\nu_1 a+\nu_2 b)/(\nu_1+\nu_2))+f(0)}{2}-f\left(\frac{\nu_1 a+\nu_2 b}{2(\nu_1+\nu_2)}\right)\right)\right)\\ &\ge \nu_1 f(a)+\nu_2 f(b)+(1-(\nu_1+\nu_2))f(0)\end{aligned}$$

where $R=\max\{\nu_1,\nu_2\}$ and $R'=\max\{\nu_1+\nu_2,1-(\nu_1+\nu_2)\}$.

Proof. We prove the first inequality. By the first inequality in (2.17), we have

$$\begin{aligned}&f(\nu_1 a+\nu_2 b)\\ &=f\left((\nu_1+\nu_2)\left(\frac{\nu_1}{\nu_1+\nu_2}a+\frac{\nu_2}{\nu_1+\nu_2}b\right)+(1-(\nu_1+\nu_2))\cdot 0\right)\\ &\le(\nu_1+\nu_2)f\left(\frac{\nu_1}{\nu_1+\nu_2}a+\frac{\nu_2}{\nu_1+\nu_2}b\right)+(1-(\nu_1+\nu_2))f(0)\\ &\quad -2r'\left(\frac{f((\nu_1 a+\nu_2 b)/(\nu_1+\nu_2))+f(0)}{2}-f\left(\frac{\nu_1 a+\nu_2 b}{2(\nu_1+\nu_2)}\right)\right)\\ &\le \nu_1 f(a)+\nu_2 f(b)+(1-(\nu_1+\nu_2))f(0)\\ &\quad -2\left(r\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)-r'\left(\frac{f((\nu_1 a+\nu_2 b)/(\nu_1+\nu_2))+f(0)}{2}-f\left(\frac{\nu_1 a+\nu_2 b}{2(\nu_1+\nu_2)}\right)\right)\right)\end{aligned}$$

which completes the proof.

Likewise, the second inequality can be obtained by utilizing the second inequality in (2.17); we omit its proof. □

Remark 2.8. Let the assumptions of Proposition 2.1 hold. If $f(0)=0$, then

$$\begin{aligned}&f(\nu_1 a+\nu_2 b)\\ &\le \nu_1 f(a)+\nu_2 f(b)\\ &\quad -2\left(r\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)-r'\left(\frac{1}{2}f\left(\frac{\nu_1 a+\nu_2 b}{\nu_1+\nu_2}\right)-f\left(\frac{\nu_1 a+\nu_2 b}{2(\nu_1+\nu_2)}\right)\right)\right)\end{aligned}$$

and

$$\begin{aligned}&f(\nu_1 a+\nu_2 b)\\ &\quad +2\left(R\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)-R'\left(\frac{1}{2}f\left(\frac{\nu_1 a+\nu_2 b}{\nu_1+\nu_2}\right)-f\left(\frac{\nu_1 a+\nu_2 b}{2(\nu_1+\nu_2)}\right)\right)\right)\\ &\ge \nu_1 f(a)+\nu_2 f(b).\end{aligned}$$

From the definition of convexity for function $f:[0,\infty)\to\mathbb{R}$, with $f(0)\le 0$, we deduce

$$f(tx)=f((1-t)0+tx)\le(1-t)f(0)+tf(x)\le tf(x)$$

for all $t\in[0,1]$. We will study the case when $t\ge 1$.

Lemma 2.2. *Let $f:[0,\infty)\to\mathbb{R}$ be a convex function. If $f(0)\le 0$, then for any $x\ge 0$,*

$$tf\left(x\right)\le f\left(tx\right),$$

where $t\ge 1$.

Proof. We consider $a,b\in[0,\infty)$ and $a\ge b$. Notice that if $\nu\ge 0$, then $a\le(1+\nu)a-\nu b$ and we deduce by the convexity of f,

$$\begin{aligned}f\left(a\right)&=f\left(\frac{1}{1+\nu}\left(\left(1+\nu\right)a-\nu b\right)+\frac{\nu}{1+\nu}b\right)\\&\le\frac{1}{1+\nu}f\left(\left(1+\nu\right)a-\nu b\right)+\frac{\nu}{1+\nu}f\left(b\right)\end{aligned}$$

i.e.,

$$\left(1+\nu\right)f\left(a\right)-\nu f\left(b\right)\le f\left(\left(1+\nu\right)a-\nu b\right).$$

The case $b=0$ implies

$$\left(1+\nu\right)f\left(a\right)\le\left(1+\nu\right)f\left(a\right)-\nu f\left(0\right)\le f\left(\left(1+\nu\right)a\right)$$

or

$$tf\left(x\right)\le f\left(tx\right);\ t\ge 1,$$

as desired. □

Remark 2.9. Inequality (2.16) represents an improvement of the result from the above lemma when $t=2$ and $x=a+b$.

If $f:(0,\infty)\to\mathbb{R}$, with $\lim\limits_{x\to 0}f(x)\le 0$, then using Lemma 2.2 the function $\frac{f(x)}{x}$ is increasing.

Interestingly, the above discussion can lead to new types of inequalities for convex functions.

Theorem 2.5. *Let $f:[0,\infty)\to\mathbb{R}$ be a convex function with $f(0)\le 0$. Then for any $a,b>0$,*

$$f\left(\frac{1}{a+b}\right)\le\frac{ab}{\left(a+b\right)^2}\left(f\left(\frac{1}{a}\right)+f\left(\frac{1}{b}\right)\right).$$

Proof. We can write by Lemma 2.2,

$$\begin{aligned}f\left(\frac{1}{a}\right)+f\left(\frac{1}{b}\right)&=f\left(\frac{a+b}{a}\frac{1}{a+b}\right)+f\left(\frac{a+b}{b}\frac{1}{a+b}\right)\\&\ge\frac{a+b}{a}f\left(\frac{1}{a+b}\right)+\frac{a+b}{b}f\left(\frac{1}{a+b}\right)\\&=\frac{\left(a+b\right)^2}{ab}f\left(\frac{1}{a+b}\right).\end{aligned}$$

After the rearrangement of the terms, we obtain

$$f\left(\frac{1}{a+b}\right)\le\frac{ab}{\left(a+b\right)^2}\left(f\left(\frac{1}{a}\right)+f\left(\frac{1}{b}\right)\right),$$

as desired. □

If we replace a and b by $1/a$ and $1/b$, in Theorem 2.5, we get:

Corollary 2.2. *Let $f:[0,\infty)\to\mathbb{R}$ be a convex function with $f(0)\le 0$. Then for any $a,b>0$,*

$$f\left(\frac{ab}{a+b}\right)\le\frac{ab}{(a+b)^2}\left(f(a)+f(b)\right).$$

Similarly to Theorem 2.5 and Corollary 2.2, we have the following $n-$tuple result.

Proposition 2.2. *Let $f:[0,\infty)\to\mathbb{R}$ be a convex function with $f(0)\le 0$. Then for any $a_i>0$ $(i=1,2,\ldots,n)$,*

$$\sum_{i=1}^{n}f(a_i)\ge n^2 f\left(\frac{1}{\sum_{i=1}^{n}\frac{1}{a_i}}\right).$$

Proof. By the use of Lemma 2.2, we have

$$\begin{aligned}
&f\left(\frac{1}{a_1}\right)+\cdots+f\left(\frac{1}{a_n}\right)\\
&=f\left(\frac{a_1+\cdots+a_n}{a_1}\frac{1}{a_1+\cdots+a_n}\right)+\cdots+f\left(\frac{a_1+\cdots+a_n}{a_n}\frac{1}{a_1+\cdots+a_n}\right)\\
&\ge\left(\frac{a_1+\cdots+a_n}{a_1}+\cdots+\frac{a_1+\cdots+a_n}{a_n}\right)f\left(\frac{1}{a_1+\cdots+a_n}\right)\\
&=\left(\sum_{i=1}^{n}a_i\right)\left(\sum_{i=1}^{n}\frac{1}{a_i}\right)f\left(\frac{1}{\sum_{i=1}^{n}a_i}\right)\ge n^2 f\left(\frac{1}{\sum_{i=1}^{n}a_i}\right),
\end{aligned}$$

where the last inequality is due to Cauchy-Schwarz inequality. We obtain the result by replacing a_i with $\frac{1}{a_i}$ in the above inequality. This completes the proof. □

From the proof of the above proposition, we find that

$$\frac{\sum_{i=1}^{n}f(a_i)}{\sum_{i=1}^{n}a_i}\ge\frac{f\left(\frac{1}{\sum_{i=1}^{n}\frac{1}{a_i}}\right)}{\sum_{i=1}^{n}\frac{1}{a_i}}.$$

Before expressing the following result, we remind the reader that a function f is called logarithmically convex if $\log f$ is convex.

Theorem 2.6. *Let $f:[0,\infty)\to(0,\infty)$ be a log-convex function, with $f(0)\le 1$. Then for any $a,b>0$,*

$$f\left(\frac{ab}{a+b}\right)\le\left(f(a)f(b)\right)^{\frac{ab}{(a+b)^2}}.$$

Proof. The assumption on f, together with Corollary 2.2, gives

$$\begin{aligned}
\log f\left(\frac{ab}{a+b}\right)&\le\frac{ab}{(a+b)^2}\left(\log f(a)+\log f(b)\right)\\
&=\frac{ab}{(a+b)^2}\left(\log f(a)f(b)\right)\\
&=\log\left(f(a)f(b)\right)^{\frac{ab}{(a+b)^2}}.
\end{aligned}$$

Consequently,

$$f\left(\frac{ab}{a+b}\right)\le\left(f(a)f(b)\right)^{\frac{ab}{(a+b)^2}},$$

as desired. □

The following result provides an improvement and a reverse for the inequality in Lemma 2.2.

Theorem 2.7. *Let $f : [0,\infty) \to \mathbb{R}$ be a convex function and $t \geq 1$. If $f(0) = 0$, then for any $x \in [0,\infty)$,*

$$tf(x) \leq f(tx) - (t - |2-t|)\left(\frac{f(tx)}{2} - f\left(\frac{tx}{2}\right)\right),$$

and

$$tf(x) \geq f(tx) + (t + |2-t|)\left(\frac{f(tx)}{2} - f\left(\frac{tx}{2}\right)\right).$$

Proof. We consider $a, b \in [0,\infty)$ and $a \geq b$. Notice that if $\nu \geq 0$, then $a \leq (1+\nu)a - \nu b$ and we deduce by the convexity of f

$$\begin{aligned} &f(a) \\ &= f\left(\frac{1}{1+\nu}((1+\nu)a - \nu b) + \frac{\nu}{1+\nu}b\right) \\ &\leq \frac{1}{1+\nu}f((1+\nu)a-\nu b) + \frac{\nu}{1+\nu}f(b) - 2\lambda'\left(\frac{f((1+\nu)a-\nu b)+f(b)}{2} - f\left(\frac{(1+\nu)a+(1-\nu)b}{2}\right)\right)\end{aligned}$$

where $\lambda' = \min\left\{\frac{1}{1+\nu}, \frac{\nu}{1+\nu}\right\}$. Multiplying the above inequality by $1+\nu$, we reach

$$(1+\nu)f(a) - \nu f(b) \leq f((1+\nu)a - \nu b) - 2\lambda\left(\frac{f((1+\nu)a-\nu b)+f(b)}{2} - f\left(\frac{(1+\nu)a+(1-\nu)b}{2}\right)\right)$$

where $\lambda = \min\{1,\nu\}$. Now, by replacing a and b by x and 0, respectively, we infer, when $1+\nu = t$, that

$$tf(x) \leq f(tx) - (t - |2-t|)\left(\frac{f(tx)}{2} - f\left(\frac{tx}{2}\right)\right).$$

For the counterpart, we have

$$\begin{aligned} &f(a) \\ &= f\left(\frac{1}{1+\nu}((1+\nu)a - \nu b) + \frac{\nu}{1+\nu}b\right) \\ &\geq \frac{1}{1+\nu}f((1+\nu)a-\nu b) + \frac{\nu}{1+\nu}f(b) + 2\gamma'\left(\frac{f((1+\nu)a-\nu b)+f(b)}{2} - f\left(\frac{(1+\nu)a+(1-\nu)b}{2}\right)\right)\end{aligned}$$

where $\gamma' = \max\left\{\frac{1}{1+\nu}, \frac{\nu}{1+\nu}\right\}$. Multiplying the above inequality by $1+\nu$ to get

$$(1+\nu)f(a) + \nu f(b) \geq f((1+\nu)a - \nu b) + 2\gamma\left(\frac{f((1+\nu)a-\nu b)+f(b)}{2} - f\left(\frac{(1+\nu)a+(1-\nu)b}{2}\right)\right)$$

where $\gamma = \max\{1,\nu\}$. Letting $a = x$, $b = 0$ and $1+\nu = t$, we have

$$tf(x) \geq f(tx) + (t + |2-t|)\left(\frac{f(tx)}{2} - f\left(\frac{tx}{2}\right)\right)$$

as desired. □

2.2. **Subadditive Inequalities for Superquadratic Functions.** We start this section by presenting the following result concerning the superadditivity of superquadratic functions.

Theorem 2.8. *Let $f : [0,\infty) \to \mathbb{R}$ be a superquadratic function. Then for any $a, b > 0$,*

$$f(a) + f(b) \leq f(a+b) - \frac{2a}{a+b}f(b) - \frac{2b}{a+b}f(a). \tag{2.21}$$

If f is subquadratic, then the inequality in (2.21) *is reversed.*

Proof. It follows from (1.4) and (P_1) that

$$\begin{aligned} f(\nu x) &\le (1-\nu) f(0) + \nu f(x) - \nu f((1-\nu)x) - (1-\nu) f(\nu x) \\ &\le \nu f(x) - \nu f((1-\nu)x) - (1-\nu) f(\nu x). \end{aligned} \tag{2.22}$$

Employing (2.22), we can write

$$\begin{aligned} & f(a) \\ &= f\left(\frac{a}{a+b}\cdot(a+b)\right) \\ &\le \frac{a}{a+b} f(a+b) - \frac{a}{a+b} f\left(\frac{b}{a+b}\cdot(a+b)\right) - \frac{b}{a+b} f\left(\frac{a}{a+b}\cdot(a+b)\right) \\ &= \frac{a}{a+b} f(a+b) - \frac{a}{a+b} f(b) - \frac{b}{a+b} f(a). \end{aligned} \tag{2.23}$$

Likewise,

$$f(b) \le \frac{b}{a+b} f(a+b) - \frac{b}{a+b} f(a) - \frac{a}{a+b} f(b). \tag{2.24}$$

Incorporating two inequalities (2.23) and (2.24) implies

$$f(a) + f(b) \le f(a+b) - \frac{2a}{a+b} f(b) - \frac{2b}{a+b} f(a),$$

as expected.

The second statement follows straightforwardly from the fact that if f is subquadratic, then $-f$ is superquadratic. □

As noted in the introduction, if a function f is superquadratic and nonnegative, it is convex. Consequently, in this case, inequality (2.21) is an improvement of inequality (1.5) for convex functions.

Remark 2.10. The inequality in Theorem 2.8 can be written as

$$\frac{a+3b}{a+b} f(a) + \frac{3a+b}{a+b} f(b) \le f(a+b).$$

The following theorem presents the external version of inequality (1.4).

Theorem 2.9. *Let $f : [0,\infty) \to \mathbb{R}$ be a superquadratic function and let $a, b \ge 0$.*

(i) *If $\nu > 0$ and $(1+\nu)a - \nu b \ge 0$, then*

$$(1+\nu) f(a) - \nu f(b) \le f((1+\nu)a - \nu b) - \nu f\left(\frac{|a-b|}{1+\nu}\right) - f\left(\frac{\nu|a-b|}{1+\nu}\right). \tag{2.25}$$

(ii) *If $\nu < -1$ and $(1+\nu)a - \nu b \ge 0$, then*

$$(1+\nu) f(a) - \nu f(b) \le f((1+\nu)a - \nu b) + (1+\nu) f\left(-\frac{|a-b|}{\nu}\right) - f\left(\frac{(1+\nu)|a-b|}{\nu}\right). \tag{2.26}$$

Moreover, inequalities (2.25) *and* (2.26) *hold in the reversed direction if f is subquadratic.*

Proof. We first consider the case $\nu > 0$. By (1.4), we obtain

$$\begin{aligned} & f(a) \\ &= f\left(\frac{1}{1+\nu}((1+\nu)a - \nu b) + \frac{\nu}{1+\nu} b\right) \\ &\le \frac{1}{1+\nu} f((1+\nu)a - \nu b) + \frac{\nu}{1+\nu} f(b) - \frac{\nu}{1+\nu} f\left(\frac{|a-b|}{1+\nu}\right) - \frac{1}{1+\nu} f\left(\frac{\nu|a-b|}{1+\nu}\right). \end{aligned}$$

This finishes the proof of assertion (i).

Now, we consider the case $\nu < -1$. Again, by (1.4), we can write

$$\begin{aligned} & f(b) \\ & = f\left(-\frac{1}{\nu}\left((1+\nu)a - \nu b\right) + \frac{1+\nu}{\nu}a\right) \\ & \le -\frac{1}{\nu}f\left((1+\nu)a - \nu b\right) + \frac{1+\nu}{\nu}f(a) - \frac{1+\nu}{\nu}f\left(-\frac{|a-b|}{\nu}\right) + \frac{1}{\nu}f\left(\frac{(1+\nu)|a-b|}{\nu}\right). \end{aligned}$$

This completes the proof of (ii). □

We infer the following if we replace ν by $-\nu$ in Theorem 2.9.

Corollary 2.3. *Let $f : [0,\infty) \to \mathbb{R}$ be a superquadratic function and let $a, b \ge 0$.*

(i) *If $\nu < 0$ and $(1-\nu)a + \nu b \ge 0$, then*

$$(1-\nu)f(a) + \nu f(b) \le f\left((1-\nu)a + \nu b\right) + \nu f\left(\frac{|a-b|}{1-\nu}\right) - f\left(\frac{\nu|a-b|}{\nu-1}\right).$$

(ii) *If $\nu > 1$ and $(1-\nu)a + \nu b \ge 0$, then*

$$(1-\nu)f(a) + \nu f(b) \le f\left((1-\nu)a + \nu b\right) + (1-\nu)f\left(\frac{|a-b|}{\nu}\right) - f\left(\frac{(\nu-1)|a-b|}{\nu}\right).$$

The following inequalities can be expressed as well:

Theorem 2.10. *Let $f : [0,\infty) \to \mathbb{R}$ be a superquadratic function. Then for any $a, b > 0$,*

$$\frac{(a+b)^2}{ab}f\left(\frac{1}{a+b}\right) + \frac{a+b}{a}f\left(\frac{a}{(a+b)^2}\right) + \frac{a+b}{b}f\left(\frac{b}{(a+b)^2}\right) \le f\left(\frac{1}{a}\right) + f\left(\frac{1}{b}\right).$$

The opposite inequality holds if f is subquadratic.

Proof. It follows, from Theorem 2.9 (i), that

$$\nu f(x) \le f(\nu x) - (\nu-1)f\left(\frac{x}{\nu}\right) - f\left(\frac{(\nu-1)x}{\nu}\right), \tag{2.27}$$

for any $\nu > 1$ and $x > 0$. Here we used the fact that $-(\nu-1)f(0) \ge 0$.

Using (2.27), we reach

$$\begin{aligned} f\left(\frac{1}{a}\right) & = f\left(\frac{a+b}{a}\cdot\frac{1}{a+b}\right) \\ & \ge \frac{a+b}{a}f\left(\frac{1}{a+b}\right) + \frac{b}{a}f\left(\frac{a}{a+b}\cdot\frac{1}{a+b}\right) + f\left(\frac{b}{a+b}\cdot\frac{1}{a+b}\right) \\ & = \frac{a+b}{a}f\left(\frac{1}{a+b}\right) + \frac{b}{a}f\left(\frac{a}{(a+b)^2}\right) + f\left(\frac{b}{(a+b)^2}\right). \end{aligned} \tag{2.28}$$

Likewise,

$$f\left(\frac{1}{b}\right) \ge \frac{a+b}{b}f\left(\frac{1}{a+b}\right) + \frac{a}{b}f\left(\frac{b}{(a+b)^2}\right) + f\left(\frac{a}{(a+b)^2}\right). \tag{2.29}$$

Adding two inequalities (2.28) and (2.29) together, we reach

$$\frac{(a+b)^2}{ab}f\left(\frac{1}{a+b}\right) + \frac{a+b}{a}f\left(\frac{a}{(a+b)^2}\right) + \frac{a+b}{b}f\left(\frac{b}{(a+b)^2}\right) \le f\left(\frac{1}{a}\right) + f\left(\frac{1}{b}\right),$$

as expected. □

If we replace a and b by $1/a$ and $1/b$, respectively, in Theorem 2.10, we get the following form.

Corollary 2.4. *Let $f : [0,\infty) \to \mathbb{R}$ be a superquadratic function. Then for any $a, b > 0$,*

$$\frac{(a+b)^2}{ab} f\left(\frac{ab}{a+b}\right) + \frac{a+b}{b} f\left(\frac{ab^2}{(a+b)^2}\right) + \frac{a+b}{a} f\left(\frac{a^2 b}{(a+b)^2}\right) \le f(a) + f(b).$$

The following result presents a reverse of inequality (1.4).

Theorem 2.11. *Let $f : [0,\infty) \to \mathbb{R}$ be a superquadratic function and let $a, b \ge 0$.*

(i) *If $0 \le \nu \le 1/2$, then*

$$(1-\nu) f(a) + \nu f(b) - 2(1-\nu)\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right) + (1-2\nu) f\left(\frac{|a-b|}{4(1-\nu)}\right) + f\left(\frac{(1-2\nu)|a-b|}{4(1-\nu)}\right)$$
$$f((1-\nu)a + \nu b).$$

(ii) *If $1/2 \le \nu \le 1$, then*

$$(1-\nu) f(a) + \nu f(b) - 2\nu\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right) + (2\nu-1) f\left(\frac{|b-a|}{4\nu}\right) + f\left(\frac{2\nu-1}{4\nu}|b-a|\right)$$
$$\le f((1-\nu)a + \nu b).$$

The opposite inequalities hold if f is subquadratic.

Proof. First suppose that $0 \le \nu \le 1/2$. In this case, we have

$$\begin{aligned}
&f((1-\nu)a + \nu b) \\
&= f\left((1+(1-2\nu))\frac{a+b}{2} - (1-2\nu) b\right) \\
&\ge (1+(1-2\nu)) f\left(\frac{a+b}{2}\right) - (1-2\nu) f(b) + (1-2\nu) f\left(\frac{|a-b|}{4(1-\nu)}\right) + f\left(\frac{(1-2\nu)|a-b|}{4(1-\nu)}\right) \\
&= (1-\nu) f(a) + \nu f(b) - 2(1-\nu)\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right) + (1-2\nu) f\left(\frac{|a-b|}{4(1-\nu)}\right) + f\left(\frac{(1-2\nu)|a-b|}{4(1-\nu)}\right),
\end{aligned}$$

thanks to part (i) of Theorem 2.9. This confirms the inequality in (i).

Now, assume that $1/2 \le \nu \le 1$. In this circumstance, we have again by the part (i) of Theorem 2.9,

$$\begin{aligned}
&f((1-\nu)a + \nu b) \\
&= f\left((1+(2\nu-1))\frac{a+b}{2} - (2\nu-1) a\right) \\
&\ge (1+(2\nu-1)) f\left(\frac{a+b}{2}\right) - (2\nu-1) f(a) + (2\nu-1) f\left(\frac{|b-a|}{4\nu}\right) + f\left(\frac{(2\nu-1)|b-a|}{4\nu}\right) \\
&= (1-\nu) f(a) + \nu f(b) - 2\nu\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right) + (2\nu-1) f\left(\frac{|b-a|}{4\nu}\right) + f\left(\frac{(2\nu-1)|b-a|}{4\nu}\right).
\end{aligned}$$

This finalizes the proof of the inequality in (ii). □

We close this paper with the following corollary, which is a complementary result to Theorem 2.8.

Corollary 2.5. *Let $f : [0,\infty) \to \mathbb{R}$ be a superquadratic function and let $a, b > 0$.*

(i) *If $a \le b$, then*

$$f(a+b) - \frac{4b}{a+b}\left(\frac{f(a+b)}{2} - f\left(\frac{a+b}{2}\right)\right) + 2\left(\frac{b-a}{a+b} f\left(\frac{(a+b)^2}{4b}\right) + f\left(\frac{b^2-a^2}{4b}\right)\right)$$
$$\le f(a) + f(b).$$

(ii) $b \le a$, *then*

$$f(a+b) - \frac{4a}{a+b}\left(\frac{f(a+b)}{2} - f\left(\frac{a+b}{2}\right)\right) + 2\left(\frac{a-b}{a+b} f\left(\frac{(a+b)^2}{4a}\right) + f\left(\frac{a^2-b^2}{4a}\right)\right)$$
$$\le f(a) + f(b).$$

If f is subquadratic, then the inequalities hold in the reversed direction.

Proof. From Theorem 2.11 (i), we conclude that

(2.30)
$$\nu f(x) - 2(1-\nu)\left(\frac{f(x)}{2} - f\left(\frac{x}{2}\right)\right) + (1-2\nu) f\left(\frac{x}{4(1-\nu)}\right) + f\left(\frac{(1-2\nu)x}{4(1-\nu)}\right) \le f(\nu x),$$

whenever $0 \le \nu \le 1/2$. Also, by Theorem 2.11 (ii), we get

$$\nu f(x) - 2\nu\left(\frac{f(x)}{2} - f\left(\frac{x}{2}\right)\right) + (2\nu-1) f\left(\frac{x}{4\nu}\right) + f\left(\frac{(2\nu-1)x}{4\nu}\right) \le f(\nu x), \tag{2.31}$$

provided that $1/2 \le \nu \le 1$.

We first prove the case $a \le b$. From the inequality (2.30), we receive

$$\frac{a}{a+b} f(a+b) - \frac{2b}{a+b}\left(\frac{f(a+b)}{2} - f\left(\frac{a+b}{2}\right)\right) + \frac{b-a}{a+b} f\left(\frac{(a+b)^2}{4b}\right) + f\left(\frac{b^2-a^2}{4b}\right) \le f(a).$$

On the other hand, by the inequality (2.31), we have

$$\frac{b}{a+b} f(a+b) - \frac{2b}{a+b}\left(\frac{f(a+b)}{2} - f\left(\frac{a+b}{2}\right)\right) + \frac{b-a}{a+b} f\left(\frac{(a+b)^2}{4b}\right) + f\left(\frac{b^2-a^2}{4b}\right) \le f(b).$$

Adding these two inequalities, we reach the expected inequality.

Now we are going to establish the case $b \le a$. From the inequality (2.31), we reach

$$\frac{a}{a+b} f(a+b) - \frac{2a}{a+b}\left(\frac{f(a+b)}{2} - f\left(\frac{a+b}{2}\right)\right) + \frac{a-b}{a+b} f\left(\frac{(a+b)^2}{4a}\right) + f\left(\frac{a^2-b^2}{4a}\right) \le f(a).$$

The inequality (2.30) also indicates

$$\frac{b}{a+b} f(a+b) - \frac{2a}{a+b}\left(\frac{f(a+b)}{2} - f\left(\frac{a+b}{2}\right)\right) + \frac{a-b}{a+b} f\left(\frac{(a+b)^2}{4a}\right) + f\left(\frac{a^2-b^2}{4a}\right) \le f(b).$$

Merging these two inequalities implies the desired inequality. □

References

[1] Abramovich, S.; Jameson, G.; Sinnamon, G. Inequalities for averages of convex and superquadratic functions. *J. Inequal. Pure and Appl. Math.* **5** (2004), no. 4 article 91, 14 pp.

[2] Abramovich, S.; Jameson, G.; Sinnamon, G. Refining Jensen's inequality. *Bull. Math. Sc. Math. Roum.* **47 (95)** (2004), no. (1-2), 3–14.

[3] Abramovich, S.; Banić, S.; Matić, M. Superquadratic functions in several variables. *J. Math. Anal. Appl.* **327** (2007), no. 2, 1444–1460.

[4] Abramovich, S.; Barić, J.; Pečarić, J. Fejér and Hermite–Hadamard type inequalities for superquadratic functions. *J. Math. Anal. Appl.* **344** (2008), no. 2, 1048–1056.

[5] Barić, J.; Matković, A.; Pečarić, J. A variant of the Jensen-Mercer operator inequality for superquadratic functions. *Math Comput Model.* **51** (2010), no. (9-10), 1230–1239.

[6] Beckenbach, E. F. Superadditivity inequalities. *Pacific J. Math.* **14** (1964), no. 2, 421–438.

[7] Bruckner, A. Some relations between locally superadditive functions and convex functions. *Proc. Amer. Math. Soc.* **15** (1964), 61–65.

[8] Bruckner, A.; Ostrow, E. Some function classes related to the class of convex functions. *Pacific J. Math.* **12** (1962), 1203–1215.

[9] Cerone, P.; Dragomir, S. S. *Midpoint-type rules from an inequalities point of view, Handbook of Analytic-Computational Methods in Applied Mathematics*. Editor: G. Anastassiou, CRC Press, New York, (2000), 135–200.
[10] Dragomir, S. S. Bounds for the normalised Jensen functional. *Bull. Austral. Math. Soc.* **3** (2006), 471–478.
[11] Gilányi, A.; Troczka-Pawelec, K. On two different concepts of subquadracity. *Inequalities and applications* 2010, 209–216, no. 161, Birkhäuser/Springer, Basel (2012).
[12] Gilányi, A.; Troczka-Pawelec, K. Regularity of weakly subquadratic functions. *J. Math. Anal. Appl.* **382** (2011), 814–821.
[13] Gümüş, I. H.; Moradi, H. R.; Sababheh, M. Further subadditive matrix inequalities. *Math. Inequal. Appl.* **23** (2020), no. 3, 1127–1134.
[14] Marshall, A. W.; Proschan, F. An inequality for convex functions involving majorization. *J. Math. Anal. Appl.* **12** (1965), 87–90.
[15] Mercer, A. M. A variant of Jensen's inequality. *J. Inequal. Pure Appl. Math.* **4** (2003), no. 4, Article 73, 2 pp.
[16] Moradi, H. R.; Furuichi, S.; Sababheh, M. A sharper bound for the Jensen's operator inequality. *J. Nonlinear Convex Anal.* **22** (2021), no. 4, 747–758.
[17] Moradi, H. R.; Heydarbeygi, Z.; Sababheh, M. Subadditive inequalities for operators. *Math. Inequal. Appl.* **23** (2020), no. 1, 317–327.
[18] Moradi, H. R.; Sababheh, M.; Furuichi, S. On the Operator Hermite-Hadamard Inequality. *Complex Anal. Oper. Theory* **15** (2021), no. 8, Paper No. 122, 13 pp.
[19] Nasiri, L.; Zardadi, A.; Moradi, H. R. Refining and reversing Jensen's inequality. *Oper. Matrices.* **16** (2022), no. 1, 19–27.
[20] Niculescu, C. P.; Persson, L.-E. *Convex functions and their applications*. New York: Springer, 2006.
[21] Oguntuase, A.; Persson, L.-E. Refinement of Hardy's inequalities via superquadratic and subquadratic functions. *J. Math. Anal. Appl.* **339** (2008), 1305–1312.
[22] Roberts, A. W.; Varberg, D. E. *Convex functions*. Academic Press, New York, 1973.
[23] Sababheh, M.; Furuichi, S.; Moradi, H. R. Composite convex functions. *J. Math. Inequal.* **15** (2021), no. 3, 1267–1285.
[24] Sababheh, M.; Moradi, H. R.; Furuichi, S. Integrals refining convex inequalities. *Bull. Malaysian Math. Sci. Soc.* **43** (2020), no. 3, 2817–2833.
[25] Sababheh, M.; Moradi, H. R. Radical convex functions. *Mediterr. J. Math.* **18** (2021), no. 4, Paper No. 137, 15 pp.
[26] Smajdor, W. *Subadditive and Subquadratic set-valued functions* Sci. Publicaions of the University of Silezia, 889, Katowice (1987).
[27] Wright, E. M. An inequality for convex functions. *Amer. Math. Monthly.* **61** (1954), no. 9, 620–622.

DEPARTMENT OF MATHEMATICS
PAYAME NOOR UNIVERSITY (PNU)
P.O. BOX, 19395-4697, TEHRAN, IRAN
Email address: hrmoradi@mshdiau.ac.ir

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
TRANSILVANIA UNIVERSITY OF BRAŞOV
IULIU MANIU 50, BRAŞOV, 500091, ROMANIA
Email address: minculete.nicusor@unitbv.ro

DEPARTMENT OF INFORMATION SCIENCE
COLLEGE OF HUMANITIES AND SCIENCES, NIHON UNIVERSITY
SETAGAYA-KU, TOKYO, JAPAN
Email address: furuichi.shigeru@nihon-u.ac.jp

DEPARTMENT OF MATHEMATICS
PRINCESS SUMAYA UNIVERSITY FOR TECHNOLOGY
AL JUBAIHA, AMMAN 11941, JORDAN
Email address: sababheh@psut.edu.jo