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Stability and bifurcation analysis of a four-dimensional economic model

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ABSTRACT. In this work a four dimensional financial system, depending on five parameters is analyzed. The system was obtained by adding a new feed-back control variable to a well-known 3D financial system, modelling the evolution of the interest rate, the investment demand and the inflation rate. The stability of economical relevant equilibria is established. All the Hopf singularities were analyzed and the existence of supercritical, subcritical, and degenerated Hopf bifurcations was proved. Corresponding to them we found stable limit cycles, saddle type limit cycles. In addition, for certain parameters strata, one of the equilibria becomes a center, an approximate two-dimensional center manifold is determined and isolated periodic solutions are emphasized numerically. A double-Hopf degenerated bifurcation is also found. The addition of the feed-back led to obtain new possibilities to stabilize the economic environment, either to a new stable equilibrium state or to stable periodic behavior.

1. INTRODUCTION

Starting with the three-dimensional financial system reported in [8]

$$\left\{\begin{array}{l} \dot{x} = z + x\left(y - a\right), \\ \dot{y} = 1 - by - x^2, \\ \dot{z} = -x - cz, \end{array}\right. \tag{1.1}$$

where $x = x\left(t\right)$ is the real interest rate (calculated as the difference between the nominal interest rate and the inflation rate), $y = y\left(t\right)$ is the investment demand, $z = z\left(t\right)$ is the inflation rate, $a \in \mathbb{R}$ is the saving amount, $b \geq 0$ is the cost per investment, $c > 0$ is the elasticity of the demand on the commercial market, a new four-dimensional system was introduced in [4]:

$$\left\{\begin{array}{l} \dot{x} = z + x\left(y - a\right), \\ \dot{y} = 1 - by - x^2, \\ \dot{z} = -x - cz + u, \\ \dot{u} = -dxy - ku - mz, \end{array}\right. \tag{1.2}$$

where u denotes a control input and economically state intervention to balance the economic environment. Apart from the study of stability of equilibria and of periodic solutions emerging through Hopf bifurcations, the authors designed an electronic circuit of the system, in order to realize simulation outputs and oscilloscope outputs.

System (1.1) have been widely investigated, with emphasis on stability, Hopf bifurcation ([8], [9], [10]), periodic solutions, chaos, global dynamics [1], [3], [6], [7], [11], [12], [21], [22]. Other 4D systems, obtained by adding a new equation to system (1.1) with certain economic significance, were proposed and analysed in [13], [14].

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In the present paper, we use a feed-back control function $u(t) = u(0)e^{\int_0^t (n-dy(t))dt}$, depending only on the investment demand y(t) and actioning as an external forcing on the second equation in system (1.1). Here n and d are real parameters, illustrating the relation between the investment demand and the government regulations.

As $\dot{u} = nu - dyn$, the new model is given by

$$(1.3) \qquad \begin{cases} \dot{x} = z + x\,(y-a)\,, \\ \dot{y} = 1 - by - x^2 - u, \\ \dot{z} = -x - cz, \\ \dot{u} = nu - dyu, \end{cases}$$

In this paper we investigate local stability of equilibria and analyze various bifurcations corresponding to them. The system may have three, four, or an infinity of equilibria. Three of these equilibria, denoted by $P_1 = (0, \frac{1}{b}, 0, 0)$, $P_{3,4} = \left(\pm\sqrt{\alpha}, \frac{ac+1}{c}, \mp\frac{1}{c}\sqrt{\alpha}, 0\right)$, with $\alpha = 1 - \frac{b(1+ac)}{c} > 0$, correspond to the equilibria of the 3-dimensional system (1.1). We establish the topological type of these equilibria and prove the existence of Hopf and Bautin bifurcations at P_1, for certain values of the parameters, leading to the existence of stable or unstable limit cycles.

The other equilibrium, denoted by $P_2 = (0, \frac{n}{d}, 0, 1 - \frac{b}{d}n)$, has no correspondent in the equilibria of system (1.1) and it is the source of a rich dynamic behavior. Most of the results in our research refer to this equilibrium point. First, we establish the topological type of P_2 and investigate the existence of Hopf bifurcations. Depending on the parameters values, these bifurcations may be subcritical, supercritical, or degenerate. By computing the first two Lyapunov coefficients, we find parameter strata corresponding to Bautin bifurcations. Also, we determine strata where the first three Lyapunov coefficients are zero, thus the equilibrium point could be a nonlinear centre. We compute the corresponding center manifold and use numerical simulations [2] to illustrate the theoretical results. Finally, we prove the existence of a double-Hopf degenerate bifurcation, by using a technique in [5].

The paper is organized as follows. In Section 2, the stability of equilibria is established, and the corresponding Hopf and Bautin bifurcations are analyzed. In Section 3 the degenerate Hopf singularity P_2 is investigated, while in Section 4 a degenerate Hopf-Hopf bifurcation at P_2 is emphasized. Finally, some conclusions are given.

2. Local analysis

If $b \neq 0$, $d \neq 0$ and $\alpha > 0$ then system (1.3) has 4 equilibrium points: $P_1 = (0, \frac{1}{b}, 0, 0)$, $P_2 = (0, \frac{n}{d}, 0, 1 - \frac{b}{d}n)$, $P_3 = (\sqrt{\alpha}, \frac{ac+1}{c}, -\frac{1}{c}\sqrt{\alpha}, 0)$ and $P_4 = (-\sqrt{\alpha}, \frac{ac+1}{c}, \frac{1}{c}\sqrt{\alpha}, 0)$,where $\alpha = 1 - \frac{b(1+ac)}{c} > 0$.

In addition, if $b \neq 0$ and $\frac{ac+1}{c} = \frac{n}{d}$, there also exist the equilibria

$$Q_s = \left(s, \frac{ac+1}{c}, -\frac{s}{c}, \alpha - s^2\right), s \in \mathbb{R},$$

defining a curve $\Gamma = \left\{\left(s, \frac{ac+1}{c}, -\frac{s}{c}, \alpha - s^2\right), s \in \mathbb{R}\right\} \subset \mathbb{R}^4$.

If $b = 0$, $d \neq 0$, then system (1.3) has 3 equilibrium points: $P_2 = (0, \frac{n}{d}, 0, 1)$, $P_3 = (1, \frac{ac+1}{c}, -\frac{1}{c}, 0)$ and $P_4 = (-1, \frac{ac+1}{c}, \frac{1}{c}, 0)$. Also, if $d = 0$, only equilibria P_1, P_3, P_4 exist.

Consider the Jacobian matrix of system (1.3) at an equilibrium point $P_0 = (x_0, y_0, z_0, u_0)$, namely:

$$J(x_0, y_0, z_0, u_0) = \begin{pmatrix} y_0 - a & x_0 & 1 & 0 \\ -2x_0 & -b & 0 & -1 \\ -1 & 0 & -c & 0 \\ 0 & -du_0 & 0 & n - dy_0 \end{pmatrix}. \tag{2.4}$$

The translation $\tau(P_0)$ defined as

$$x_1 = x - x_0, x_2 = y - y_0, x_3 = z - z_0, x_4 = u - u_0, \tag{2.5}$$

brings the equilibrium P_0 at the origin of the system

$$\dot{x} = J(x_0, y_0, z_0, u_0)\, x + \frac{1}{2} B(x, x),$$

where

$$B(x, y) = \begin{pmatrix} x_1 y_2 + y_1 x_2 \\ -2x_1 y_1 \\ 0 \\ -d(x_2 y_4 + y_2 x_4) \end{pmatrix}. \tag{2.6}$$

2.1. **Stability and Hopf bifurcation at P_1.** The equilibrium $P_1 = (0, \frac{1}{b}, 0, 0)$ exists as $b > 0$. At P_1, the characteristic equation of the Jacobian matrix $J_1 = J(P_1)$ reads

$$(\lambda + b)\left(\lambda + \frac{d}{b} - n\right)\left[\lambda^2 + \lambda\left(a + c - \frac{1}{b}\right) + 1 + ac - \frac{c}{b}\right] = 0, \tag{2.7}$$

thus the eigenvalues satisfy

$$\begin{aligned} \lambda_1 &= -b, \quad \lambda_2 = n - \frac{d}{b}, \\ \lambda_3 \lambda_4 &= 1 + ac - \frac{c}{b}, \quad \lambda_3 + \lambda_4 = \frac{1}{b} - a - c. \end{aligned}$$

The following results are easily obtained:

Theorem 2.1. *The equilibrium P_1 is a hyperbolic attractor if and only if $bn < d$, $a + c > \frac{1}{b}$, $1 + ac - \frac{c}{b} > 0$. For the other values of the parameters P_1 is either a saddle or a nonhyperbolic equilibrium.*

Proof. As $b > 0$, we have $\lambda_1 < 0$. Thus P_1 is a hyperbolic attractor if and only if $\lambda_2 < 0$, $Re(\lambda_{3,4}) < 0$, hence the conditions in the theorem. □

Theorem 2.2. *The equilibrium P_1 is non-hyperbolic in one of the following situations:*

(1) if (i) $d = bn$, *and* $1 + ac - \frac{c}{b} \neq 0, b(a + c) \neq 1$; $(P_1 \equiv P_2)$; *or* (ii) $d = bn, b(a + c) = 1$, *and* $1 + ac - \frac{c}{b} < 0$, $(P_1 \equiv P_2)$; *or* (iii) $1 + ac - \frac{c}{b} = 0, b(a + c) \neq 1$, *and* $d \neq bn$, $(P_1 \equiv P_3 = P_4 \in \Gamma)$; *then P_1 is a fold singularity;*

(2) if $d \neq bn$, $b(a + c) = 1$, and $1 + ac - \frac{c}{b} > 0$, then P_1 is a Hopf singularity, with $\lambda_{3,4}(0) = \pm i\omega_1$, with $\omega_1 = \sqrt{1 - c^2}$;

(3) if $d = bn$, $b(a + c) = 1$, and $1 + ac - \frac{c}{b} > 0$, then P_1 is a fold-Hopf singularity;

(4) if (i) $d = bn$, $1 + ac - \frac{c}{b} = 0$, *and* $b(a + c) \neq 1$ *or* (ii) $1 + ac - \frac{c}{b} = 0, b(a + c) = 1$, *and $d \neq bn$, then P_1 is a double-zero singularity;*

(5) if $d = bn$, $1 + ac - \frac{c}{b} = 0$, and $b(a + c) = 1$ then P_1 is a triple-zero singularity.

Theorem 2.3. *A Hopf bifurcation takes place at the equilibrium P_1, when parameters transversally cross the stratum $\frac{1}{b} = a + c$, $1 + ac - \frac{c}{b} > 0$ and $d \neq bn$, provided*

$$l_1(0) = \frac{1}{2\omega_1}\left[\frac{2c-b}{b^2+4\omega_1^2} - \frac{2}{b}\right] \neq 0.$$

Proof. We may consider $\beta = \frac{1}{b} - a - c$ as the bifurcation parameter. For $b(a+c) = 1$, $1 + ac - \frac{c}{b} > 0$, $d \neq bn$, we have $\beta = 0$, and the Jacobian matrix J_1 has a pair of purely imaginary eigenvalues, $\lambda_{3,4}(0) = \pm i\omega_1$, with $\omega_1 = \sqrt{1-c^2}$. Close to $\beta = 0$, we have $Re(\lambda_{3,4}(\beta)) = \frac{1}{2}\beta$, thus the transversality condition from the Hopf bifurcation Theorem [5] is satisfied. The second condition to verify is $l_1(0) \neq 0$, that is the first Lyapunov coefficient is nonzero at the bifurcation value $\beta = 0$.

In order to compute the Lyapunov coefficient, we use the projection method [5] and the formulae deduced in [17], [18].

As $\beta = 0$, perform the translation $\tau(P_1)$, bringing the equilibrium P_1 at the origin of the system

$$\dot{x} = Ax + \frac{1}{2}B(x,x),$$

with $A = J_1$ and B given in (2.6).

Two complex eigenvectors satisfying the conditions $Aq = i\omega_1 q$, $A^T p = -i\omega_1 p$ and $\langle p, q\rangle = 1$ can be chosen as $q = \begin{pmatrix} 1 & 0 & i\omega_1 - c & 0 \end{pmatrix}^T$ and $p = \frac{i}{2\omega_1}\begin{pmatrix} c - i\omega_1 & 0 & 1 & 0 \end{pmatrix}^T$. The complex vectors involved in the computation of the first Lyapunov coefficient read $h_{11} = \begin{pmatrix} 0 & -\frac{2}{b} & 0 & 0 \end{pmatrix}^T$ and $h_{20} = \begin{pmatrix} 0 & -\frac{2}{b+2i\omega_1} & 0 & 0 \end{pmatrix}^T$. Thus, using the formula (5.39) in [5], we obtain:

$$l_1(0) = \frac{1}{2\omega_1}\left[\frac{2c-b}{b^2+4\omega_1^2} - \frac{2}{b}\right].$$

Consequently, if $\frac{2c-b}{b^2+4\omega_1^2} - \frac{2}{b} \neq 0$, the first Lyapunov coefficient at $\beta = 0$ is nonzero. □

Remark 2.1. If $\frac{2c-b}{b^2+4\omega_1^2} - \frac{2}{b} < 0$, the Hopf bifurcation at P_1 is supercritical, that is a limit cycle appears, locally attractive on the center manifold, while if $\frac{2c-b}{b^2+4\omega_1^2} - \frac{2}{b} > 0$, the Hopf bifurcation is subcritical, that is the limit cycle is repelling on the center manifold. In addition, as $bn - d < 0$, the center manifold is attractive.

In Figure 1, projections of the attractive limit cycle existing for the parameters $a = 0.6, b = 1, c = 0.2, d = 3, n = -1$ on the planes (x,z), (y,z), (x,y), and (x,u) are represented.

As $l_1(0) = 0$, we get $b = \frac{1}{3}\left(c + \sqrt{25c^2-24}\right)$. The following result holds.

Theorem 2.4. *As $l_1(0) = 0$, the second Lyapunov coefficient reads*

$$l_2(0) = \frac{-324\left(-108 + 45c^2 + 168c^4 - 100c^6 + \left(9c - 24c^3 + 20c^5\right)\sqrt{25c^2-24}\right)}{\left(c + \sqrt{25c^2-24}\right)^3\left(6 - 5c^2 + c\sqrt{25c^2-24}\right)^3}. \tag{2.8}$$

A Bautin bifurcation could take place at the equilibrium P_1, when parameters transversally cross the stratum $\frac{1}{b} = a + c$, $1 + ac - \frac{c}{b} > 0$ and $d \neq bn$, provided

$$l_2(0) \neq 0.$$

Proof. Using the notations in [17], the complex vectors involved in the computation of the second Lyapunov coefficient corresponding to the Hopf bifurcation at P_1 read

$h_{21} = \frac{2(3b+4i\omega_1)}{b(b+2i\omega_1)\left(1-c^2+3\omega_1^2\right)}\begin{pmatrix} i\omega_1 - c & 0 & c^2+\omega_1^2 & 0 \end{pmatrix}^T$,

$h_{30} = \frac{6}{(b+2i\omega_1)\left(-1+c^2+9\omega_1^2\right)}\begin{pmatrix} c+3i\omega_1 & 0 & -1 & 0 \end{pmatrix}^T$,

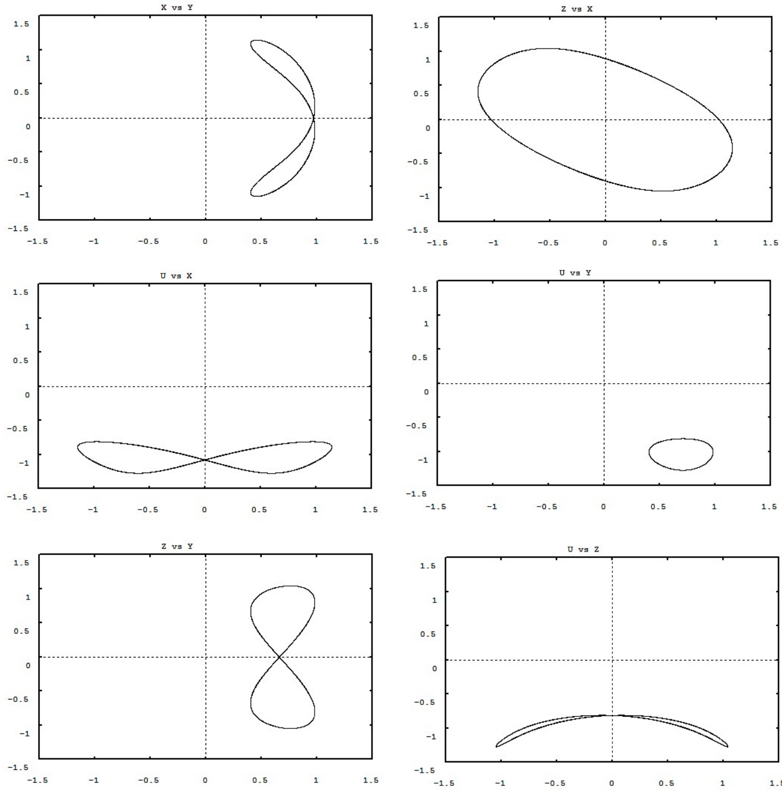

FIGURE 1. Projections of the limit cycle corresponding to P_1 on different planes, parameters: $a = 0.6, b = 1, c = 0.2, d = 3, n = -1$.

$$h_{31} = \frac{3}{(b+2i\omega_1)^2}\begin{pmatrix} 0 & \frac{\left(\frac{-4}{b}-\frac{2}{b+2i\omega_1}\right)(\omega_1-ic)}{\omega_1} - \frac{4(c-i\omega_1)(3b+4i\omega_1)}{b\left(c^2-1-3\omega_1^2\right)} - \frac{4(c+3i\omega_1)}{\left(c^2-1+9\omega_1^2\right)} & 0 & 0 \end{pmatrix}^T,$$

$$h_{22} = \begin{pmatrix} 0 & \frac{4\left(12b^3c-2bc\left(c^2-1-19\omega_1^2\right)+b^2\left(3c^2-3-17\omega_1^2\right)+8\omega_1^2\left(c^2-1-3\omega_1^2\right)\right)}{b^3\left(b^2+4\omega_1^2\right)\left(1-c^2+3\omega_1^2\right)} & 0 & 0 \end{pmatrix}^T.$$

Replacing these expressions into formula (29) in [18] for the second Lyapunov coefficient, we obtain the expression (2.8). □

2.2. **Stability and Hopf bifurcation at P_2.** At $P_2 = (0, \frac{n}{d}, 0, 1 - \frac{bn}{d})$, the characteristic equation associated with the Jacobian matrix $J_2 = J(P_2)$ reads

$$\left(\lambda^2 + b\lambda + bn - d\right)\left[\lambda^2 - \lambda\left(\frac{n}{d} - a - c\right) + 1 + ac - \frac{nc}{d}\right] = 0.$$

Thus the eigenvalues of J_2 satisfy

$$\begin{aligned} \lambda_1\lambda_2 &= bn - d, \quad \lambda_1 + \lambda_2 = -b, \\ \lambda_3\lambda_4 &= 1 + ac - \frac{nc}{d}, \quad \lambda_3 + \lambda_4 = \frac{n}{d} - a - c. \end{aligned}$$

Analysing the signs of these eigenvalues, the following results are easily obtained:

Theorem 2.5. *The equilibrium P_2 is a hyperbolic attractor if and only if $b > 0$, $d < bn$, $(a+c) > \frac{n}{d}$, $1+ac-\frac{nc}{d} > 0$. For all the other values of the parameters, P_2 is either a saddle or nonhyperbolic.*

Theorem 2.6. *As $b > 0$, the equilibrium P_2 is non-hyperbolic in one of the following situations:*

(1) *if (i)* $bn = d$, $\frac{n}{d} \neq a + c$ *and* $1 + ac \neq \frac{nc}{d}$ $(P_2 \equiv P_1)$; *or (ii)*$bn \neq d$, $\frac{n}{d} \neq a + c$ *and* $1 + ac = \frac{n}{d}c$; $(P_2 \in \Gamma)$ *or (iii)* $bn = d$, $\frac{n}{d} = a + c$ *and* $1 + ac < \frac{n}{d}c$ $(P_2 \in \Gamma)$ *then* P_2 *is a fold singularity;*

(2) *if* $bn \neq d$, $\frac{n}{d} = a + c$ *and* $1 + ac > \frac{n}{d}c$ *then* P_2 *is a Hopf singularity, with* $\lambda_{3,4} = \pm i\omega_1$, $\omega_1 = \sqrt{1 - c^2}$.

(3) *if* $bn = d$, $\frac{n}{d} = a + c$ *and* $1 + ac > \frac{n}{d}c$, *then* P_2 *is a fold-Hopf singularity.*

(4) *if (i)* $bn = d$, $\frac{n}{d} \neq a + c$ *and* $1 + ac = \frac{n}{d}c$ *or (ii)* $bn \neq d$, $\frac{n}{d} = a + c$ *and* $1 + ac = \frac{n}{d}c$, *then* P_2 *is a double-zero singularity.*

(5) *if* $bn = d$, $\frac{n}{d} = a + c$ *and* $1 + ac = \frac{n}{d}c$ *then* P_2 *is a triple-zero singularity.*

Theorem 2.7. *As* $b = 0$, *the equilibrium* P_2 *is non-hyperbolic in one of the following situations:*

(1) *if (i)* $d < 0$, $\frac{n}{d} \neq a + c$ *and* $1 + ac \neq \frac{n}{d}c$ *or (ii)* $bn > d < 0$, $\frac{n}{d} = a + c$ *and* $1 + ac < \frac{n}{d}c$, *the equilibrium* P_2 *is a Hopf singularity with* $\lambda_{1,2} = \pm i\omega_2$, $\omega_2 = \sqrt{-d}$, $d < 0$.

(2) *if* $d > 0$, $\frac{n}{d} = a + c$ *and* $1 + ac > \frac{n}{d}c$ *then* P_2 *is a Hopf singularity, with* $\lambda_{3,4} = \pm i\omega_1$, $\omega_1 = \sqrt{1 - c^2}$.

(3) *if* $d < 0$, $\frac{n}{d} = a + c$ *and* $1 + ac > \frac{n}{d}c$, *then* P_2 *is a double Hopf singularity, with* $\lambda_{1,2} = \pm i\omega_2$ *and* $\lambda_{3,4} = \pm i\omega_1$.

(4) *If* $d > 0$, $\frac{n}{d} \neq a + c$ *and* $1 + ac = \frac{n}{d}c$ *then* P_2 *is a fold singularity.*

(5) *If* $d > 0$, $\frac{n}{d} = a + c$ *and* $1 + ac = \frac{n}{d}c$ *then* P_2 *is a double-zero singularity.*

(6) *If* $d < 0$, $\frac{n}{d} \neq a + c$ *and* $1 + ac = \frac{n}{d}c$, *then* P_2 *is a fold-Hopf singularity with* $\lambda_{1,2} = \pm i\omega_2$.

(7) *If* $d < 0$, $\frac{n}{d} = a + c$ *and* $1 + ac = \frac{n}{d}c$, *then* P_2 *is a double–zero Hopf singularity.*

In the next theorem the type of the Hopf bifurcation corresponding to P_2 is deduced.

Theorem 2.8. *A Hopf bifurcation takes place at the equilibrium* P_2, *when parameters transversally cross the stratum* $b > 0, d \neq bn$, $\frac{n}{d} - a - c = 0$, *and* $1 + ac - \frac{nc}{d} > 0$, *provided*

$$c\left(4\omega_1^2 + d - bn\right) - 2b\omega_1^2 \neq 0,$$

with $\omega_1 = \sqrt{1 - c^2}$.
In addition, if $c\left(4\omega_1^2 + d - bn\right) - 2b\omega_1^2 < 0$, *the Hopf bifurcation at* P_2 *is supercritical, while if* $c\left(4\omega_1^2 + d - bn\right) - 2b\omega_1^2 > 0$, *the Hopf bifurcation is subcritical.*

Proof. We may consider the bifurcation parameter $\beta = \frac{n}{d} - a - c$, and the bifurcation value $\beta = 0$. For $b > 0, d \neq bn$, $\frac{n}{d} - a - c = 0$, and $1 + ac - \frac{nc}{d} > 0$, the Jacobian matrix J_2 has a pair of purely imaginary eigenvalues, $\lambda_{3,4}(0) = \pm i\omega_1$. Close to $\beta = 0$, we have $Re(\lambda_{3,4}(\beta)) = \frac{1}{2}\beta$, thus the transversality condition $\frac{\partial Re(\lambda_{3,4}(\beta))}{\partial \beta}\,|_{\beta=0} \neq 0$, from the Hopf bifurcation Theorem, is satisfied.

At $\beta = 0$, the transformation $\tau(P_2)$, brings the equilibrium P_2 at the origin of the system

$$\dot{x} = Ax + \frac{1}{2}B\left(x, x\right),$$

where $A = J_2$ and B is given in (2.6).

Two complex eigenvectors satisfying the conditions $Aq = i\omega_1 q$, $A^T p = -i\omega_1 p$ and $\langle p, q\rangle = 1$ can be chosen as $q = \begin{pmatrix} 1 & 0 & i\omega_1 - c & 0 \end{pmatrix}^T$ and $p = \frac{i}{2\omega_1}\begin{pmatrix} c - i\omega_1 & 0 & 1 & 0 \end{pmatrix}^T$, while $h_{11} = \begin{pmatrix} 0 & 0 & 0 & -2 \end{pmatrix}^T$, and $h_{20} = \frac{1}{4\omega_1^2 + d - bn - 2bi\omega_1}\begin{pmatrix} 0 & 4i\omega_1 & 0 & -2\left(d - bn\right) \end{pmatrix}^T$. Thus, using formula (5.39) in [5], the first Lyapunov coefficient at $\beta = 0$ is obtained into the form:

$$l_1 = \frac{1}{\omega_1}\frac{c\left(4\omega_1^2 + d - bn\right) - 2b\omega_1^2}{(4\omega_1^2 + d - bn)^2 + 4b^2\omega_1^2}.$$

The Hopf bifurcation condition $l_1(0) \neq 0$ is satisfied iff $c\left(4\omega_1^2 + d - bn\right) - 2b\omega_1^2 \neq 0$. □

In Figure 2, projections of the attractive limit cycle corresponding to the specified parameters on the planes (y, x), (x, z), (x, u), (y, u) and on the spaces (x, y, z), (x, y, u) are represented, while in Figure 3, the time series of the four variables corresponding to this cycle are given. Remark that in this case a periodic control u determines periodic behaviour for the real interest rate x, for the investment demand y, and for the inflation rate z.

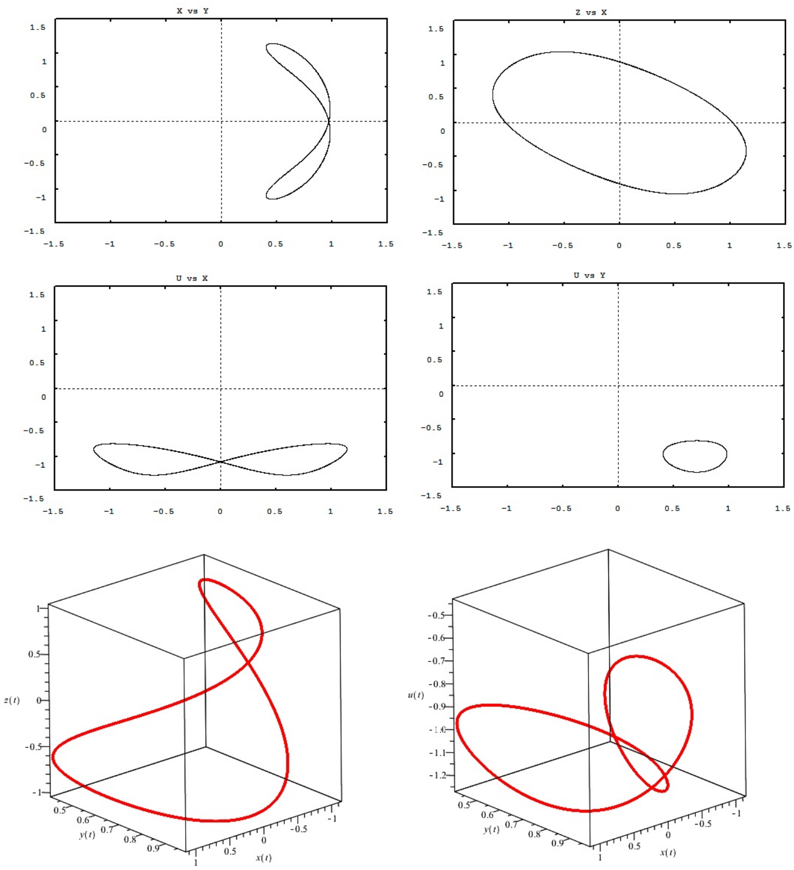

FIGURE 2. Projections of the limit cycle corresponding to P_2, parameters: $a = 0.1, b = 2, c = 0.5, d = 1.4, n = 1$.

As $l_1(0) = 0$, we get $n = \frac{d}{b} + \frac{4(1-c^2)}{b} - \frac{2(1-c^2)}{c}$. The following result holds.

Theorem 2.9. *As $l_1(0) = 0$, the second Lyapunov coefficient reads*

$$l_2(0) = \frac{c^3\left(-2c + b\left(2 + d\right)\right)}{2b^3\left(c^2 - 1\right)}. \tag{2.9}$$

A Bautin bifurcation could take place at the equilibrium P_2, when parameters transversally cross the stratum $b > 0, d \neq bn$, $\frac{n}{d} - a - c = 0$, and $1 + ac - \frac{nc}{d} > 0$, provided

$$l_2(0) \neq 0.$$

Proof. Using the notations in [18], the complex vectors involved in the computation of the second Lyapunov coefficient corresponding to the Hopf bifurcation at P_2 read:

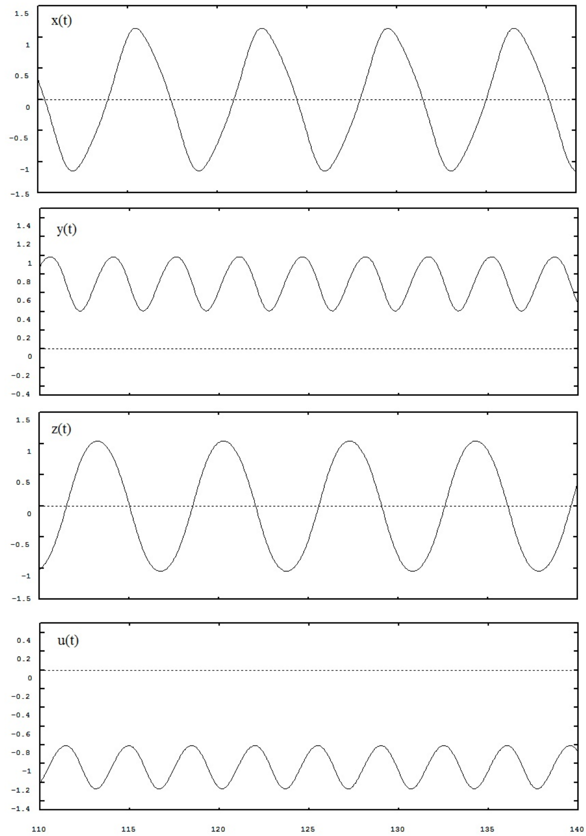

FIGURE 3. Time series for the variables on the limit cycle in Figure 2.

$$
\begin{aligned}
h_{21} &= \tfrac{4i\omega_1}{\left(1-c^2+3\omega_1^2\right)\left(d-b(n+2i\omega_1)+4\omega_1^2\right)} \begin{pmatrix} c-i\omega_1 & 0 & -(c^2+\omega_1^2) & 0 \end{pmatrix}^T,\\
h_{30} &= \tfrac{12i\omega_1}{\left(d-b(n+2i\omega_1)+4\omega_1^2\right)\left(-1+c^2+9\omega_1^2\right)} \begin{pmatrix} -c-3i\omega_1 & 0 & 1 & 0 \end{pmatrix}^T,\\
h_{31} &= \tfrac{6}{\left(d-b(n+2i\omega_1)+4\omega_1^2\right)^2} \begin{pmatrix} 0 & h_{31}^2 & 0 & h_{31}^4 \end{pmatrix}^T, h_{22} = \begin{pmatrix} 0 & h_{22}^2 & 0 & h_{22}^4 \end{pmatrix}^T, \text{where}\\
h_{31}^2 &= 4id\omega_1 + \tfrac{8\omega_1^2(c-i\omega_1)}{c^2-1-3\omega_1^2} - \tfrac{2(c+i\omega_1)\left(bn-d+4\omega_1^2\right)}{d-b(n+2i\omega_1)+4\omega_1^2} + \tfrac{8\omega_1^2(c+3i\omega_1)}{c^2-1+9\omega_1^2},\\
h_{31}^4 &= 4d\omega_1\,(2\omega_1-ib) + \tfrac{4(d-bn)\omega_1(ic+\omega_1)}{c^2-1-3\omega_1^2} - \tfrac{2(d-bn)(c+i\omega_1)(b+4i\omega_1)}{d-b(n+2i\omega_1)+4\omega_1^2} + \tfrac{4i\omega_1(d-bn)(c+3i\omega_1)}{c^2-1+9\omega_1^2},\\
h_{22}^2 &= \tfrac{8\left(-2b\omega_1^2+c\left(d-bn+4\omega_1^2\right)\right)}{(d-bn)\left(d^2-8bn\omega_1^2+16\omega_1^4+b^2\left(n^2+4\omega_1^2\right)+d\left(-2bn+8\omega_1^2\right)\right)},\\
h_{22}^4 &= \tfrac{16\omega_1}{c^2-1-3\omega_1^2}\left[\tfrac{ic+\omega_1}{d-b(n+2i\omega_1)+4\omega_1^2} + \tfrac{-ic+\omega_1}{d-b(n-2i\omega_1)+4\omega_1^2}\right] - bh_{22}^2.
\end{aligned}
$$

As $l_1 = 0$, we get $n = \frac{d}{b} + \frac{4\left(1-c^2\right)}{b} - \frac{2\left(1-c^2\right)}{c}$. In this case, the second Lyapunov coefficient reads (2.9). □

Theorem 2.10. *A subcritical Hopf bifurcation takes place at the equilibrium P_2, when parameters transversally cross the stratum $b = 0, d > 0, \frac{n}{d} - a - c = 0$, and $1 + ac - \frac{nc}{d} > 0$.*

Proof. We may consider the bifurcation parameter $\beta = \frac{n}{d} - a - c$, and the bifurcation value $\beta = 0$. For $b = 0, d > 0, \frac{n}{d} - a - c = 0$, and $1 + ac - \frac{nc}{d} > 0$, and the Jacobian matrix

J_2 has a pair of purely imaginary eigenvalues, $\lambda_{3,4}(0) = \pm i\omega_1$. Close to $\beta = 0$, we have $Re(\lambda_{3,4}(\beta)) = \frac{1}{2}\beta$, thus the transversality condition $\frac{\partial Re(\lambda_{3,4}(\beta))}{\partial \beta}|_{\beta=0} \neq 0$, from the Hopf bifurcation Theorem, is satisfied.

Computations similar to those in Theorem 2.8 lead to the following expression for the Lyapunov coefficient at the bifurcation value:

$$l_1(0) = \frac{1}{\omega_1} \frac{c}{(4\omega_1^2 + d)}.$$

As the parameters c, d are positive, it follows that $l_1(0) > 0$. According to the Hopf bifurcation Theorem, a subcritical Hopf bifurcation takes place. □

Remark 2.2. As a consequence, a repulsive limit cycle appears on the center manifold, close to the bifurcation stratum, with $\frac{n}{d} - a - c < 0$.

2.3. **Stability and Hopf bifurcation at P_3 and P_4.** The equilibria P_3 and P_4 exist in the hypothesis $c - b - abc \geq 0$. If $c - b - abc = 0$, they both coincide with P_1.

Theorem 2.11. *Assume $c - b - abc > 0$. Denote by $d_0 = \frac{c}{ac+1}n$. The following assertions are valid.*

1) If $d < d_0$, then P_3 is a saddle or Hopf singularity.

2) If $d > d_0$ then P_3 is an attractor or a saddle or a Hopf singularity.

3) If $d = d_0$, then P_3 is a fold or a fold-Hopf singularity.

Proof. Denote by $y_0 = a + \frac{1}{c} > 0$. The first eigenvalue of P_3 is $\lambda_1 = -(d - d_0)\, y_0$, while the other three are the roots of

$$\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 = 0 \tag{2.10}$$

where

$$c_2 = a + b + c - y_0, \quad c_1 = 2\alpha + b(a - y_0) + c(a + b - y_0) + 1, \quad c_0 = 2(c - b - abc).$$

Notice that $c_0 > 0$, $c_2 = b + c - \frac{1}{c}$ and $c_1 = bc - 2ab + 2 - \frac{3b}{c}$. Thus $\lambda_2\lambda_3\lambda_4 = -c_0 < 0$.

1) If $d < d_0$, then $\lambda_1 > 0$, thus, when P_3 is hyperbolic, it is a saddle.

2) If $d > d_0$, then $\lambda_1 < 0$, thus, when P_3 is hyperbolic, it is either saddle or an attractor.

Using the Hurwitz conditions, the equation (2.10) has all eigenvalues with negative real parts if and only if $c_2 > 0$, $c_2 c_1 > c_0$.

3) As $d = d_0$, then $\lambda_1 = 0$, thus P_3 is either fold singularity or a fold-Hopf singularity. □

Remark 2.3. If $d > d_0$, $c_1 > 0$ and $c_2 c_1 = c_0$, the system has a Hopf singularity at P_3.

Indeed, (2.10) reads in this case $(\lambda + c_2)\left(c_1 + \lambda^2\right) = 0$, thus, $\pm i\sqrt{c_1}$ are two purely complex roots. In addition, the other two eigenvalues are both negative, $\lambda_1 = -(d - d_0)\frac{ac+1}{c} < 0$ and $\lambda_2 = -c_2 < 0$, whenever $d > d_0$. The existence of a Hopf bifurcation at P_3 can be analyzed as the one for P_1 or P_2 above. Note that if a supercritical Hopf bifurcation occurs, the stable limit cycle is attractive not only on the center manifold.

As the characteristic equation associated to the Jacobi matrix at P_4 coincides with the one of P_3, the topological types of the two equilibria P_3 and P_4 are the same.

3. Degenerate Hopf singularity P_2

Theorem 3.12. *As $b = 0$, $d < 0$, and (i) $\frac{n}{d} - a - c \neq 0$,$1 + ac - \frac{nc}{d} \neq 0$, or (ii) $\frac{n}{d} - a - c = 0$, $1 + ac - \frac{nc}{d} < 0$, the equilibrium P_2 is a degenerate Hopf singularity, of order at least 3.*

In addition, a local two-dimensional center manifold is given by

$$x = 0, z = 0,$$

up to sixth order terms.

Proof. For $b = 0$, $d < 0$, and (i) or (ii), the Jacobian matrix J_2 has a pair of purely imaginary eigenvalues, $\lambda_{1,2}(0) = \pm i\omega_2$, with $\omega_2 = \sqrt{-d}$, while the other two eigenvalues have nonzero real parts. Thus, there exists a two-dimensional local center manifolds T^c through P_2.

Perform the transformations $x_1 = x$, $x_2 = y - \frac{n}{d}$, $x_3 = z$, $x_4 = u - 1$, to bring the equilibrium P_2 at the origin of the system

$$\dot{x} = f(x), \tag{3.11}$$

where $f(x) = Ax + \frac{1}{2}B(x,x)$, with $A = \begin{pmatrix} \frac{n}{d} - a & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & -c & 0 \\ 0 & -d & 0 & 0 \end{pmatrix}$.

Choosing the two complex eigenvectors, satisfying the conditions $Aq = \lambda_1 q$, $A^T p = -\lambda_1 p$, and $\langle p, q\rangle = 1$, as $q = \begin{pmatrix} 0 & 1 & 0 & -i\omega_2 \end{pmatrix}^T$ and $p = \begin{pmatrix} 0 & \frac{1}{2} & 0 & -\frac{i}{2\omega_2} \end{pmatrix}^T$ and using the formulae in [18], we obtain

$$h_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^T, \quad h_{20} = \begin{pmatrix} 0 & -\frac{2}{3}i\omega_2 & 0 & \frac{4}{3}d \end{pmatrix}^T.$$

Consequently, we get

$$G_{21} = \langle p, B(\bar{q}, h_{20}) + 2B(q, h_{11})\rangle = -\frac{id^2}{3\omega_2}, \tag{3.12}$$

thus, the first Lyapunov coefficient reads:

$$l_1 = \frac{1}{2} Re\,(G_{21}) = 0. \tag{3.13}$$

We may now compute the second Lyapunov coefficient. We get

$$h_{30} = \begin{pmatrix} 0 & \frac{3}{4}d & 0 & -\frac{9}{4}id\omega_2 \end{pmatrix}^T, \quad h_{21} = \begin{pmatrix} 0 & -\frac{1}{6}d & 0 & -\frac{1}{6}id\omega_2 \end{pmatrix}^T$$

As $l_1 = 0$, using the coefficients of forth order term, given by the formulae in [18], we obtain

$$h_{40} = \begin{pmatrix} 0 & -\frac{1}{10}id\omega_2 & 0 & \frac{2}{5}d^2 \end{pmatrix}^T, h_{31} = \begin{pmatrix} 0 & -\frac{1}{2}id\omega_2 & 0 & d^2 \end{pmatrix}^T, h_{22} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^T.$$

Thus,

$$G_{32} = \frac{id^3}{12\omega_2},$$

and the second Lyapunov coefficient has the expression:

$$l_2 = \frac{1}{12} Re\,(G_{32}) = 0.$$

Next, we have

$$\begin{aligned} h_{32} &= \begin{pmatrix} 0 & -\frac{1}{24}d^2 & 0 & -\frac{1}{24}id^2\omega_2 \end{pmatrix}^T, h_{41} = \begin{pmatrix} 0 & \frac{349}{240}d^2 & 0 & -\frac{349}{80}id^2\omega_2 \end{pmatrix}^T, \\ h_{33} &= \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^T, h_{42} = \begin{pmatrix} 0 & -\frac{31}{36}id^2\omega_2 & 0 & \frac{31}{18}d^3 \end{pmatrix}^T. \end{aligned}$$

Thus,

$$G_{43} = \frac{21id^4}{8\omega_2},$$

and the third Lyapunov coefficient is:

$$l_3 = \frac{1}{144} Re\,(G_{43}) = 0.$$

Consequently, on the 2-dimensional center manifold W^c_{loc} the system reads

$$w' = i\omega_0 w + O(|w|^8). \tag{3.14}$$

Thus, the equilibrium in origin is a degenerate Hopf singularity, of order at least three.

Using the real variable x, the center manifold can be parameterized by $w \in \mathbb{C}$, as

$$x = H(w, \bar{w}), \tag{3.15}$$

where H is given by

$$H(w,\bar{w}) = wq + \bar{w}\bar{q} + \sum_{2\le j+k\le n} \frac{1}{j!k!} h_{jk} w^j \bar{w}^k + O\left(|w|^{n+1}\right), \tag{3.16}$$

with $h_{jk} \in \mathbb{C}$, $h_{jk} = \bar{h}_{kj}$,[18].

Substituting in (3.15) the expressions determined for q and $h_{jk,}$ we find for the center manifold the equations

$$\begin{aligned} x_1 &= 0 + O\left(|w|^7\right), \\ x_3 &= 0 + O\left(|w|^7\right). \end{aligned}$$

□

If all Lyapunov coefficients are zero, the 2-dimensional system (3.14) has a family of closed orbits around a nonlinear center, corresponding to a family of closed orbits for the 4-dimensional system, situated on the center manifold.

In Figure 4 the phase portrait corresponding to the center manifold (y, u) is represented for some values of the parameters. For $u > 0$ all the trajectories are closed and surround the equilibrium point P_2. For other values of the parameters the closed orbits in the (y, u) plane and in the (x, y, u) space are represented in Figure 5.

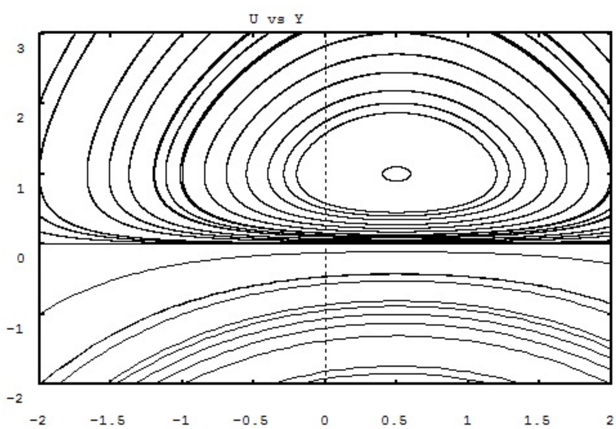

FIGURE 4. Orbits on the center manifold $x = 0, z = 0$; parameters: $a = 1, b = 0, c = 0.4, d = -1, n = -0.5$.

Taking into account the effect of the other stable or unstable manifolds of the equilibrium P_2, these orbits could either attract or repel orbits through points close to the center manifold. For instance, as $\frac{n}{d} - a - c < 0$, and $1 + ac - \frac{nc}{d} > 0$, the equilibrium P_2 has a 2-dimensional stable manifold W^s_{loc}. Thus, the closed orbits on the center manifold W^c_{loc} attract trajectories close to W^c_{loc}. In Figure 6 three such trajectories are plotted for parameters $a = 1, b = 0, c = 1, d = -1, n = 0$.

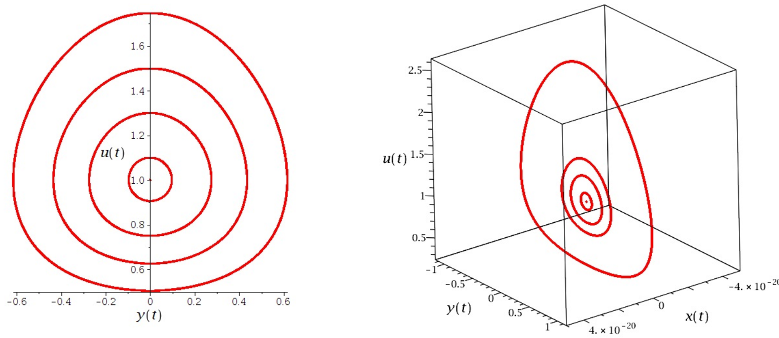

FIGURE 5. Closed orbits on the center manifold in the (y, u) plane (left) and in the (x, y, u) space (right), parameters: $a = 1, b = 0, c = 1, d = -1, n = 0$; time $t \in [400, 500]$

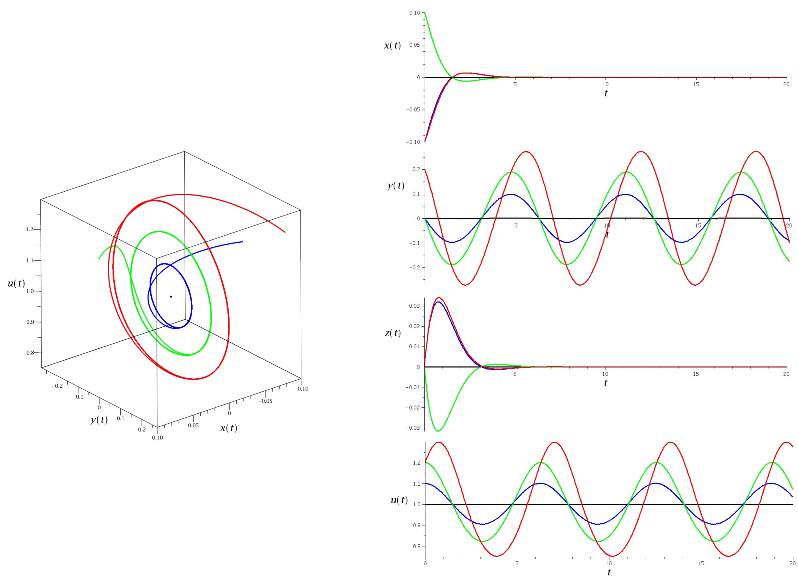

FIGURE 6. Three orbits in the (x, y, u) space attracted by closed orbits situated on the center manifold (y, u); time series for these orbits, parameters: $a = 1, b = 0, c = 1, d = -1, n = 0$.

Unlike the situation in Figure 3, a periodic control u determines the stability of the real interest rate x and of the inflation rate z, while the investment demand y is periodic.

4. DEGENERATE HOPF-HOPF BIFURCATION AT P_2

Theorem 4.13. *A degenerate Hopf-Hopf bifurcation takes place at the equilibrium P_2, when parameters satisfy $b = 0, d < 0, \frac{n}{d} - a - c = 0$, and $1 + ac - \frac{nc}{d} > 0$.*

Proof. Consider the bifurcation parameter $\beta = (\beta_1, \beta_2)\,, \beta_1 = \frac{n}{d} - a - c, \beta_2 = -b$, and the bifurcation value $\beta = (0, 0)$. For $b = 0, d < 0, \frac{n}{d} - a - c = 0$, and $1 + ac - \frac{nc}{d} > 0$, the Jacobian matrix J_2 has two pairs of purely imaginary eigenvalues, $\Lambda_{1,4}(0) = \pm i\omega_1$, $\Lambda_{2,3} = \pm i\omega_2$, with $\omega_1^2 = 1 - c^2$, $\omega_2^2 = -d$, and $Re(\Lambda_{1,4}(\beta)) = \frac{1}{2}\beta_1$, $Re(\Lambda_{2,3}(\beta)) = \frac{1}{2}\beta_2$.

At $\beta = 0$, the transformation $\tau(P_2)$, brings the equilibrium P_2 at the origin of the system

$$\dot{x} = Ax + F(x), \tag{4.17}$$

where

$$A = \begin{pmatrix} c & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & -c & 0 \\ 0 & -d & 0 & 0 \end{pmatrix}, \quad F(x) = \begin{pmatrix} x_1x_2 \\ -x_1^2 \\ 0 \\ -dx_2x_4 \end{pmatrix}.$$

Following the lines in [5], p. 351, at $\beta = 0$, we choose the complex eigenvectors q_1, q_2, p_1, p_2 satisfying the conditions $Aq_1 = i\omega_1 q_1, Aq_2 = i\omega_2 q_2$, $A^T p_1 = -i\omega_1 p_1, A^T p_2 = -i\omega_2 p_2$, normalized such that $\langle p_1, q_1\rangle = 1$, $\langle p_2, q_2\rangle = 1$. These vectors can be chosen as $q_1 = \begin{pmatrix} 1 & 0 & i\omega_1 - c & 0 \end{pmatrix}^T$, $q_2 = \begin{pmatrix} 0 & 1 & 0 & -i\omega_2 \end{pmatrix}^T$,
$p_1 = \frac{i}{2\omega_1}\begin{pmatrix} c - i\omega_1 & 0 & 1 & 0 \end{pmatrix}^T$, and $p_2 = \begin{pmatrix} 0 & \frac{1}{2} & 0 & -\frac{i}{2\omega_2} \end{pmatrix}^T$.

Changing to the complex coordinates z_1, z_2, given by

$$x = z_1q_1 + \bar{z}_1\bar{q}_1 + z_2q_2 + \bar{z}_2\bar{q}_2 = \begin{pmatrix} z_1 + \bar{z}_1 \\ z_2 + \bar{z}_2 \\ (i\omega_1 - c)\, z_1 - (c + i\omega_1)\, \bar{z}_1 \\ -i\omega_2\,(z_2 - \bar{z}_2) \end{pmatrix}$$

we find

$$\begin{aligned} z_1 \;&:= \langle p_1, x\rangle = -\frac{i}{2\omega_1}(c + i\omega_1)\, x_1 - \frac{i}{2\omega_1}x_3, \\ z_2 \;&:= \langle p_2, x\rangle = \frac{1}{2}x_2 + \frac{i}{2\omega_2}x_4, \end{aligned}$$

and, at $\beta = 0$, system (4.17) takes the complex form

$$\begin{cases} \dot{z}_1 = i\omega_1 z_1 + g(z_1, \bar{z}_1, z_2, \bar{z}_2), \\ \dot{z}_2 = i\omega_2 z_2 + h(z_1, \bar{z}_1, z_2, \bar{z}_2), \end{cases} \tag{4.18}$$

where

$$\begin{aligned} g(z_1, \bar{z}_1, z_2, \bar{z}_2) &= \langle p_1, F(z_1q_1 + \bar{z}_1\bar{q}_1 + z_2q_2 + \bar{z}_2\bar{q}_2)\rangle, \\ h(z_1, \bar{z}_1, z_2, \bar{z}_2) &= \langle p_2, F(z_1q_1 + \bar{z}_1\bar{q}_1 + z_2q_2 + \bar{z}_2\bar{q}_2)\rangle. \end{aligned}$$

We obtain

$$\begin{aligned} g(z_1, \bar{z}_1, z_2, \bar{z}_2) &= \gamma\,(z_1 + \bar{z}_1)(z_2 + \bar{z}_2), \\ h(z_1, \bar{z}_1, z_2, \bar{z}_2) &= -\frac{1}{2}(z_1 + \bar{z}_1)^2 - \frac{d}{2}\left(z_2^2 - \bar{z}_2^2\right), \end{aligned}$$

with

$$\gamma = -\frac{i(c + i\omega_1)}{2\omega_1} = \frac{1}{2} - \frac{ic}{2\omega_1}.$$

In the hypothesis $k\omega_1 \neq l\omega_2$, for $k, l \in N$, $k + l \leq 5$, the Poincaré normal form of (4.18) in complex coordinates (w_1, w_2) near $\beta = 0$ reads [5]

$$\begin{cases} \dot{w}_1 = \Lambda_1(\beta)\, w_1 + G_{2100}(\beta)\, w_1 |w_1|^2 + G_{1011}(\beta)\, w_1 |w_2|^2 + G_{3200}(\beta)\, w_1 |w_1|^4 \\ \qquad + G_{2111}(\beta)\, w_1 |w_1|^2 |w_2|^2 + G_{1022}(\beta)\, w_1 |w_2|^4 + O\left(\|(w_1, \bar{w}_1, w_2, \bar{w}_2)\|^6\right), \\ \dot{w}_2 = \Lambda_2(\beta)\, w_2 + H_{1110}(\beta)\, w_2 |w_1|^2 + H_{0021}(\beta)\, w_2 |w_2|^2 + H_{2210}(\beta)\, w_2 |w_1|^4 \\ \qquad + H_{1121}(\beta)\, w_2 |w_1|^2 |w_2|^2 + H_{0032}(\beta)\, w_2 |w_2|^4 + O\left(\|(w_1, \bar{w}_1, w_2, \bar{w}_2)\|^6\right), \end{cases} \tag{4.19}$$

Using the formulae in [5], p. 353, we find the following expressions for the relevant third order terms of the Poincaré normal form of system (4.18) at $\beta = 0$:

$$\begin{aligned}
G_{2100}(0) &= \frac{i\gamma}{2}\left(\frac{1}{2\omega_1+\omega_2}+\frac{1}{2\omega_1-\omega_2}\right),\\
G_{1011}(0) &= -i\gamma^2\left(\frac{1}{2\omega_1+\omega_2}+\frac{1}{2\omega_1-\omega_2}\right),\\
H_{1110}(0) &= \frac{i}{\omega_2}(d+\gamma+\bar{\gamma})-i\left(\frac{\gamma}{2\omega_1+\omega_2}-\frac{\bar{\gamma}}{2\omega_1-\omega_2}\right),\\
H_{0021}(0) &= -\frac{id^2}{6\omega_2}.
\end{aligned}$$

We deduce

$$\begin{aligned}
Re\left(G_{2100}(0)\right) &= \frac{c}{4\left(1-c^2\right)+d},\\
Re\left(G_{1011}(0)\right) &= -\frac{2c}{4\left(1-c^2\right)+d},\\
Re\left(H_{1110}(0)\right) &= -\frac{2c}{4\left(1-c^2\right)+d},\\
Re\left(H_{0021}(0)\right) &= 0.
\end{aligned} \tag{4.20}$$

Thus, condition (HH.4) in [5], Lemma 8.14, is not satisfied and the Hopf-Hopf bifurcation corresponding to the equilibrium point P_2 is degenerated. □

A more detailed study of the phenomena involved by the presence of this degenerate Hopf-Hopf bifurcation may be further performed using techniques as in [15], [16], [19], [20].

5. Conclusions

The 4D system designed by us, starting with the 3D Ma & Chen financial model, exhibits a rich dynamics: stable equilibria, hyperbolic saddles, non-hyperbolic points of fold, Hopf, double-zero, fold-Hopf, Hopf-Hopf, triple-zero, double-zero-Hopf, nonlinear center type, stable limit cycles, unstable limit cycles.

Compared to the initial 3D system, we found that by adding the feed-back control function, a new equilibrium point appears, and so new possibilities to stabilize the economic environment to a stable equilibrium state or to stable periodic behavior.

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