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# Replacing units by unipotents

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ABSTRACT. A unit $u$ is called *unipotent* if $u-1$ is nilpotent. As special classes of stable range one elements, replacing in the definitions units by unipotents, strong left (or right) stable range one elements, unipotent left (or right) stable range elements, strong left (or right) unipotent stable range one elements and unipotent-regular elements are defined. All these type of elements (and the corresponding rings, for some of these) are studied with special emphasis to matrices.

## 1. INTRODUCTION

In Ring Theory, there are plenty of notions whose definitions involve units (see for example [11] and [12]). When such notions are too general or too hard to describe, an easy way of obtaining some more restrictive notions is replacing the units by unipotent elements. In this paper this is what we do for stable range one elements, unit stable range one elements and unit-regular elements.

All rings we consider are associative with identity. For a ring $R$, $U(R)$ denotes the group of all the units, $N(R)$ the set of all nilpotent elements, $J(R)$ the Jacobson radical and $ureg(R)$ the set of all the *unit-regular* elements of $R$ (i.e., elements which have a unit inner inverse). By $Id(R)$ we denote the set of all idempotents of the ring $R$. Recall that a unit $u$ is called *unipotent* if $u-1$ is nilpotent and an element $a$ of a ring $R$ has *left stable range one* if whenever $Ra+Rb=R$ for some $b\in R$, there exists $r\in R$ such that $a+rb$ is a unit.

Equivalently, for every $x\in R$ there is $y\in R$ such that $a+y(1-xa)$ is a unit. Right stable range 1 is defined symmetrically. We denote by $sr1(R)$ the stable range elements of $R$. The element $y$ generally depends on $a$ and $x$ and will be called a *unitizer* for $a$ with respect to $x$.

The stable range 1 for elements (and rings) were specialized requiring $y$ to be an idempotent or else requiring $y$ to be a unit. This way the *idempotent (resp. unit) stable range 1 elements (and rings)* were defined and studied.

In this paper, we define and study some other specializations.

**Definition 1.1.** An element $a\in R$ has *strong left stable range one* if for every $x\in R$ there exists $y\in R$ such that $a+y(1-xa)$ is a unipotent.

An element $a\in R$ has *unipotent left stable range one* if for every $x\in R$ there exists a unipotent $y$ such that $a+y(1-xa)$ is a unit.

An element $a\in R$ has *strong unipotent left stable range one* if for every $x\in R$ there exists a unipotent $y$ such that $a+y(1-xa)$ is a unipotent.

*Strong (unipotent) right stable range one* elements are defined symmetrically.

Just taking $y=0$, shows that *unipotent elements* have left (and right) strong unipotent stable range 1. Clearly, elements that have strong unipotent stable range one have also strong (or unipotent) stable range one.

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An element $a \in R$ is called *unipotent-regular* if there exists a unipotent $u$ such that $a = aua$. Clearly, unipotent-regular elements are unit-regular.

Examples of such elements will appear all over this paper. As general reference on stable range one elements and rings we mention [5].

In Section 2 we study strong stable range one elements with special emphasis to matrices. In Section 3 we study (strong) unipotent stable range one elements, again with special emphasis to matrices. Finally, in Section 4 we characterize the unipotent-regular rings and the unipotent regular $2 \times 2$ matrices over Prüfer domains.

By $E_{ij}$ we denote the square matrix with all entries zero excepting the $(i, j)$-entry which is 1.

## 2. Strong stable range one elements

Note that *inverses of unipotent elements are unipotent*.

In order to prove some of our results, we also need (finite) products of unipotents to be unipotent. To that end we first recall the following result from [14]:

**Theorem 2.1.** *Let $R$ be a ring with the set of nilpotents $N(R)$. The following are equivalent:*
*(i) $N(R)$ is additively closed,*
*(ii) $N(R)$ is multiplicatively closed and $R$ satisfies Köthe's conjecture,*
*(iii) $N(R)$ is closed under the operation $x \circ y = x + y - xy$,*
*(iv) $N(R)$ is a subring of $R$.*
*(v) $N(R)$ is closed under the operation $x * y = x + y + xy$.*

Such rings were called *NR* rings in [6]. Moreover

**Corollary 2.1.** [14] *Let $R$ be a unital ring. Then $R$ is NR if and only if $1+N(R)$ is a multiplicative subgroup of $U(R)$.*

That is, we can also add

*(vi) In an NR ring, finite products of unipotents are also unipotent.*

Indeed, $(1 + t)(1 + s) = 1 + s + t + st = 1 + s * t \in 1 + N(R)$ iff for every $s, t \in N(R)$, $s * t \in N(R)$, i.e., precisely (v) above.

Examples of NR rings include the so-called *NI* rings (i.e., the upper nilradical equals $N(R)$) and so semiprime, or 2-primal (the lower nilradical equals $N(R)$, in particular commutative) rings and *Armendariz* rings (i.e., whenever polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{i=0}^{n} b_i x^i$ satisfy $f(x)g(x) = 0$ then $a_i b_j = 0$ for every $i$ and $j$) and so commutative PIDs and their factor rings, or reduced rings.

### 2.1. Simple properties.

Recall that $Id(R), U(R) \subseteq ureg(R) \subseteq sr1(R)$.

**Proposition 2.1.** *Idempotents have strong stable range one.*

*Proof.* A unitizer for an idempotent $e$ is its complementary idempotent $1 - e$. In detail, suppose $xe + b = 1$ and denote the *unipotent* (unit) $v = 1 - (1 - e)xe$. Then

$$e - v = e - 1 + (1 - e)xe = (1 - e)(xe - 1) = -(1 - e)b$$

and so $e + (1 - e)b = v$ is unipotent, as desired. □

For unit-regular elements, we can also indicate a unitizer independent of $x$, that is

**Proposition 2.2.** *Unit-regular elements have (left and right) stable range one. For $a = aua$ with $u \in U(R)$, we can choose $y = u^{-1} - a$ (independent of $x$).*

*Proof.* For an arbitrary $x$ of $R$ we show that $a + (u^{-1} - a)(1 - xa) \in U(R)$.

It suffices to replace $a$ with $auu^{-1}$:
$a + (u^{-1} - a)(1 - xa) = auu^{-1} + (u^{-1} - auu^{-1})(1 - xauu^{-1}) =$
$= auu^{-1} + (1 - au)u^{-1}(1 - xauu^{-1}) = u^{-1} - (1 - au)u^{-1}xauu^{-1} =$
$= [1 - (1 - au)u^{-1}x(au)]u^{-1} \in U(R)$, since $au$ is idempotent and $(1 - au)u^{-1}x(au)$ is zerosquare. □

**Remark 2.1.** In contrast with the idempotents (see previous proposition), here $[1 - (1 - au)u^{-1}x(au)]u^{-1}$ is a product of an unipotent and a unit which may *not* be unipotent. Actually, unit-regular elements may *not* have strong stable range one. Moreover, *units (which are not unipotents) may not have strong stable range one.*

As an example, in any ring with $2 \notin N(R)$, $-1$ (is a unit but) has not strong stable range one. This follows from the invariance result below.

**Lemma 2.1.** *Assume $R$ is an NR ring.*
*(i) If $a$ has strong stable range one and $v$ is unipotent then $va$ has strong stable range one.*
*(ii) In any ring $R$,$-a$ has strong stable range one whenever $a$ has it if and only if $2 \in N(R)$.*
*(iii) Strong stable range one elements are invariant to conjugations with unipotents.*
*(iv) If $a$ has strong stable range one and $u$ is unipotent then $au$ has strong stable range one.*
*(v) Strong stable range one elements are invariant to unipotent equivalences.*

*Proof.* (i) Suppose $(xv)a + b = 1$. There exists $y$ such that $a + yb \in 1 + N(R)$. Multiplying from left by $v$ we get $va + (vy)b \in 1 + N(R)$ since products of unipotents in NR rings are also unipotent.

(ii) If 2 is nilpotent then $-1$ is unipotent and the claim follows from (i). Conversely, suppose $-a$ has strong stable range one whenever $a$ has it. Since 1 (is a unipotent and) has strong stable range one (in any ring), it follows that $-1$ has strong stable range one, that is, for every $x$ there is $y$ such that $-1 + y(1 + x)$ is unipotent. Taking $x = -1$ shows that $-1$ is unipotent. Hence $2 \in N(R)$.

(iii) For every $x$ there is a $y$ such that $a + y(1 - xa) \in 1 + N(R)$. Then for any unipotent $u$, $u^{-1}[a + y(1 - xa)]u \in 1 + N(R)$ (since the ring is NR) but we can write this as $u^{-1}au + u^{-1}yu[1 - (u^{-1}xu)(u^{-1}au)]$, as desired.

(iv) If $a$ has strong stable range one and $u \in 1 + N(R)$ then $u^{-1}au$ has strong stable range one, by (iii). Then by (i), $u(u^{-1}au) = au$ has strong stable range one.

(v) Follows from (i) and (iv). □

**Proposition 2.3.** *A unit has strong stable range one if and only if it is unipotent.*

*Proof.* The condition is clearly sufficient ($y = 0$). Conversely, suppose $u \in U(R)$ has unipotent stable range one. Then for every $x$ there is $y$ such that $u + y(1 - xu)$ is unipotent. Taking $x = u^{-1}$ shows that $u$ must be a unipotent. □

Since in any ring *a regular element has stable range one if and only if it is unit-regular* (see Theorem 3.5, [10]), it was easy to foresee the following result.

**Theorem 2.2.** *Over any ring, a regular element is unipotent-regular whenever it has strong stable range one. If $R$ is an NR ring, any unipotent-regular element has strong stable range one.*

*Proof.* A standard proof (see [8], Th. 4.12) works. Suppose $a = axa$. There exists $y$ such that $a + y(1 - xa) = 1 + t$ for some $t \in N(R)$. Denote by $u = (1 + t)^{-1}$. Thus $a = axa = au(a + y(1 - xa))xa = auaxa = aua$ with unipotent $u$. Hence $a$ is unipotent-regular.

In the opposite direction, suppose $a = aua$ with unipotent $u$. As already seen in the proof of Proposition 2.2:

$$a + (u^{-1} - a)(1 - xa) = [1 - (1 - au)u^{-1}x(au)]u^{-1}$$

where the parenthesis [] is unipotent, and so is $u^{-1}$, if $a$ is unipotent-regular. Hence $a$ has strong stable range one (here we need the NR hypothesis: a product of unipotents is also unipotent). □

We address unipotent-regular rings and matrices in the last section.

2.2. $2 \times 2$ **matrices with strong stable range one.** We mention that we preferred $a+y(xa-1)$ instead of $a+y(1-xa)$ (which amounts just to change the sign of $y$), (at least) when discussing strong stable range one matrices, in order to diminish the number of minus signs in computations and to be able to use the computations from [4]. However, since the negative of an unipotent may not be unipotent, this cannot be done for (strong) unipotent stable range one elements or matrices.

In [4] (Th. 5), over commutative rings, the $2 \times 2$ matrices which have stable range one were characterized as follows: $sr(A) = 1$ if and only if for every $X$ there is $Y$ such that $A + Y(XA - I_2)$ is a unit if and only if $\det[A + Y(XA - I_2)]$ is a unit. The computation finally gave

$$\det(Y)(\det(X)\det(A) - \mathrm{Tr}(XA) + 1) + \det(A(\mathrm{Tr}(XY) + 1)) - \mathrm{Tr}(A adj(Y))$$

is a unit (where $adj(Y)$ denotes the adjugate of $Y$).

The characterization of the strong stable range one $2 \times 2$ matrices is slightly different.

**Theorem 2.3.** *Let $A \in \mathbb{M}_2(R)$ for a commutative ring $R$. Then $A$ has strong stable range one if and only if for every $X \in \mathbb{M}_2(R)$ there exists $Y \in \mathbb{M}_2(R)$ such that*

$$\det(Y)(\det(X)\det(A) - \mathrm{Tr}(XA) + 1) + \det(A(\mathrm{Tr}(XY) + 1)) - \mathrm{Tr}(A adj(Y)) = 1$$
$$\mathrm{Tr}(A) - \mathrm{Tr}(Y) + \mathrm{Tr}(YXA) = 2$$

*Proof.* First notice that $A + Y(XA - I_2)$ is unipotent if and only if $A - I_2 + Y(XA - I_2)$ is nilpotent, that is, has zero trace and zero determinant. We equivalently write these two conditions as follows

$\mathrm{Tr}[A - I_2 + Y(XA - I_2)] = \mathrm{Tr}[A + Y(XA - I_2)] - 2 = 0$ and so

$$\mathrm{Tr}(A) - \mathrm{Tr}(Y) + \mathrm{Tr}(YXA) = 2,$$

and $\det[A - I_2 + Y(XA - I_2)] = \det[A + Y(XA - I_2)] - \mathrm{Tr}[A + Y(XA - I_2)] + 1 = 0$, or else

$$\det[A + Y(XA - I_2)] = 1.$$

Here, for the determinant, we can use the computation performed in [4] and this gives the first condition in the statement (not any unit but 1). □

Moreover, since using the properties of determinants, the properties of the trace and the commutativity of the base ring, it is readily seen that changing $A$, $X$, $Y$ into transposes (since transposes of nilpotent matrices are nilpotent, it follows that transposes of unipotent matrices are unipotent) and reversing the order of the products does not change the condition in the previous theorem, we also get

**Corollary 2.2.** *Let $R$ be a commutative ring and $A \in \mathbb{M}_2(R)$. Then $A$ has strong left stable range one if and only if $A$ has strong right stable range one.*

Therefore, in the sequel we remove the adjective "left" (or "right") and discuss only about (left) strong stable range one $2 \times 2$ matrices over commutative rings.

**Proposition 2.4.** *Assume $R$ is a commutative ring. A $2 \times 2$ matrix $U$ is unipotent if and only if* $\det(U) = 1$ *and* $\mathrm{Tr}(U) = 2$.

*Proof.* Indeed, if $U = I_2 + T$, with nilpotent $T$, then $\det(I_2 + T) = 1 + \mathrm{Tr}(T) + \det(T) = 1$ and $\mathrm{Tr}(I_2 + T) = 2$. Conversely, if $\det(U) = 1$ and $\mathrm{Tr}(U) = 2$, by Cayley-Hamilton's theorem, $U^2 - 2U + I_2 = (U - I_2)^2 = 0_2$. Hence $U = I_2 + T$ with $T^2 = 0_2$. □

2.3. **Applications to special matrices.** In this section we browse some of the examples in [4].

**1**. In $\mathbb{M}_n(R)$, all matrices $rE_{ij}$ have stable range one. However, *this fails for strong stable range one*.

For a start, $E_{11}$ has strong stable range one, as any idempotent has it.

**Proposition 2.5.** *Let $R$ be a commutative ring and $r \in R$, $r \notin \{\pm 1\}$. The matrix $rE_{11}$ has not strong stable range one whenever $r + 1 \notin U(R)$.*

*Proof.* We use Theorem 2.3. Since $\det(rE_{11}) = 0$, the conditions reduce to $\det(Y)(1 - \mathrm{Tr}(rXE_{11}) - \mathrm{Tr}(rE_{11}adj(Y) = 1$ and $r - \mathrm{Tr}(Y) + \mathrm{Tr}(rYXE_{11}) = 2$. By computation, the second becomes $w = (ra - 1)x + rcy$, and using this, the first becomes $(w + 1)^2 = [(ra - 1)z + rcw]y$.

We choose $a = 2r + 1$ and $c = r + 1$. Then $w = (r + 1)[(2r - 1)x + ry]$ and $(w + 1)^2 = [(ra - 1)z + rcw]y$. Denote $k = (2r - 1)x + ry$ and so $w = (r + 1)k$. Replacing in the second equality gives $(r+1)[(2r-1)yz+2(r+1)y-(r+1)k^2-2k] = 1$ with no solution if $r + 1 \notin U(R)$. □

**Corollary 2.3.** *Over $\mathbb{Z}$, the multiples $nE_{11}$ have not strong stable range one, excepting $n \in \{0, 1\}$.*

*Proof.* As for $n = -1$, we use Lemma 2.1, (ii). □

**2**. If $R$ is a Bézout ring (the sum of any two principal ideals is again a principal ideal) then $A = \begin{bmatrix} r & s \\ 0 & 0 \end{bmatrix}$ has stable range one. *This fails for strong stable range one* even over commutative rings.

**Proposition 2.6.** *Let $R$ be a commutative ring. Then $A = 2E_{11} + E_{12} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbb{M}_2(R)$ has not strong stable range one if $3 \notin U(R)$.*

*Proof.* As $\det(A) = 0$, the characterization conditions become $\det(Y)(1 - \mathrm{Tr}(XA)) - \mathrm{Tr}(Aadj(Y) = 1$ and $\mathrm{Tr}(Y) = \mathrm{Tr}(YXA) = 0$. By computation the second gives $(2a - 1)x + 2cy + az + (c - 1)w = 0$ and the first gives $(xw - yz)(1 - 2a - c) - (2w - z) = 1$.

Take $a = -1$, $c = 3$. Then the conditions amount to $z - 2w = 1$ and $-3(x - 2y) - z + 2w = -3(x - 2y) - 1 = 0$ with no solutions if 3 is not a unit. □

On the contrary, $E_{11} + E_{12} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ has strong stable range one, being idempotent.

**3**. A matrix $A \in \mathbb{M}_2(\mathbb{Z})$ has stable range one if and only if $\det(A) \in \{-1, 0, 1\}$, that is, units or zero determinant matrices.

By Proposition 2.3, the only units which have strong stable range one are the unipotents. Using again Theorem 2.3 and Proposition 2.4 we obtain

**Theorem 2.4.** *An integral matrix $A$ has strong stable range one if and only if it is unipotent or has zero determinant. In the first case,* $\mathrm{Tr}(A) = 2$ *and* $\det(A) = 1$ *and in the second case for every $X$ there exists $Y$ such that* $\det(Y)(1 - \mathrm{Tr}(XA)) - \mathrm{Tr}(AadjY) = 1$ *and* $\mathrm{Tr}(A) - \mathrm{Tr}(Y) + \mathrm{Tr}(YXA) = 2$.

As already mentioned, for example $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$ has stable range one but not strong stable range one.

**Remark 2.2.** Notice that in any matrix, adding a multiple of one row (or column) to another row (resp. column) is an elementary operation which can be performed, up to a unipotent equivalence, by multiplications with unipotents. Indeed, such row operations can be performed by left multiplications with matrices obtained performing the same row operations on a copy of the identity matrix, which is a unipotent matrix. Symmetrically, such column operations amount to right multiplications with unipotent matrices.

**4**. Recall from [4] that *nilpotent integral* $2 \times 2$ *matrices* have stable range one.

**Proposition 2.7.** *The nilpotent matrix* $E_{12}$ *has strong stable range one over any commutative ring.*

*Proof.* We check that if $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $Y = \begin{bmatrix} a-2 & 0 \\ 1 & 0 \end{bmatrix}$ is a suitable unitizer. We use Theorem 2.3. Since $\det(E_{12}) = \mathrm{Tr}(E_{12}) = 0$, for every $X$ we need an $Y$ with $\det(Y)(1 - \mathrm{Tr}(XE_{12})) - \mathrm{Tr}(E_{12}adjY) = 1$ and $\mathrm{Tr}(YXE_{12}) = 2 + \mathrm{Tr}(Y)$.

These reduce to $z[ax + (c-1)y + 1] = (x+1)^2$ and $az + (c-1)w = x + 2$. A solution is $x = a - 2$, $z = 1$ and $y = w = 0$. Indeed, $E_{12} + Y(XE_{12} - I_2) = E_{12} + \begin{bmatrix} a-2 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & a \\ 0 & c-1 \end{bmatrix} = \begin{bmatrix} 2-a & (a-1)^2 \\ -1 & a \end{bmatrix}$. We get $\begin{bmatrix} 1-a & (a-1)^2 \\ -1 & a-1 \end{bmatrix}$, by subtracting $I_2$, which is nilpotent (zero determinant and zero trace). $\square$

It was harder to prove that $2E_{12}$ has strong stable range one, that is, to find a suitable unitizer.

As in the case above, since $2XE_{12} - I_2$ depends only on $a$ and $c$, so is also the suitable unitizer $Y = \begin{bmatrix} 2c^2 + 2ac - 3c - 1, & 2c^2 + 2a(c-1) - 5c + 2 \\ c & c-1 \end{bmatrix}$. In order to keep the fluency of our exposition, the details are given in the Annex.

Actually, more can be proved.

**Theorem 2.5.** *Let* $R$ *be a commutative ring and* $r \in R$. *Then the* $2 \times 2$ *nilpotent matrix* $rE_{12}$ *has strong stable range one.*

*Proof.* Roughly speaking, in the proof given in the Annex, we just replace 2 by $r$. A suitable unitizer for $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ turns out to be

$$Y = \begin{bmatrix} rc^2 + rac - (r+1)c - 1 & rc^2 + ra(c-1) - (2r+1)c + r \\ c & c-1 \end{bmatrix}.$$

$\square$

These nilpotents are important since *over any Bézout domain, every nilpotent* $2 \times 2$ *matrix is similar to some* $rE_{12}$ (see Annex for a proof). However, we cannot infer that over any Bézout domain, every $2 \times 2$ nilpotent matrix has strong stable range one, since the proof has a unitizer which may not be unipotent.

Unipotent similarity will be addressed elsewhere.

## 3. Unipotent stable range one elements

As the previous subsection was depending on [4], this section depends on results recently obtained on unit stable range one elements in [1].

We first recall the definitions. An element $a \in R$ has *unipotent left stable range 1* if for every $x \in R$ there exists a *unipotent* $u$ such that $a+u(1-xa)$ is a unit and *strong unipotent left stable range 1* if for every $x \in R$ there exists a *unipotent* $u$ such that $a+u(1-xa)$ is a *unipotent*.

Secondly, we recall the results in [1].

By left multiplication with $-u^{-1}$ (and change of notation), notice that $a$ has *(strong) unipotent left stable range one* if and only if for every $x \in R$, there is a unipotent $u$ such that

$$(u+x)a+1$$

is a unit (resp. unipotent).

Recall (well-known as the "*Jacobson"s Lemma*") that for any unital ring $R$ and elements $\alpha, \beta \in R$, $1+\alpha\beta$ is a unit if and only if $1+\beta\alpha$ is a unit. Using the last equivalent definition, it follows that *the unipotent stable range one (for elements) is a left-right symmetric property*. The *strong unipotent stable range one (for elements) is also a left-right symmetric property* because the Jacobson's Lemma holds also for unipotents (indeed, it reduces to $\alpha\beta \in N(R)$ implies $\beta\alpha \in N(R)$ which holds: $(\alpha\beta)^n = 0$ implies $(\beta\alpha)^{n+1} = 0$). Hence, in the sequel we discuss (strong) unipotent left stable range one elements removing the adjective "left".

An element $a \in R$ satisfies the *strong GM* (Goodearl-Menal) *condition* if for every $x$ there exists a unit $u$ such that $x-u$, $a-u^{-1}$ are unipotents and satisfies the *(strong) unipotent GM condition* if for every $x \in R$, there exists a unipotent $u$ such that both $x-u$, $a-u^{-1}$ are units (resp. unipotents). Notice that $a-u^{-1}$ is a unit if and only if $ua-1$ is a unit, which is a special case of the unit stable range one definition, for $x=0$.

Analogous results to the results in [1] can be obtained *mutatis mutandis* in the unipotent stable range one case. However, for strong unipotent stable range one, the results are essentially different.

In order not to lengthen the exposition we just state some of the results concerning unipotent stable range one elements and matrices and prove the results concerning strong unipotent stable range one elements and matrices.

**Lemma 3.2.** *If $a$ satisfies the (strong) unipotent GM condition then $a$ has the (strong) unipotent stable range one.*

**Lemma 3.3.** *Over NR rings (strong) unipotent stable range one elements are invariant to unipotent equivalences.*

**Example 3.1.** Over any ring, $rE_{12}$ has unipotent stable range one.

For every $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we can take the unipotent unitizer $U = \begin{bmatrix} 1 & 0 \\ -c & 1 \end{bmatrix}$. Then $(U+X)rE_{12} - I_2 = \begin{bmatrix} -1 & (a+1)r \\ 0 & -1 \end{bmatrix}$ is a unit.

**Theorem 3.6.** *(i) Let $R$ be a commutative ring and $A \in \mathbb{M}_2(R)$. Then $A$ has unipotent stable range 1 if and only if for any $X \in \mathbb{M}_2(R)$ there exists a unipotent $U \in \mathbb{M}_2(R)$ such that*

$$\det((U+X)A) + \mathrm{Tr}((U+X)A) + 1$$

*is a unit of $R$.*

*(ii) Let $R$ be a commutative ring and $A \in \mathbb{M}_3(R)$. Then $A$ has unipotent stable range 1 if and only if for any $X \in \mathbb{M}_3(R)$ there exists a unipotent $U \in \mathbb{M}_3(R)$ such that*

$$\det((U+X)A) + \mathrm{Tr}(adj(U+X)A) + \mathrm{Tr}((U+X)A) + 1$$

*is a unit of $R$.*

**Theorem 3.7.** *(i) Let $R$ be a commutative ring and $A \in \mathbb{M}_2(R)$. Then $A$ has strong unipotent stable range 1 if and only if for any $X \in \mathbb{M}_2(R)$ there exists a unipotent $U \in \mathbb{M}_2(R)$ such that* $\det[(U+X)A] = \mathrm{Tr}[(U+X)A] = 0$.

*(ii) Let $R$ be a commutative ring and $A \in \mathbb{M}_3(R)$. Then $A$ has strong unipotent stable range 1 if and only if for any $X \in \mathbb{M}_3(R)$ there exists a unipotent $U \in \mathbb{M}_3(R)$ such that* $\det[(U+X)A] = \mathrm{Tr}[(U+X)A] = \mathrm{Tr}[(U+X)A]^2 = 0$.

*Proof.* (i) Indeed, $(U+X)A + I_2$ is unipotent if and only if $(U+X)A$ is nilpotent if and only if $\det[(U+X)A] = \mathrm{Tr}[(U+X)A] = 0$.

(ii) Again, $(U+X)A + I_3$ is unipotent if and only if $(U+X)A$ is nilpotent if and only if $\det[(U+X)A] = \mathrm{Tr}[(U+X)A] = \mathrm{Tr}[(U+X)A]^2 = 0$. □

Finally

**Theorem 3.8.** *A $2 \times 2$ matrix $A$ over a commutative ring satisfies the (strong) unipotent left GM condition if and only if for every $X$ there is a unipotent $U$ such that $\det(X) + 1 - \mathrm{Tr}(adj(X)U)$ and $\det(A) - \mathrm{Tr}(UA) + 1$ are units (resp. unipotents) of $R$.*

According to Proposition 2.4, here $\det(U) = 1$ and $\mathrm{Tr}(U) = 2$. In the strong unipotent *left* case, this amounts to nilpotent $\det(X) - \mathrm{Tr}(adj(Y)U)$ and $\det(A) - \mathrm{Tr}(UA)$.

## 4. Reduced rings

As reduced rings have only one unipotent element, namely 1, the definitions we have introduced are drastically affected. In particular this happens for domains and even more special, for division rings. Recall that reduced rings are *Abelian* (i.e., have only central idempotents) and so *Dedekind finite* (i.e., one-sided inverses are two-sided; *DF*, for short).

**Proposition 4.8.** *Let $R$ be a reduced ring.*

*(i) An element $a \in R$ has strong stable range one if and only if $a \in J(R)$.*

*(ii) An element $a \in R$ has unipotent stable range one if and only if $a \in J(R)$.*

*(iii) The only element which has strong unipotent stable range one is 0.*

*Proof.* (i) An element $a$ has strong stable range one if and only if for every $x$ there is $y$ such that $a + y(1 - xa) = 1$. Taking $x = 1$ shows that $(1-y)(1-a) = 1$ and so (using DF), $1 - a \in U(R)$. Hence, for every $x \in R$, there is $y$ such that $y(1 - xa) \in U(R)$. Therefore, since the ring is DF, $1 - xa \in U(R)$ for every $x \in R$. Finally, $a \in J(R)$. Conversely, if $a \in J(R)$ then for every $x$, $1 - xa$ is invertible and so we can choose as unitizer $y = (1-a)(1-xa)^{-1}(1-xa)$.

(ii) An element $a$ has unipotent stable range one if and only if for every $x \in R$, $a + (1 - xa) \in U(R)$. Equivalently, $1 + Ra \subseteq U(R)$. By Jacobson's Lemma, we also have $1 + aR \subseteq U(R)$ and so $a \in J(R)$, the Jacobson radical (the largest ideal $I$ with $1 + I \subseteq U(R)$). Conversely, if $a \in J(R)$ then $1 - za$ is (left) invertible for every $z$. Taking $z = x - 1$ gives a unit $a + (1 - xa)$, as desired.

(iii) An element $a$ has strong unipotent stable range one if and only if for every $x \in R$, $a + (1 - xa) = 1$. Taking $x = 0$ shows that $a = 0$. The converse is obvious. □

## 5. Unipotent-regular rings and $2 \times 2$ matrices

Coming back to unit-regular elements, some of these *have strong stable range one*: the ones we have called *unipotent-regular*, i.e., elements $a \in R$ for which there is $t \in N(R)$ such that $a = a(1+t)a$. This follows from Theorem 2.2.

As in the case of unit-regular elements, *an element is unipotent-regular iff it is a product of an idempotent and a unipotent*. Indeed $[a(1+t)]^2 = a(1+t) = e$ implies $a = e(1+s)$, for $s \in N(R)$. Actually $s = -t + t^2 - ... + (-1)^{n-1}t^{n-1}$ whenever $t^n = 0$.

The so-called *UU rings* (rings with only unipotent units) were defined and studied in [2]. Their study was further developed in [7].

We have the following result.

**Theorem 5.9.** *A ring is unipotent-regular if and only if it is Boolean.*

*Proof.* First observe that a unit is unipotent-regular if and only if it is unipotent. Indeed, as inverses of unipotents are unipotent, one way is obvious. Conversely, suppose $u \in U(R)$ and $u = e(1+t)$ for an idempotent $e$ and a nilpotent $t$. Then $eu = u$ and so $e = 1$. Hence $u = 1 + t$, as claimed.

As a consequence, every unipotent-regular ring is UU. It remains only to recall from [7] (see Theorem 4.1, (5) $\Leftrightarrow$ (3)) that $R$ is a regular UU ring iff $R$ is a Boolean ring. $\square$

As for matrix rings, since *for any ring $R \neq 0$ and any integer $n \geq 2$, $\mathbb{M}_n(R)$ is not a UU ring* (see [2]), we have the following result

**Proposition 5.9.** *For any ring $R \neq 0$ and any integer $n \geq 2$, $\mathbb{M}_n(R)$ is not unipotent-regular.*

In what follows we determine the $2 \times 2$ unipotent-regular matrices over a Prüfer domain.

First notice that *only zero determinant $2 \times 2$ matrices can be unipotent-regular*. Indeed, this follows at once since

$$\det(E(I_2 + T)) = \det(E)\det(I_2 + T) = 0 \cdot 1 = 0,$$

for any idempotent $E$ and nilpotent $T$.

Next, since in our characterization we use Prüfer domains, recall that a *Prüfer domain* is a semihereditary integral domain. Equivalently, an integral domain $R$ is a Prüfer domain if every nonzero finitely generated ideal of $R$ is invertible. Fields, PIDs and Bézout domains are Prüfer domains but UFDs may not be Prüfer.

In the next theorem we intend to use the Kronecker (Rouché) - Capelli theorem for compatible linear systems. As early as 1971 we recall from [3] the following characterization.

**Theorem 5.10.** *Let $R$ be an integral domain, $A$ a matrix of rank $r$ over $R$ and $\mathbf{x}$ and $\mathbf{b}$ column vectors over $R$. The condition $D_r(A) = D_r[A, \mathbf{b}]$ is necessary and sufficient for the system $A\mathbf{x} = \mathbf{b}$ to be solvable if and only if $R$ is a Prüfer domain.*

Here the ideal $D_t(A)$ generated by the $t \times t$ minors of the matrix is called the *$t$-th determinantal ideal* of $A$ and we put $D_0 = 1$. As customarily, $[A, \mathbf{b}]$ denotes the augmented matrix.

**Theorem 5.11.** *A (zero determinant) matrix $A = [a_{ij}]_{1 \leq i,j \leq 2}$ over a Prüfer domain is unipotent-regular if and only if there exist $a, c$ with $c \mid a(1-a)$ such that $c\mathrm{row}_1(A) = a\mathrm{row}_2(A)$ and if $bc = a(1-a)$ then $(a_{11} - a)^2, (a_{12} - b)^2$ and $(a_{11} - a)(a_{12} - b)$ are divisible by $ba_{11} - aa_{12}$. The divisibilities are equivalent with $(a_{21} - c)^2, (a_{22} + a - 1)^2$ and $(a_{21} - c)(a_{22} + a - 1)$ being divisible by $(1-a)a_{21} - ca_{22}$.*

We discuss separately the cases $a \in \{0, 1\}$, so below we assume $a, b, c \neq 0$ and $a \neq 1$.

*Proof.* Over any integral domain a unipotent-regular $2 \times 2$ matrix is of form $E(I_2+T) = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix} \begin{bmatrix} 1+x & y \\ z & 1-x \end{bmatrix}$ with $a(1-a) = bc$ and $x^2+yz = 0$. Denoting $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the equality $A = E(I_2+T)$ amounts to the system

$$\begin{array}{lcl} a(1+x)+bz & = & a_{11} \\ b(1-x)+ay & = & a_{12} \\ c(1+x)+(1-a)z & = & a_{21} \\ (1-a)(1-x)+cy & = & a_{22} \end{array}$$

We write the linear system as follows

$$\begin{array}{lcl} ax+bz & = & a_{11}-a \\ -bx+ay & = & a_{12}-b \\ cx+(1-a)z & = & a_{21}-c \\ -(1-a)x+cy & = & a_{22}+a-1 \end{array}.$$

The four equations form a linear system with 3 unknowns and 4 equations whose augmented matrix is

$$\begin{bmatrix} a & 0 & b & a_{11}-a \\ -b & a & 0 & a_{12}-b \\ c & 0 & 1-a & a_{21}-c \\ a-1 & c & 0 & a_{22}+a-1 \end{bmatrix}.$$

An easy computation shows that the system matrix

$$\begin{bmatrix} a & 0 & b \\ -b & a & 0 \\ c & 0 & 1-a \\ a-1 & c & 0 \end{bmatrix}$$

has rank 2, as $a(1-a) = bc$.

Since the $3 \times 3$ minors of the system matrix are zero, so is the determinant of the augmented matrix.

Another easy computation shows that the remaining twelve $3 \times 3$ minors of the augmented matrix are zero iff $crow_1(A) = arow_2(A)$.

Thus, in order to find a solution we select (say) the first two equations i.e., $ax+bz = a_{11}-a$, $-bx+ay = a_{12}-b$. Then $x = \frac{a_{11}-a-bz}{a}$ and $y = \frac{b(a_{11}-a-bz)+a(a_{12}-b)}{a^2}$ and replacing in $x^2+yz = 0$, $x = -\frac{(a_{11}-a)(a_{12}-b)}{ba_{11}-aa_{12}}$, $y = -\frac{(a_{12}-b)^2}{ba_{11}-aa_{12}}$ and $z = \frac{(a_{11}-a)^2}{ba_{11}-aa_{12}}$. Hence, the existence of this solution requires the divisibilities in the statement. □

**The case** $a = 1$. As $a(1-a) = bc$, at least one of $b, c$ must be zero and (say $c = 0$) $E = \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}$. Then $a_{21} = a_{22} = 0$ are necessary conditions for a matrix $A = [a_{ij}]_{1 \leq i,j \leq 2}$ to be unipotent-regular. As in the previous proof, $x = a_{11}-bz-1$, $y = a_{12}-b+bx = a_{12}-b+b(a_{11}-bz-1)$ and $x^2+yz = 0$ gives $x = -\frac{(a_{11}-1)(a_{12}-b)}{ba_{11}-a_{12}}$, $y = -\frac{(a_{12}-b)^2}{ba_{11}-a_{12}}$ and $z = \frac{(a_{11}-1)^2}{ba_{11}-a_{12}}$ with $(a_{11}-1)^2, (a_{12}-b)^2$ and $(a_{11}-1)(a_{12}-b)$ divisible by $ba_{11}-a_{12}$.

The case $b = 0$ follows by transpose.

**The case** $a = 0$. Again at least one of $b, c$ must be zero and (say) $E = \begin{bmatrix} 0 & b \\ 0 & 1 \end{bmatrix}$. The first two equations of the linear system are $bz = a_{11}$, $b(1-x) = a_{12}$. Therefore if both $a_{11}, a_{12}$ are divisible by $b$,we get $x = 1 - \frac{a_{12}}{b}$, $z = \frac{a_{11}}{b}$ and arbitrary $y$.

**Remark 5.3.** If $R$ is not an integral domain, we don't have a known form for $2 \times 2$ idempotent or nilpotent matrices and so the above proof is not suitable.

Same for $n \times n$ matrices with $n \geq 3$.

In view of Theorem 5.9, we could wonder whether there exist unipotent-regular matrices which are not idempotent. Such matrices do exist.

**Example 5.2.** The zero determinant integral matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is *not idempotent but is unipotent-regular*.

The rows are dependent so we can take $a = k$, $c = 2k$ for any $k$. To choose $b$, from $k(1-k) = 2kb$ we need $2b = 1 - k$.

For $k = 1$, that is $a = 1$, $c = 2$, $c$ divides $a(1-a) = 0$. Then $b = 0$ and $0^2, 2^2$ and $0 \cdot 2$ are divisible by $a_{12} = 2$. Indeed, $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is an idempotent-unipotent product.

The decomposition is unique. Since $2b = 1 - k$, $k$ must be odd, say $k = 2l - 1$. Then $b - 2a = 3 - 5l$ should divide $2(1-l)^2$, $(1-l)^2$ and $4(1-l)^2$. Over $\mathbb{Z}$, this amounts to a quadratic Diophantine equation $l^2 + 5lm - 2l - 3m + 1 = 0$ which has only one solution: $(l, m) = (1, 0)$. Hence $k = 1$.

## 6. Annex

**1**. *The details in proving that $2E_{12}$ has strong stable range one.*

For $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we are looking for $Y = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ such that $2E_{12} + Y(XE_{12} - I_2)$ is unipotent (i.e., subtracting $I_2$ we get a zero trace and zero determinant matrix). Since

$$2E_{12} + Y(XE_{12} - I_2) = \begin{bmatrix} -x-1 & 2ax + (2c-1)y + 2 \\ -z & 2az + (2c-1)w - 1 \end{bmatrix},$$

the conditions are $2az + (2c-1)w = x + 2$ and (using this) $(x+1)^2 = z[2ax + (2c-1)y + 2]$.

The first equation is a linear (Diophantine) equation with the obvious solution $(-2 + 2az + (2c-1)w, z, w)$. Replacing in the second equation gives

$$(2c-1)yz = 1 + (2c-1)^2w^2 - 3(c-1)w + 2a(2c-1)zw - 2z$$

from which we have to find $y$ (depending of $z, w$).

In order to get $y$ two divisibilities would be sufficient: $z \mid [(2c-1)w - 1]^2$ and $2c - 1 \mid (-1 + 2az)^2 - z[2a(2az-2) + 2 = 1 - 2z$.

For the second, $z = c$ is suitable and thus the first reduces to $c \mid (w+1)^2$. For this, $w = c - 1$ is suitable.

This way we obtain $y = 2c^2 + 2a(c-1) - 5c + 2$ and so $(2c^2 + 2ac - 3c - 1, 2c^2 + 2a(c-1) - 5c + 2, c, c-1)$ is a solution which gives a suitable unitizer (as claimed in Section 3).

**2**. *The proof of the auxiliary result.*

**Proposition 6.10.** *Every nonzero nilpotent $2 \times 2$ matrix over a (commutative) Bézout domain $R$ is similar to $rE_{12}$, for some $r \in R$.*

*Proof.* We are looking for an invertible matrix $U=(u_{ij})$ such that $TU=U(rE_{12})$ with $T=\begin{bmatrix} x & y \\ z & -x \end{bmatrix}$ and $x^2+yz=0$.

Recall that every Bézout domain is a GCD (greatest common divisors exist) domain.

Let $d=\gcd(x;y)$ and denote $x=dx_1$, $y=dy_1$ with $\gcd(x_1;y_1)=1$. Then $d^2x_1^2=-dy_1z$ and since $\gcd(x_1;y_1)=1$ implies $\gcd(x_1^2;y_1)=1$, it follows $y_1$ divides $d$. Set $d=y_1y_2$ and so $T=\begin{bmatrix} x_1y_1y_2 & y_1^2y_2 \\ -x_1^2y_2 & -x_1y_1y_2 \end{bmatrix}=y_2\begin{bmatrix} x_1y_1 & y_1^2 \\ -x_1^2 & -x_1y_1 \end{bmatrix}=y_2T'$.

Since $\gcd(x_1;y_1)=1$ there exist $s,t\in R$ such that $sx_1+ty_1=1$. Take $U=\begin{bmatrix} y_1 & s \\ -x_1 & t \end{bmatrix}$ which is invertible (indeed, $U^{-1}=\begin{bmatrix} t & -s \\ x_1 & y_1 \end{bmatrix}$). One can check $T'U=\begin{bmatrix} 0 & y_1 \\ 0 & -x_1 \end{bmatrix}=UE_{12}$, so $r=y_2$. $\square$

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