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Self-adaptive CQ-type algorithms for the split feasibility problem involving two bounded linear operators in Hilbert spaces

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ABSTRACT. In this article, we consider and investigate a split convex feasibility problem involving two bounded linear operators in Hilbert spaces. We introduce a self-adaptive CQ-type algorithm by selecting the stepsize which is independent of the operator norms and establish a strong convergence result of the proposed algorithm under some mild control conditions. Moreover, we propose a self-adaptive relaxed CQ-type algorithm for solving the problem constrained by sub-level sets of convex functions. A numerical example and an application in compressed sensing are also given to illustrate the convergence behaviour of our proposed algorithms. Our results in this paper improve and generalize some existing results in the literature.

1. INTRODUCTION

Let C and Q be two nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. The *split feasibility problem* (shortly, SFP) is to find a point

$$x \in C \quad \text{such that} \quad Ax \in Q, \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator. The SFP is the first instance of the split inverse problem (referred to [13, Sect. 2]), which was first introduced by Censor and Elfving [11] in Euclidean spaces. The SFP model can be applied to solving many mathematical problems such as the constrained least-squares problem, the linear split feasibility problem, and the linear programming problem and it can be used in real-world applications, for example, in signal processing, in image recovery, in intensity-modulated therapy, in pattern recognition and in data prediction (see [3, 5, 10, 12, 20, 22]). Consequently, the SFP has been widely studied and various methods for solving such a problem have been invented and developed by many authors, see [2, 9, 17, 24, 25, 35, 36, 37, 38, 41, 43, 44] and the references therein. One of the powerful methods for approximating solutions of (1.1) is known as the *CQ algorithm* introduced by Byrne [2] as follows:

$$\begin{cases} x_1 \in H_1, \\ x_{k+1} = P_C\left(x_k - \lambda A^*(I - P_Q)Ax_k\right), \quad k \geq 1, \end{cases} \tag{1.2}$$

where $\lambda \in \left(0, 2/\|A\|^2\right)$, P_C and P_Q are the metric projections onto C and Q, respectively, and A^* stands for the adjoint operator of A. After that, various kinds of the split inverse problem, which are generalizations of the SFP were introduced and studied, see [4, 12, 13, 14, 28, 32] for instance.

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In this paper, we focus on a generalization of the SFP (1.1) in which two bounded linear operators $A, B : H_1 \to H_2$ are involved – Finding a point

$$x \in C \quad \text{such that} \quad Ax \in Q \ \text{ and } Bx \in \tilde{Q}, \tag{1.3}$$

where $C \subseteq H_1$ and $Q, \tilde{Q} \subseteq H_2$ are nonempty closed convex subsets. We call (1.3) the *two-operator split feasibility problem* (two-operator SFP), see [28, 32] for the general versions of this problem. The two-operator SFP (1.3) can be reduced to the convexly constrained linear problem ([15, 29]) involving two linear operators, that is, finding a point $x \in C$ such that $Ax = y, Bx = \tilde{y}$ in H_2.

In 2019, Kangtunyakarn [21] studied the two-operator SFP (1.3) in case that $Q = \tilde{Q}$ and introduced a viscosity-based algorithm with a given contraction $f : C \to C$ as follows:

(1.4)

$$\begin{cases} x_1 \in C, \\ x_{k+1} = \beta_k f(x_k) + \delta_k x_k + \gamma_k P_C\Big[x_k - \dfrac{\lambda}{2}\Big(A^*(I - P_Q)Ax_k + B^*(I - P_Q)Bx_k\Big)\Big], \quad k \geq 1, \end{cases}$$

where $\lambda \in \big(0, 2/\max\{\|A\|^2, \|B\|^2\}\big)$ and $\{\beta_k\}, \{\delta_k\}$, and $\{\gamma_k\}$ are real sequences in $(0,1)$. A strong convergence theorem of (1.4) was proved under some suitable conditions on the control sequences, see [21, Theorem 3.1].

It is noted that the parameters λ in (1.2) and in (1.4) depend on the norms of bounded linear operators, so these algorithms have a drawback in the sense that the implementation of them requires to calculate or estimate the operator norms, which is not an easy task in general practice (see [25, Subsection 6.1.2] for instance). To overcome this, in [2, Proposition 4.1], it was presented a helpful method for estimating operator (matrix) norms but its conditions seem restrictive. López et al. [25] proposed an alternative way that is to select the stepsize λ_k which does not need any prior knowledge of the operator norm for replacing the parameter λ in (1.2) as follows:

$$\lambda_k := \frac{\mu_k \|(I - P_Q)Ax_k\|^2}{2\,\|A^*(I - P_Q)Ax_k\|^2}, \tag{1.5}$$

where $\mu_k \in (0, 4)$. We can see that the choice of the stepsize λ_k in (1.5) is independent of the operator norm $\|A\|$. This stepsize was widely employed in optimization methods and was also modified for use in fixed point methods, see [8, 18, 19, 27, 33, 35]. The CQ algorithm with the self-adaptive stepsize defined by (1.5) [25, Algorithm 3.1] guarantees only weak convergence for the SFP (1.1), see [25, Theorem 3.5]. However, strong convergence gives more desirable theoretical result in the setting of Hilbert spaces. To get strong convergence, Vinh et al. [35] employed a modification of the CQ algorithm ([37, Algorithm 4.1]) with the stepsize (1.5) for solving the SFP (1.1) as follows:

$$\begin{cases} x_1 \in H_1, \\ x_{k+1} = P_C\Big[(1 - \beta_k)\big(x_k - \lambda_k A^*(I - P_Q)Ax_k\big)\Big], \quad k \geq 1, \end{cases} \tag{1.6}$$

where the stepsize λ_k is defined by (1.5) and $\{\beta_k\} \subset (0,1)$. They proved that the sequence generated by (1.6) converges strongly to the minimum-norm solution to (1.1) under some suitable control conditions, see [35, Theorem 3.1].

Here, the above review leads us to the following natural questions.

1. Can we design a CQ-type algorithm whose stepsize does not depend on the operator norm $\|A\|$ or $\|B\|$ to solve the two-operator SFP (1.3)?
2. How do we adapt the algorithm designed from Question 1 to be a strongly convergent method?

Motivated and inspired by the above questions and the results of Kangtunyakarn [21], López et al. [25], and Vinh et al. [35], we aim to invent a self-adaptive CQ-type algorithm whose stepsize does not depend on any operator norms for solving the two-operator SFP in the setting of Hilbert spaces. Moreover, we will prove that the sequence generated by the proposed algorithm converges strongly to the minimum-norm solution. The rest of the paper is organized as follows. In Sect. 2, some basic facts and useful lemmas for proving our main results are given. Our main result is in Sect. 3. In this section, we introduce a self-adaptive CQ-type algorithm using the stepsize which is independent of the bounded linear operator norms for finding a solution of (1.3). A strong convergence theorem of the proposed algorithm is analyzed and established. In Sect. 4, we propose a self-adaptive relaxed CQ-type algorithm for solving the two-operator SFP in case of sub-level sets of convex functions and also prove its strong convergence result. Finally, in Sect. 5, we provide numerical experiments of our proposed algorithms in the setting of a Euclidean space and in the signal recovery problem with two different blurring operations, and also compare the efficiency of our algorithms with that of some methods depending on the operator norms.

2. Preliminaries

Throughout this paper, we suppose that H, H_1 and H_2 are real Hilbert spaces with inner products $\langle\cdot,\cdot\rangle$ and the induced norms $\|\cdot\|$ (in particular, in Euclidean spaces, $\|\cdot\|_1$ denotes the l_1-norm and $\|\cdot\|_2$ denotes the Euclidean norm). The notation I stands for the identity operator on a Hilbert space. Let $\{x_k\}$ be a sequence in H. Weak and strong convergence of $\{x_k\}$ to $x \in H$ are denoted by $x_k \rightharpoonup x$ and $x_k \to x$, respectively. The set of all weak-cluster points of $\{x_k\}$ is denoted by $\omega_w(x_k)$.

Let $f : H \to \mathbb{R}$ be a function and $x \in H$. We say that f is weakly lower semi-continuous at x if for every sequence $\{x_k\} \subset H$, $x_k \rightharpoonup x$ implies $f(x) \leq \liminf_{k\to\infty} f(x_k)$. A subdifferential ∂f of f at x is defined by

$$\partial f(x) = \{u \in H : f(x) + \langle u, z - x\rangle \leq f(z),\ \forall z \in H\}.$$

The function f is said to be subdifferentiable at x if $\partial f(x) \neq \emptyset$. One can see that if f is subdifferentiable at x, then f is weakly lower semi-continuous at x. We denote the gradient of f by ∇f if f is differentiable.

Let K be a nonempty closed convex subset of H. Recall that the metric projection P_K from H onto K assigns to each $x \in H$ the unique point $P_K x$ in K satisfying $\|x - P_K x\| = \inf_{z\in K} \|x - z\|$. Some properties of the metric projection are listed below.

Lemma 2.1. *The metric projection P_K has the following properties:*

(1) $\langle x - P_K x, z - P_K x\rangle \leq 0, \quad \forall x \in H, \forall z \in K;$
(2) $\langle x - P_K x, x - z\rangle \geq \|x - P_K x\|^2, \quad \forall x \in H, \forall z \in K;$
(3) *P_K is firmly nonexpansive, i.e.,*

$$\|P_K x - P_K y\|^2 \leq \|x - y\|^2 - \|(x - P_K x) - (y - P_K y)\|^2, \quad \forall x, y \in H,$$

in particular,

$$\|P_K x - z\|^2 \leq \|x - z\|^2 - \|x - P_K x\|^2, \quad \forall x \in H, \forall z \in K.$$

Let Q be a nonempty closed convex subset of H_2 and let $A : H_1 \to H_2$ be a bounded linear operator with the adjoint operator A^*. Define a function $f : H_1 \to \mathbb{R}$ by

$$f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2.$$

We know that f is weakly lower semi-continuous on H_1 and differentiable with the gradient $\nabla f : H_1 \to H_1$ given by

$$\nabla f(x) = A^*(I - P_Q)Ax.$$

Moreover, ∇f is Lipschitz continuous with the Lipschitz constant $\|A\|^2$, i.e.,

$$\|\nabla f(x) - \nabla f(y)\| \leq \|A\|^2 \|x - y\|, \quad \forall x, y \in H_1.$$

For more details, the reader is referred to optimization books, see [1, 31] for instance.

We end this section with the following useful lemmas for proving our strong convergence results.

Lemma 2.2 ([39]). *Let $\{t_k\}$ be a sequence of nonnegative real numbers satisfying*

$$t_{k+1} \leq (1 - \beta_k)t_k + \beta_k \delta_k, \quad \forall k \in \mathbb{N},$$

where $\{\beta_k\}$ is a sequence in $(0, 1)$ and $\{\delta_k\}$ is a sequence of real numbers such that $\sum_{k=1}^{\infty} \beta_k = \infty$ and $\limsup_{k\to\infty} \delta_k \leq 0$. Then, $\lim_{k\to\infty} t_k = 0$.

Lemma 2.3 ([26]). *Let $\{s_k\}$ be a sequence of real numbers such that there exists a subsequence $\{k_j\}$ of $\{k\}$ which satisfies $s_{k_j} < s_{k_j+1}$ for all $j \in \mathbb{N}$. Let $\{\tau(k)\}$ be a sequence of positive integers defined by*

$$\tau(k) := \max\{n \leq k : s_n < s_{n+1}\}$$

for all $k \geq k_0$ (for some k_0 large enough). Then $\{\tau(k)\}$ is a nondecreasing sequence such that $\tau(k) \to \infty$ as $k \to \infty$, and it holds that

$$s_{\tau(k)} \leq s_{\tau(k)+1} \text{ and } s_k \leq s_{\tau(k)+1}, \quad \forall k \geq k_0.$$

3. Self-adaptive CQ-type algorithm and its convergence result

This main section provides positive answers to the questions raised in the introduction section, namely that we introduce a self-adaptive CQ-type algorithm (Algorithm 1) whose stepsize does not depend on the bounded linear operator norms to solve the two-operator SFP (1.3). Subsequently, we analyze and establish strong convergence of the proposed algorithm. The following assumptions are set throughout the section:

- H_1 and H_2 are real Hilbert spaces,
- $C \subseteq H_1$ and $Q, \tilde{Q} \subseteq H_2$ are nonempty closed convex subsets,
- $A, B : H_1 \to H_2$ are two bounded linear operators,
- $\Gamma = \left\{x \in C : Ax \in Q, Bx \in \tilde{Q}\right\} \neq \emptyset$.

We begin with the following result that will be helpful in designing our algorithm.

Lemma 3.4. *Let $x^* \in C$. Then, $x^* \in \Gamma$ if and only if $\left\|A^*(I - P_Q)Ax^* + B^*(I - P_{\tilde{Q}})Bx^*\right\| = 0$.*

Proof. Let $x^* \in C$. If $x^* \in \Gamma$, then $Ax^* \in Q, Bx^* \in \tilde{Q}$ and so $(I - P_Q)Ax^* = (I - P_{\tilde{Q}})Bx^* = 0$. It is obvious that $\left\|A^*(I - P_Q)Ax^* + B^*(I - P_{\tilde{Q}})Bx^*\right\| = 0$. Conversely, we assume that

$\left\|A^*(I-P_Q)Ax^*+B^*(I-P_{\tilde{Q}})Bx^*\right\|=0$. Pick $p\in\Gamma$. By Lemma 2.1(2), we have

$$\begin{aligned}
0 &= \left\|A^*(I-P_Q)Ax^*+B^*(I-P_{\tilde{Q}})Bx^*\right\| \|x^*-p\| \\
&\geq \left\langle A^*(I-P_Q)Ax^*+B^*(I-P_{\tilde{Q}})Bx^*, x^*-p\right\rangle \\
&= \langle A^*(I-P_Q)Ax^*, x^*-p\rangle + \left\langle B^*(I-P_{\tilde{Q}})Bx^*, x^*-p\right\rangle \\
&= \langle (I-P_Q)Ax^*, Ax^*-Ap\rangle + \left\langle (I-P_{\tilde{Q}})Bx^*, Bx^*-Bp\right\rangle \\
&\geq \|(I-P_Q)Ax^*\|^2 + \|(I-P_{\tilde{Q}})Bx^*\|^2,
\end{aligned}$$

which implies that $(I-P_Q)Ax^*=(I-P_{\tilde{Q}})Bx^*=0$. Hence, $Ax^*\in Q$ and $Bx^*\in\tilde{Q}$, that is, $x^*\in\Gamma$. □

Here, our iterative algorithm for solving the two-operator SFP (1.3) is designed as follows.

Algorithm 1: Self-adaptive CQ-type algorithm for the two-operator SFP

Initialization: Take two real sequences $\{\beta_k\}\subset(0,1)$ and $\{\mu_k\}\subset(0,4)$.
Choose $x_0\in H_1$ arbitrarily. Set $x_1=P_Cx_0$ and $k=1$.

Iterative Step: Given x_k, if $\left\|A^*(I-P_Q)Ax_k+B^*(I-P_{\tilde{Q}})Bx_k\right\|=0$, then $x_{k+1}=x_k$ (in this case, x_k solves (1.3) by Lemma 3.4) and the iterative process stops. Otherwise, calculate

$$\lambda_k=\mu_k\frac{\|(I-P_Q)Ax_k\|^2+\|(I-P_{\tilde{Q}})Bx_k\|^2}{\left\|A^*(I-P_Q)Ax_k+B^*(I-P_{\tilde{Q}})Bx_k\right\|^2}, \tag{3.7}$$

$$x_{k+1}=P_C\left[(1-\beta_k)\left(x_k-\frac{\lambda_k}{2}\left(A^*(I-P_Q)Ax_k+B^*(I-P_{\tilde{Q}})Bx_k\right)\right)\right]. \tag{3.8}$$

Update $k:=k+1$ and return to Iterative Step.

For the sake of simplicity, we let $g:H_1\to\mathbb{R}$ be defined by

$$g(x):=\frac{1}{4}\left(\|(I-P_Q)Ax\|^2+\|(I-P_{\tilde{Q}})Bx\|^2\right) \tag{3.9}$$

with the gradient given by

$$\nabla g(x)=\frac{1}{2}\Big(A^*(I-P_Q)Ax+B^*(I-P_{\tilde{Q}})Bx\Big),\quad x\in H_1.$$

Note that (1.3) is equivalent to the problem of finding $x\in C$ such that $g(x)=0$. In other words, (3.7) and (3.8) can be rewritten in the form of the following modified gradient-projection method:

$$x_{k+1}=P_C\Big[(1-\beta_k)\big(x_k-\lambda_k\nabla g(x_k)\big)\Big],\ \text{where } \lambda_k=\frac{\mu_k g(x_k)}{\|\nabla g(x_k)\|^2}.$$

To verify the convergence of Algorithm 1, the following two lemmas are required.

Lemma 3.5. *Let $\{x_k\}$ be a sequence generated by Algorithm 1. If $\nabla g(x_k) \neq 0$, then the following two inequalities hold for all $x^* \in \Gamma$,*

$$\|x_{k+1} - x^*\|^2 \leq \beta_k \|x^*\|^2 + (1-\beta_k)\|x_k - x^*\|^2 - \mu_k(4-\mu_k)(1-\beta_k)\frac{g^2(x_k)}{\|\nabla g(x_k)\|^2} \tag{3.10}$$

and

$$\begin{aligned}\|x_{k+1} - x^*\|^2 \leq (1-\beta_k)\|x_k - x^*\|^2 + \beta_k\Big[&\beta_k\|x^*\|^2 + 2(1-\beta_k)\langle x_k - x^*, -x^*\rangle \\ &+ 2(1-\beta_k)\lambda_k \langle \nabla g(x_k), x^*\rangle\Big].\end{aligned} \tag{3.11}$$

Proof. Let $x^* \in \Gamma$. Using (3.8) and Lemma 2.1(3), we have

$$\begin{aligned}\|x_{k+1} - x^*\|^2 &= \Big\|P_C\Big[(1-\beta_k)\big(x_k - \lambda_k \nabla g(x_k)\big)\Big] - P_C x^*\Big\|^2 \\ &\leq \big\|(1-\beta_k)\big(x_k - \lambda_k\nabla g(x_k)\big) - x^*\big\|^2 \\ &= \big\|\beta_k(-x^*) + (1-\beta_k)\big(x_k - \lambda_k \nabla g(x_k) - x^*\big)\big\|^2 \qquad (3.12)\\ &\leq \beta_k\|x^*\|^2 + (1-\beta_k)\left\|x_k - \lambda_k\nabla g(x_k) - x^*\right\|^2. \qquad (3.13)\end{aligned}$$

By Lemma 2.1(2), we get

$$\begin{aligned}\langle \nabla g(x_k), x_k - x^*\rangle &= \frac{1}{2}\Big\langle A^*(I-P_Q)Ax_k + B^*(I-P_{\tilde{Q}})Bx_k, x_k - x^*\Big\rangle \\ &= \frac{1}{2}\Big[\langle A^*(I-P_Q)Ax_k, x_k - x^*\rangle + \langle B^*(I-P_{\tilde{Q}})Bx_k, x_k - x^*\rangle\Big] \\ &= \frac{1}{2}\Big[\langle (I-P_Q)Ax_k, Ax_k - Ax^*\rangle + \langle (I-P_{\tilde{Q}})Bx_k, Bx_k - Bx^*\rangle\Big] \\ &\geq \frac{1}{2}\Big[\|(I-P_Q)Ax_k\|^2 + \|(I-P_{\tilde{Q}})Bx_k\|^2\Big] = 2g(x_k). \qquad (3.14)\end{aligned}$$

Now using (3.7) and (3.14), we obtain

$$\begin{aligned}\|x_k - \lambda_k\nabla g(x_k) - x^*\|^2 &= \|x_k - x^*\|^2 + \lambda_k^2\left\|\nabla g(x_k)\right\|^2 - 2\lambda_k\left\langle \nabla g(x_k), x_k - x^*\right\rangle \\ &\leq \|x_k - x^*\|^2 + \lambda_k^2\left\|\nabla g(x_k)\right\|^2 - 4\lambda_k g(x_k) \\ &= \|x_k - x^*\|^2 + \frac{\mu_k^2 g^2(x_k)}{\|\nabla g(x_k)\|^2} - \frac{4\mu_k g^2(x_k)}{\|\nabla g(x_k)\|^2} \\ &= \|x_k - x^*\|^2 - \mu_k(4-\mu_k)\frac{g^2(x_k)}{\|\nabla g(x_k)\|^2}. \qquad (3.15)\end{aligned}$$

Consequently, substituting (3.15) into (3.13) yields

$$\|x_{k+1} - x^*\|^2 \leq \beta_k\|x^*\|^2 + (1-\beta_k)\|x_k - x^*\|^2 - \mu_k(4-\mu_k)(1-\beta_k)\frac{g^2(x_k)}{\|\nabla g(x_k)\|^2}$$

and (3.10) is obtained. We next show that (3.11) is true. From (3.15), we also have

$$\begin{aligned}\|x_k - \lambda_k\nabla g(x_k) - x^*\|^2 &\leq \|x_k - x^*\|^2 - \mu_k(4-\mu_k)\frac{g^2(x_k)}{\|\nabla g(x_k)\|^2} \\ &\leq \|x_k - x^*\|^2. \qquad (3.16)\end{aligned}$$

By using (3.12) and (3.16), we obtain

$$\begin{aligned}
\|x_{k+1}-x^*\|^2 &\le \big\|\beta_k(-x^*)+(1-\beta_k)\big(x_k-\lambda_k\nabla g(x_k)-x^*\big)\big\|^2\\
&=\beta_k^2\|x^*\|^2+(1-\beta_k)^2\,\|x_k-\lambda_k\nabla g(x_k)-x^*\|^2\\
&\quad+2\beta_k(1-\beta_k)\,\langle x_k-\lambda_k\nabla g(x_k)-x^*,-x^*\rangle\\
&\le \beta_k^2\|x^*\|^2+(1-\beta_k)^2\,\|x_k-x^*\|^2+2\beta_k(1-\beta_k)\,\langle x_k-x^*,-x^*\rangle\\
&\quad+2\beta_k(1-\beta_k)\lambda_k\,\langle\nabla g(x_k),x^*\rangle\\
&\le (1-\beta_k)\|x_k-x^*\|^2+\beta_k\Big[\beta_k\|x^*\|^2+2(1-\beta_k)\langle x_k-x^*,-x^*\rangle\\
&\quad+2(1-\beta_k)\lambda_k\,\langle\nabla g(x_k),x^*\rangle\Big].
\end{aligned}$$

This completes the proof. □

Lemma 3.6. *The sequence $\{x_k\}$ generated by Algorithm 1 is bounded.*

Proof. If $\nabla g(x_m)=0$ for some $m\in\mathbb{N}$, then $x_k=x_m$ for all $k>m$ and hence $\{x_k\}$ is bounded. Assume that $\nabla g(x_k)\neq 0$ for all $k\in\mathbb{N}$. Let $x^*\in\Gamma$. Using (3.10), we get

$$\begin{aligned}
\|x_{k+1}-x^*\|^2 &\le \beta_k\|x^*\|^2+(1-\beta_k)\|x_k-x^*\|^2-\mu_k(4-\mu_k)(1-\beta_k)\frac{g^2(x_k)}{\|\nabla g(x_k)\|^2}\\
&\le \beta_k\|x^*\|^2+(1-\beta_k)\|x_k-x^*\|^2\\
&\le \max\left\{\|x^*\|^2,\|x_k-x^*\|^2\right\}.
\end{aligned}$$

By mathematical induction, we deduce that

$$\|x_{k+1}-x^*\|^2 \;\le\; \max\{\|x^*\|^2,\|x_1-x^*\|^2\},\quad \forall k\in\mathbb{N},$$

it follows that $\{x_k\}$ is bounded. □

Now, we are ready to prove a strong convergence theorem of Algorithm 1.

Theorem 3.1. *The sequence $\{x_k\}$ generated by Algorithm 1 converges strongly to a solution x^* to (1.3) provided that the control sequences $\{\beta_k\}$ and $\{\mu_k\}$ satisfy the following conditions:*

(C1) (1) $\lim\limits_{k\to\infty}\beta_k=0$ *and* (2) $\sum\limits_{k=1}^{\infty}\beta_k=\infty$;

(C2) $\inf\limits_{k}\mu_k(4-\mu_k)>0$.

Proof. If $\nabla g(x_m)=0$ for some $m\in\mathbb{N}$, then the result is obtained directly by Lemma 3.4. So, we assume that $\nabla g(x_k)\neq 0$ for all $k\in\mathbb{N}$. Let $x^*:=P_\Gamma 0$. Using (3.10), we get

$$\|x_{k+1}-x^*\|^2 \le \beta_k\|x^*\|^2+(1-\beta_k)\|x_k-x^*\|^2-\mu_k(4-\mu_k)(1-\beta_k)\frac{g^2(x_k)}{\|\nabla g(x_k)\|^2},$$

it follows that

$$\mu_k(4-\mu_k)(1-\beta_k)\frac{g^2(x_k)}{\|\nabla g(x_k)\|^2}\le \beta_k\|x^*\|^2+\|x_k-x^*\|^2-\|x_{k+1}-x^*\|^2. \tag{3.17}$$

The rest of the proof into will be divided into two cases:

Case 1. Assume that there exists $k_0\in\mathbb{N}$ such that $\{\|x_k-x^*\|\}_{k\ge k_0}$ is either nonincreasing or nondecreasing. In this case, $\{\|x_k-x^*\|\}$ is convergent because it is bounded. It follows that $\|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 \to 0$ as $k \to \infty$. Then, in view of (3.17) with (C1)(1) and (C2), we obtain

$$\lim_{k\to\infty} \frac{g^2(x_k)}{\|\nabla g(x_k)\|^2} = 0, \tag{3.18}$$

which implies that

$$\lim_{k\to\infty} \lambda_k \|\nabla g(x_k)\| = \lim_{k\to\infty} \frac{g(x_k)}{\|\nabla g(x_k)\|} = 0. \tag{3.19}$$

Let $y_k = (1-\beta_k)\big(x_k - \lambda_k \nabla g(x_k)\big)$. Consider

$$\|x_k - y_k\| = \|(1-\beta_k)\lambda_k \nabla g(x_k) + \beta_k x_k\| \le \lambda_k \|\nabla g(x_k)\| + \beta_k \|x_k\|,$$

it follows from (3.19) and (C1)(1) that

$$\lim_{k\to\infty} \|x_k - y_k\| = 0. \tag{3.20}$$

By the same computation as the proof of (3.10), we get

$$\|y_k - x^*\|^2 \le \beta_k \|x^*\|^2 + (1-\beta_k)\|x_k - x^*\|^2. \tag{3.21}$$

For $x \in H_1$, we let

$$h(x) := \|(I - P_C)x\|^2,\ g^A(x) := \frac{1}{2}\|(I-P_Q)Ax\|^2 \text{ and } g^B(x) := \frac{1}{2}\|(I - P_{\tilde{Q}})Bx\|^2.$$

Using Lemma 2.1(3) and (3.21), we have

$$\begin{aligned} h(y_k) &= \|y_k - P_C y_k\|^2 \\ &\le \|y_k - x^*\|^2 - \|P_C y_k - x^*\|^2 \\ &\le \beta_k \|x^*\|^2 + \|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2, \end{aligned}$$

which implies that

$$\lim_{k\to\infty} h(y_k) = 0. \tag{3.22}$$

Since ∇g^A and ∇g^B are Lipschitz continuous with coefficients $\|A\|^2$ and $\|B\|^2$, respectively, one is able to show that ∇g is Lipschitz continuous with a coefficient $L := \max\{\|A\|^2, \|B\|^2\}$. Thus, we have

$$\|\nabla g(x_k)\| = \|\nabla g(x_k) - \nabla g(x^*)\| \le L\|x_k - x^*\|, \quad \forall k \in \mathbb{N}.$$

By the boundedness of $\{x_k - x^*\}$, the above inequality yields that $\{\nabla g(x_k)\}$ is bounded. This together with (3.18) implies that $g(x_k) \to 0$ as $k \to \infty$ and hence

$$\lim_{k\to\infty} g^A(x_k) = \lim_{k\to\infty} g^B(x_k) = 0. \tag{3.23}$$

We next show that $\omega_w(x_k) \subseteq \Gamma$. Since $\{x_k\}$ is bounded, $\omega_w(x_k) \neq \emptyset$. Let $\hat{x} \in \omega_w(x_k)$. Then, there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $x_{k_j} \rightharpoonup \hat{x}$. Since g^A is weakly lower semi-continuous on H_1, it follows from (3.23) that

$$0 \le g^A(\hat{x}) \le \liminf_{j\to\infty} g^A(x_{k_j}) = 0.$$

Hence, $g^A(\hat{x}) = 0$, that is, $A\hat{x} \in Q$. Similarly, by using the weakly lower semicontinuity of g^B and (3.23), we get $g^B(\hat{x}) = 0$, that is, $B\hat{x} \in \tilde{Q}$. Since $x_{k_j} \rightharpoonup \hat{x}$, it also follows from (3.20) that $y_{k_j} \rightharpoonup \hat{x}$. By using the weakly lower semicontinuity of h and (3.22), we then deduce that $\hat{x} \in C$. Therefore, $\hat{x} \in \Gamma$ and this means that $\omega_w(x_k) \subseteq \Gamma$. Since $x^* = P_\Gamma 0$, it follows from Lemma 2.1(1) that

$$\limsup_{k\to\infty} \langle x_k - x^*, -x^* \rangle = \max_{\hat{x}\in\omega_w(x_k)} \langle \hat{x} - x^*, -x^* \rangle \le 0. \tag{3.24}$$

Now using (3.11), we have

$$\begin{aligned}\|x_{k+1}-x^*\|^2 &\le (1-\beta_k)\|x_k-x^*\|^2+\beta_k\Big[\beta_k\|x^*\|^2+2(1-\beta_k)\langle x_k-x^*,-x^*\rangle \\ &\quad +2(1-\beta_k)\lambda_k\left\langle \nabla g(x_k),x^*\right\rangle\Big] \\ &\le (1-\beta_k)\|x_k-x^*\|^2+\beta_k\Big[\beta_k\|x^*\|^2+2(1-\beta_k)\langle x_k-x^*,-x^*\rangle \\ &\quad +2(1-\beta_k)\lambda_k\left\|\nabla g(x_k)\right\|\,\|x^*\|\Big] \\ &= (1-\beta_k)\|x_k-x^*\|^2+\beta_k\delta_k, \quad \forall k\in\mathbb{N}, \qquad (3.25)\end{aligned}$$

where

$$\delta_k := \beta_k\|x^*\|^2+2(1-\beta_k)\langle x_k-x^*,-x^*\rangle+2(1-\beta_k)\lambda_k\left\|\nabla g(x_k)\right\|\,\|x^*\|.$$

Using (C1)(1), (3.19), and (3.24), we get $\limsup_{k\to\infty}\delta_k\le 0$. Recall from (C1)(2) that $\sum_{k=1}^{\infty}\beta_k=\infty$. Consequently, by applying Lemma 2.2 to (3.25), we immediately obtain that $x_k\to x^*$ as $k\to\infty$.

Case 2. Assume that $\{\|x_k-x^*\|\}$ is not monotone. There exists a subsequence $\{k_j\}$ of $\{k\}$ such that $\|x_{k_j}-x^*\|<\|x_{k_j+1}-x^*\|$ for all $j\in\mathbb{N}$. Define a positive interger sequence $\tau(k)$ by

$$\tau(k):=\max\{n\le k:\|x_n-x^*\|<\|x_{n+1}-x^*\|\}$$

for all $k\ge k_0$ (for some k_0 large enough). By Lemma 2.3, $\{\tau(k)\}$ is nondecreasing such that $\tau(k)\to\infty$ as $k\to\infty$ and

$$\left\|x_{\tau(k)}-x^*\right\|^2-\left\|x_{\tau(k)+1}-x^*\right\|^2\le 0 \qquad (3.26)$$

for all $k\ge k_0$. From (3.17) and (3.26), we have

$$\begin{aligned}\mu_{\tau(k)}\left(4-\mu_{\tau(k)}\right)\left(1-\beta_{\tau(k)}\right)\frac{g^2\left(x_{\tau(k)}\right)}{\left\|\nabla g\left(x_{\tau(k)}\right)\right\|^2} &\le \beta_{\tau(k)}\|x^*\|^2+\left\|x_{\tau(k)}-x^*\right\|^2-\left\|x_{\tau(k)+1}-x^*\right\|^2 \\ &\le \beta_{\tau(k)}\|x^*\|^2.\end{aligned}$$

In view of the above inequity with (C1)(1) and (C2), we get

$$\lim_{k\to\infty}\frac{g^2\left(x_{\tau(k)}\right)}{\left\|\nabla g\left(x_{\tau(k)}\right)\right\|^2}=0.$$

By the same way as the proof in Case 1, we obtain

$$\limsup_{k\to\infty}\left\langle x_{\tau(k)}-x^*,-x^*\right\rangle=\max_{\hat{x}\in\omega_w\left(x_{\tau(k)}\right)}\langle\hat{x}-x^*,-x^*\rangle\le 0$$

and

$$\left\|x_{\tau(k)+1}-x^*\right\|^2\le\left(1-\beta_{\tau(k)}\right)\left\|x_{\tau(k)}-x^*\right\|^2+\beta_{\tau(k)}\delta_{\tau(k)}, \qquad (3.27)$$

where

$$\delta_{\tau(k)}:=\beta_{\tau(k)}\|x^*\|^2+2\left(1-\beta_{\tau(k)}\right)\langle x_{\tau(k)}-x^*,-x^*\rangle+2\left(1-\beta_{\tau(k)}\right)\lambda_{\tau(k)}\left\|\nabla g\left(x_{\tau(k)}\right)\right\|\,\|x^*\|$$

such that $\limsup_{k\to\infty}\delta_{\tau(k)}\le 0$. By looking at (3.27) with the fact that $\|x_{\tau(k)}-x^*\|\le\|x_{\tau(k)+1}-x^*\|$, we have $\left\|x_{\tau(k)+1}-x^*\right\|^2\le\delta_{\tau(k)}$. This implies that $\limsup_{k\to\infty}\left\|x_{\tau(k)+1}-x^*\right\|^2\le 0$. Consequently, by utilizing Lemma 2.3, we have

$$0\le\|x_k-x^*\|\le\left\|x_{\tau(k)+1}-x^*\right\|\to 0 \text{ as } k\to\infty.$$

Therefore, in both cases we conclude that $\{x_n\}$ converges strongly to $x^* = P_\Gamma 0$. The proof is complete. □

Remark 3.1. It is worth mentioning that there are some advantages of our main result as follows:

(1) If $\{x_k\}$ is a sequence generated by Algorithm 1 such that $\nabla g(x_k) \neq 0$ for all $k \in \mathbb{N}$, then $\{x_k\}$ converges to the minimum-norm solution x^* to (1.3), where $x^* = P_\Gamma 0$.
(2) The choice of stepsize λ_k defined by (3.7) depends on x_k and hence Algorithm 1 does not need to know the value of $\|A\|$ or $\|B\|$.
(3) A result in [35, Theorem 3.1] for solving the SFP is a consequence of Theorem 3.1, namely that if $A = B$ and $Q = \tilde{Q}$ in our problem, then Algorithm 1 is immediately reduced to (1.6) [35, Algorithm 3.1].

Remark 3.2. We note that the concept of choosing the stepsizes λ_k in (1.5) and (3.7) can be extended to the case of the finite familes of operators A_j and sets Q_j $(j = 1, 2, \ldots, n)$ in such a way:

$$\lambda_k = \frac{\mu_k g(x_k)}{\|\nabla g(x_k)\|^2}, \tag{3.28}$$

where $\mu_k \in (0, 4)$ and $g : H_1 \to \mathbb{R}$ is defined by $g(x) := \frac{1}{2n}\sum_{j=1}^{n} \left\|(I - P_{Q_j})A_j x\right\|^2$ with the gradient given by $\nabla g(x) = \frac{1}{n}\sum_{j=1}^{n} A_j^*(I - P_{Q_j})A_j x$. It would be interesting to modify the gradient-projection method with the stepsize (3.28) to solve the *constrained multiple-set split feasibility problem* (CMSSFP) [28] which is formulated as finding a point

$$x \in \bigcap_{i=1}^{m} C_i \ \text{ such that } \ A_j x \in Q_j,$$

where $C_i \subseteq H_1$ $(i = 1, 2, \ldots, m)$ and $Q_j \subseteq H_2$ $(j = 1, 2, \ldots, n)$ are nonempty closed convex subsets and $\{A_j : H_1 \to H_2\}$ is a finite family of bounded linear operators.

4. Self-adaptive relaxed CQ-type algorithm

Due to our main result in Sect. 3, we consider the two-operator SFP (1.3) for general closed convex subsets C, Q, and $\tilde{Q}$; however, finding the explicit forms of the metric projections P_C, P_Q, and $P_{\tilde{Q}}$ in Algorithm 1 may not be easy when these closed convex subsets are complicated. Fortunately, one of the ways for calculating the metric projection onto a sub-level set of a convex function suggested by Fukushima [16] is to compute the sequence of metric projections onto half-spaces containing such a sub-level set. By this idea, Yang [41] considered the SFP (1.1) in the case of two sub-level sets

$$C = \{x \in H_1 : f_1(x) \le 0\} \ \text{ and } \ Q = \{y \in H_2 : f_2(y) \le 0\}, \tag{4.29}$$

where $f_1 : H_1 \to \mathbb{R}$ and $f_2 : H_2 \to \mathbb{R}$ are two convex functions. Also, assume that f_1 and f_2 are subdifferentiable on H_1 and H_2, respectively, and both ∂f_1 and ∂f_2 are bounded operators (i.e., bounded on bounded sets). Yang [41] then introduced the so-called *relaxed CQ algorithm* for solving the SFP (1.1) constrained by (4.29) as follows:

$$\begin{cases} x_1 \in H_1, \\ x_{k+1} = P_{C_k}(x_k - \lambda A^*(I - P_{Q_k})Ax_k), \quad k \ge 1, \end{cases} \tag{4.30}$$

where $\lambda \in (0, 2/\|A\|^2)$ and C_k and Q_k are half-spaces given as

$$C_k = \{x \in H_1 : f_1(x_k) + \langle c_k, x - x_k\rangle \le 0\},$$

where $c_k \in \partial f_1(x_k)$ and

$$Q_k = \{y \in H_2 : f_2(Ax_k) + \langle q_k, y - Ax_k\rangle \le 0\},$$

where $q_k \in \partial f_2(Ax_k)$. It follows from the definition of the subdifferential that $C \subseteq C_k$ and $Q \subseteq Q_k$ for all $k \ge 1$. Since P_{C_k} and P_{Q_k} have closed forms (see [6, 16]), then the implementation of the relaxed CQ algorithm (4.30) is easier than that of the CQ algorithm (1.2) (in situations that P_C and P_Q have no closed forms). In addition, López et al. [25, Algorithm 4.1] modified (4.30) by using the self-adaptive stepsize λ_k (1.5). Vinh et al. [35, Algorithm 4.1] also introduced a relaxation version of the self-adaptive CQ-type algorithm (1.6) to solve this problem.

This section was motivated by the above-mentioned notions and results. We now focus on the two-operator SFP (1.3) in which closed convex subsets C, Q, and $\tilde{Q}$ are sub-level sets of convex functions. In what follows, we set the following hypotheses:

- H_1 and H_2 are real Hilbert spaces,
- $\emptyset \ne C \subseteq H_1$ and $\emptyset \ne Q, \tilde{Q} \subseteq H_2$ are given as:
$$C = \{x \in H_1 : f_1(x) \le 0\},$$
$$Q = \{y \in H_2 : f_2(y) \le 0\},$$
$$\tilde{Q} = \left\{y \in H_2 : \tilde{f}_2(y) \le 0\right\},$$
where $f_1 : H_1 \to \mathbb{R}$ and $f_2, \tilde{f}_2 : H_2 \to \mathbb{R}$ are subdifferentiable and convex functions such that their subdifferential operators are bounded,
- $A, B : H_1 \to H_2$ are two bounded linear operators,
- $\Gamma = \left\{x \in C : Ax \in Q, Bx \in \tilde{Q}\right\} \ne \emptyset$.

Let $x_k \in H_1$. Denote

$$C_k := \{x \in H_1 : f_1(x_k) + \langle c_k, x - x_k\rangle \le 0\}, \tag{4.31}$$

where $c_k \in \partial f_1(x_k)$,

$$Q_k := \{y \in H_2 : f_2(Ax_k) + \langle q_k, y - Ax_k\rangle \le 0\}, \tag{4.32}$$

where $q_k \in \partial f_2(Ax_k)$, and

$$\tilde{Q}_k := \left\{y \in H_2 : \tilde{f}_2(Bx_k) + \langle \tilde{q}_k, y - Bx_k\rangle \le 0\right\}, \tag{4.33}$$

where $\tilde{q}_k \in \partial \tilde{f}_2(Bx_k)$.

Lemma 4.7. *If there exists $x_k \in C$ such that $\left\|A^*(I - P_{Q_k})Ax_k + B^*(I - P_{\tilde{Q}_k})Bx_k\right\| = 0$, then $x_k \in \Gamma$.*

Proof. Let $x_k \in C$ be such that $\left\|A^*(I - P_{Q_k})Ax_k + B^*(I - P_{\tilde{Q}_k})Bx_k\right\| = 0$. Pick any $p \in \Gamma$. Since $Q \subseteq Q_k$ and $\tilde{Q} \subseteq \tilde{Q}_k$, then $Ap \in Q_k$ and $Bp \in \tilde{Q}_k$. By the same computation as the proof in Lemma 3.4, we get

$$\begin{aligned} 0 &= \left\|A^*(I - P_{Q_k})Ax_k + B^*(I - P_{\tilde{Q}_k})Bx_k\right\| \|x_k - p\| \\ &\ge \|(I - P_{Q_k})Ax_k\|^2 + \|(I - P_{\tilde{Q}_k})Bx_k\|^2, \end{aligned}$$

which follows that $(I-P_{Q_k})Ax_k = (I-P_{\tilde{Q}_k})Bx_k = 0$ and hence $Ax_k \in Q_k$ and $Bx_k \in \tilde{Q}_k$. By (4.32) and (4.33), we have $f_2(Ax_k) \leq 0$ and $\tilde{f}_2(Bx_k) \leq 0$. Thus, $Ax_k \in Q$ and $Bx_k \in \tilde{Q}$, i.e., $x_k \in \Gamma$. □

Using (4.31)–(4.33), a relaxation version of Algorithm 1 is presented as follows.

Algorithm 2: Self-adaptive relaxed CQ-type algorithm for the two-operator SFP

Initialization: Take two real sequences $\{\beta_k\} \subset (0,1)$ and $\{\mu_k\} \subset (0,4)$.
Choose an initial point $x_1 \in H_1$ arbitrarily and set $k=1$.

Iterative Step: Given x_k, if $\left\| A^*(I-P_{Q_k})Ax_k + B^*(I-P_{\tilde{Q}_k})Bx_k \right\| = 0$, then $x_{k+1} = x_k$ and the iterative process stops. Otherwise, calculate

$$\lambda_k = \mu_k \frac{\|(I-P_{Q_k})Ax_k\|^2 + \|(I-P_{\tilde{Q}_k})Bx_k\|^2}{\left\| A^*(I-P_{Q_k})Ax_k + B^*(I-P_{\tilde{Q}_k})Bx_k \right\|^2}, \tag{4.34}$$

$$x_{k+1} = P_{C_k}\left[(1-\beta_k)\left(x_k - \frac{\lambda_k}{2}\Big(A^*(I-P_{Q_k})Ax_k + B^*(I-P_{\tilde{Q}_k})Bx_k\Big)\right)\right]. \tag{4.35}$$

Update $k := k+1$ and go on to Iterative Step.

For the sake of simplicity, we define a function $g_k : H_1 \to \mathbb{R}$ by

$$g_k(x) := \frac{1}{4}\left(\|(I-P_{Q_k})Ax\|^2 + \|(I-P_{\tilde{Q}_k})Bx\|^2\right)$$

with the gradient given by

$$\nabla g_k(x) = \frac{1}{2}\Big(A^*(I-P_{Q_k})Ax + B^*(I-P_{\tilde{Q}_k})Bx\Big), \quad x \in H_1.$$

So, (4.34) and (4.35) become

$$\lambda_k = \frac{\mu_k g_k(x_k)}{\|\nabla g_k(x_k)\|^2} \ \text{ and } \ x_{k+1} = P_{C_k}\Big[(1-\beta_k)\big(x_k - \lambda_k \nabla g_k(x_k)\big)\Big].$$

Below we prove a strong convergence result of Algorithm 2 which extends a result in [35, Theorem 4.1].

Theorem 4.2. *Let $\{x_k\}$ be a sequence generated by Algorithm 2 with the control sequences $\{\beta_k\}$ and $\{\mu_k\}$ satisfying:*

(C1) (1) $\lim\limits_{k\to\infty} \beta_k = 0$ *and* (2) $\sum\limits_{k=1}^{\infty} \beta_k = \infty$;

(C2) $\inf\limits_k \mu_k(4-\mu_k) > 0$.

If $\nabla g_k(x_k) \neq 0$ for all $x_k \notin C$, then $\{x_k\}$ converges strongly to a point $x^* \in \Gamma$.

Proof. If $\nabla g_m(x_m) = 0$ for some $x_m \in C$, then the result is done by Lemma 4.7. So, we suppose that $\nabla g_k(x_k) \neq 0$ for all $k \in \mathbb{N}$. Let $x^* := P_\Gamma 0$. In view of the proof of Lemma 3.5 with replacing g and C by g_k and C_k, respectively, we deduce that

$$\|x_{k+1} - x^*\|^2 \leq \beta_k\|x^*\|^2 + (1-\beta_k)\|x_k - x^*\|^2 - \mu_k(4-\mu_k)(1-\beta_k)\frac{g_k^2(x_k)}{\|\nabla g_k(x_k)\|^2} \tag{4.36}$$

and

$$\|x_{k+1} - x^*\|^2 \le (1-\beta_k)\|x_k - x^*\|^2 + \beta_k\Big[\beta_k\|x^*\|^2 + 2(1-\beta_k)\langle x_k - x^*, -x^*\rangle + 2(1-\beta_k)\lambda_k \langle \nabla g_k(x_k), x^*\rangle\Big]. \tag{4.37}$$

By (4.36), we obtain that $\{x_k\}$ is bounded and

$$\mu_k(4-\mu_k)(1-\beta_k)\frac{g_k^2(x_k)}{\|\nabla g_k(x_k)\|^2} \le \beta_k\|x^*\|^2 + \|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2. \tag{4.38}$$

Now we consider the rest of the proof into two cases:

Case 1. Suppose that $\{\|x_k - x^*\|\}_{k\ge k_0}$ is either nonincreasing or nondecreasing (for some k_0). We then have $\{\|x_k - x^*\|\}$ is a convergent sequence and so $\|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 \to 0$ as $k \to \infty$. From (4.38), we get

$$\lim_{k\to\infty}\frac{g_k^2(x_k)}{\|\nabla g_k(x_k)\|^2} = 0, \tag{4.39}$$

which implies that

$$\lim_{k\to\infty}\lambda_k\|\nabla g_k(x_k)\| = \lim_{k\to\infty}\frac{g_k(x_k)}{\|\nabla g_k(x_k)\|} = 0. \tag{4.40}$$

Set $y_k = (1-\beta_k)\big(x_k - \lambda_k\nabla g_k(x_k)\big)$. By the same computation as in the proof of Theorem 3.1, we deduce that

$$\lim_{k\to\infty}\|x_k - y_k\| = 0 \tag{4.41}$$

and

$$\lim_{k\to\infty}\|(I - P_{C_k})y_k\| = 0. \tag{4.42}$$

Since $P_{C_k}y_k \in C_k$, it follows from (4.31) and using (4.41), (4.42), and the boundedness assumption on ∂f_1 that

$$\begin{aligned} f_1(x_k) &\le \langle c_k, x_k - P_{C_k}y_k\rangle \\ &= \langle c_k, x_k - y_k + y_k - P_{C_k}y_k\rangle \\ &\le \|c_k\|\left(\|x_k - y_k\| + \|(I-P_{C_k})y_k\|\right) \to 0 \text{ as } k\to\infty. \end{aligned} \tag{4.43}$$

Note that $\nabla g_k(x^*) = 0$ for all $k \in \mathbb{N}$. Since ∇g_k is Lipschitz continuous with a coefficient $L := \max\{\|A\|^2, \|B\|^2\}$, we have

$$\|\nabla g_k(x_k)\| = \|\nabla g_k(x_k) - \nabla g_k(x^*)\| \le L\|x_k - x^*\|, \quad \forall k\in\mathbb{N}.$$

So, $\{\nabla g_k(x_k)\}$ is bounded. This together with (4.39) yields that $g_k(x_k)\to 0$ as $k\to\infty$ and hence

$$\lim_{k\to\infty}\|(I - P_{Q_k})Ax_k\| = \lim_{k\to\infty}\|(I - P_{\tilde{Q}_k})Bx_k\| = 0. \tag{4.44}$$

Since $P_{Q_k}(Ax_k) \in Q_k$, it follows from (4.32) and using (4.44) and the boundedness assumption on ∂f_2 that

$$\begin{aligned} f_2(Ax_k) &\le \langle q_k, (I-P_{Q_k})Ax_k\rangle, \\ &\le \|q_k\|\,\|(I-P_{Q_k})Ax_k\| \to 0 \text{ as } k\to\infty. \end{aligned} \tag{4.45}$$

Similarly, since $P_{\tilde{Q}_k}(Bx_k) \in \tilde{Q}_k$, it follows from (4.33) and using (4.44) and the boundedness assumption on $\partial \tilde{f}_2$ that

$$\begin{aligned} \tilde{f}_2(Bx_k) &\leq \langle \tilde{q}_k, (I - P_{\tilde{Q}_k})Bx_k \rangle, \\ &\leq \|\tilde{q}_k\| \, \|(I - P_{\tilde{Q}_k})Bx_k\| \ \to 0 \ \text{ as } \ k \to \infty. \end{aligned} \tag{4.46}$$

Now let $\hat{x} \in \omega_w(x_k)$. Thus, there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $x_{k_j} \rightharpoonup \hat{x}$. By the weakly lower semicontinuity of f_1 and using (4.43), we get

$$f_1(\hat{x}) \leq \liminf_{j\to\infty} f_1(x_{k_j}) \leq 0.$$

This means that $\hat{x} \in C$. Since A and B are bounded linear operators, we also have $Ax_{k_j} \rightharpoonup A\hat{x}$ and $Bx_{k_j} \rightharpoonup B\hat{x}$. By the weakly lower semicontinuity of f_2 and $\tilde{f}_2$ and using (4.45), (4.46), we obtain

$$f_2(A\hat{x}) \leq \liminf_{j\to\infty} f_2(Ax_{k_j}) \leq 0 \ \text{ and } \ \tilde{f}_2(B\hat{x}) \leq \liminf_{j\to\infty} \tilde{f}_2(Bx_{k_j}) \leq 0,$$

which imply that $A\hat{x} \in Q$ and $B\hat{x} \in \tilde{Q}$. Hence, $\hat{x} \in \Gamma$ and so we obtain that $\omega_w(x_k) \subseteq \Gamma$. Now, using the characterization of the projection, Lemma 2.1(1) with $P_\Gamma 0 = x^*$, we have

$$\limsup_{k\to\infty} \langle x_k - x^*, -x^* \rangle = \max_{\hat{x}\in\omega_w(x_k)} \langle \hat{x} - x^*, -x^* \rangle \leq 0. \tag{4.47}$$

From (4.37), we get

$$\begin{aligned} \|x_{k+1} - x^*\|^2 &\leq (1-\beta_k)\|x_k - x^*\|^2 + \beta_k \Big[\beta_k \|x^*\|^2 + 2(1-\beta_k)\langle x_k - x^*, -x^* \rangle \\ &\qquad + 2(1-\beta_k)\lambda_k \|\nabla g_k(x_k)\| \, \|x^*\|\Big] \\ &= (1-\beta_k)\|x_k - x^*\|^2 + \beta_k \delta_k, \quad \forall k \in \mathbb{N}, \end{aligned}$$

where $\delta_k := \beta_k \|x^*\|^2 + 2(1-\beta_k)\langle x_k - x^*, -x^* \rangle + 2(1-\beta_k)\lambda_k \|\nabla g_k(x_k)\| \, \|x^*\|$. It follows from (4.40) and (4.47) that $\limsup_{k\to\infty} \delta_k \leq 0$. Finally, utilizing Lemma 2.2 with the above inequality, we can conclude that $x_k \to x^*$ as $k \to \infty$.

Case 2. Assume that $\{\|x_k - x^*\|\}$ is not monotone. Using Lemma 2.3 and following the similar argument to the proof in Case 1, one can prove that $\{x_k\}$ also converges strongly to $x^* = P_\Gamma 0$. So, we omit the proof for this case. □

5. Numerical experiments

To illustrate the convergence performance of our proposed algorithms and to support our main results, we first employ Algorithm 1 for solving (1.3) in the setting of a Euclidean space (see Example 5.1). After that, we use Algorithm 2 to solve the problem of recovering a sparse signal from a limited number of sampling with two different blurring operations (see Example 5.2). In both examples, we also compare the efficiency of our algorithms with that of some methods based on the operator norms. All the numerical experiments are completed on Apple MacBook Pro with 2 GHz Quad-Core Intel Core i5 with 16 GB memory. The program is implemented in MATLAB R2023a.

Example 5.1. Let $H_1 = H_2 = \mathbb{R}^2$ with the Euclidean norm. Consider a ball C and a half-space $Q = \tilde{Q}$ given by

$$C = \Big\{(a,b) \in \mathbb{R}^2 : \sqrt{(a-2)^2 + b^2} \leq 2\Big\} \ \text{ and } \ Q = \big\{(a,b) \in \mathbb{R}^2 : 3a + 2b \leq -3\big\}$$

and two operators $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ given by $A(a,b) = (-a,0)$ and $B(a,b) = (0,b)$ for all $(a,b) \in \mathbb{R}^2$. One can see that $\Gamma = \{x \in C : Ax, Bx \in Q\} \neq \emptyset$. We will find the minimum-norm element x^* in Γ by using our self-adaptive CQ-type algorithm, Algorithm 1. To do this, we arrange the following explicit forms of the metric projections:

$$P_C(a,b) = \begin{cases} (2,0) + \dfrac{2}{\sqrt{(a-2)^2 + b^2}}(a-2,b), & \text{if } (a,b) \notin C, \\ (a,b), & \text{otherwise,} \end{cases}$$

and

$$P_Q(a,b) = \begin{cases} (a,b) - \dfrac{3a+2b+3}{13}(3,2), & \text{if } (a,b) \notin Q, \\ (a,b), & \text{otherwise,} \end{cases}$$

for all $(a,b) \in \mathbb{R}^2$. Firstly, we test the convergence behavior of Algorithm 1 by taking $\beta_k = \frac{1}{k+1}$ and $\mu_k = \frac{2k}{k+1}$ with the starting point $x_0 = (4,2)$ as shown in Figure 1. It is observed that $x_k \to (1,-1.5) \in \Gamma$ where $\|(1,-1.5)\| = \min_{p \in \Gamma} \|p\|$.

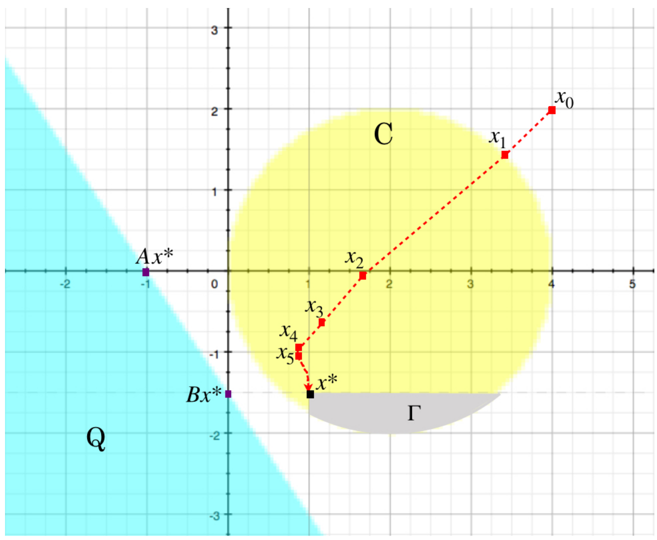

FIGURE 1. Illustration of the convergence behavior of Algorithm 1

Next, we analyze the convergence performance of Algorithm 1 by choosing different accelerating sequences $\{\mu_k\}$ and also compare with that of the following algorithms depending on the operator norms.

Algorithm 3: Let $\{x_k\}$ be a sequence generated by (3.8) where

$$\lambda_k := \lambda \in \left(0, \frac{2}{\max\{\|A\|^2, \|B\|^2\}}\right).$$

Algorithm 4: ([21]) Let $\{x_k\}$ be a sequence generated by (1.4) where $f := 0$.

Each algorithm is equipped with the parameters in Table 1.

TABLE 1. Setting parameters for each algorithm

Parameters	Algorithm 1	Algorithm 3	Algorithm 4
$\beta_k = \dfrac{1}{k+1}$	√	√	√
$\mu_k = \dfrac{\rho k}{k+1},\ 0 < \rho < 4$	√	–	–
$\delta_k = \gamma_k = \dfrac{k}{2k+2}$	–	–	√
$0 < \lambda < 2$	–	√	√

TABLE 2. Numerical experiments with the different choices of the stepsizes

	Choice of the stepsizes	k (No. of iter.)	CPU time (s)	x_k	E_k
Algorithm 1	$\rho = 0.5$	2198	0.2279830	$(0.9990240, -1.4967059)$	9.996E-07
	$\rho = 1$	1100	0.1576130	$(0.9990244, -1.4967074)$	9.987E-07
	$\rho = 2$	551	0.1091000	$(0.9990253, -1.4967104)$	9.968E-07
	$\rho = 3.5$	198	0.0890400	$(0.9991338, -1.4968926)$	8.726E-07
	$\rho = 3.9$	111	0.0785620	$(0.9993079, -1.4966262)$	9.585E-07
Algorithm 3	$\lambda = 0.5$	5919	0.5656190	$(0.9990239, -1.4967055)$	9.998E-07
	$\lambda = 1$	2960	0.3619410	$(0.9990240, -1.4967061)$	9.995E-07
	$\lambda = 1.9$	1558	0.2533760	$(0.9990241, -1.4967063)$	9.993E-07
Algorithm 4 ([21])	$\lambda = 0.5$	11837	1.1435740	$(0.9990238, -1.4967052)$	9.999E-07
	$\lambda = 1$	5919	0.6078450	$(0.9990239, -1.4967055)$	9.998E-07
	$\lambda = 1.9$	3115	0.3736660	$(0.9990238, -1.4967052)$	9.998E-07

We choose the starting point $x_0 = x_1 = (2, 2)$ and use the stopping criterion for the iterative process as: $E_k := g(x_k) < 10^{-6}$, where g is defined by (3.9). Now the comparision of the numerical experiments of Algorithms 1, 3, and 4 are shown in Table 2.

Remark 5.3. By testing the performance of Algorithms 1, 3, and 4 and from Table 2, we observe that

(1) All studied algorithms give the approximate solutions close to $(1, -1.5)$ which is the minimum-norm solution.
(2) Algorithm 1 converges the fastest and takes the least time.
(3) The choice of the stepsizes influencses the convergence behavior of all studied algorithms. Namely that if $\{\mu_k\}$ is taken close to 4 (for Algorithm 1) and λ is taken close to 2 (for Algorithms 3 and 4), then the number of iterations and the CPU time have reduction. Meanwhile, choosing different starting points has no significant impact on their convergence behavior.

Example 5.2. (*Compressed Sensing* [25, 30]). Here, we consider the problem of recovering a sparse signal $x \in \mathbb{R}^N$ from the observation of two signals $y, \tilde{y} \in \mathbb{R}^M$ $(M < N)$ via the linear equation systems:

$$y = Ax + \varepsilon \quad \text{and} \quad \tilde{y} = Bx + \tilde{\varepsilon}, \tag{5.48}$$

where $A, B : \mathbb{R}^N \to \mathbb{R}^M$ are two bounded linear observation operators (they are often ill-condition) and $\varepsilon, \tilde{\varepsilon}$ are additive noises. The problem (5.48) can be solved by using the *LASSO* technique ([34]) in the froms of the constrained least-squares problem:

$$\text{minimize} \quad \frac{1}{2}\|Ax - y\|_2^2 \quad \text{and} \quad \frac{1}{2}\|Bx - \tilde{y}\|_2^2 \tag{5.49}$$

with respect to $x \in C := \{x \in \mathbb{R}^N : \|x\|_1 \le t\}$, where $t > 0$ is a given constant. If (5.49) has a solution, we see that (5.49) is a particular case of the two-operator SFP (1.3) where $Q = \{y\}$ and $\tilde{Q} = \{\tilde{y}\}$. Since C is the closed l_1 ball in $\mathbb{R}^N$ with the radius t, we will employ the relaxation version of our self-adaptive CQ-type algorithm, Algorithm 2 to solve (5.49). Define $f_1(x) = \|x\|_1 - t$ and consider the half-space C_k denoted by (4.31). The closed form of the metric projection from $\mathbb{R}^N$ onto C_k is as follows:

$$P_{C_k}(x) = \begin{cases} x, & \text{if } f_1(x_k) + \langle c_k, x - x_k\rangle \le 0, \\ x - \dfrac{f_1(x_k) + \langle c_k, x - x_k\rangle}{\|c_k\|^2} c_k, & \text{otherwise,} \end{cases}$$

where $c_k \in \partial f_1(x_k)$ is chosen as

$$c_k^{(i)} = \begin{cases} 1, & \text{if } x_k^{(i)} > 0, \\ 0, & \text{if } x_k^{(i)} = 0, \\ -1, & \text{if } x_k^{(i)} < 0, \end{cases}$$

see [17, Section 5].

In our experiment, two sampling matrices $A, B \in \mathbb{R}^{M\times N}$ are generated randomly from normal distributions with $N = 2048$ and $M = 1024$. The sparse signal $x^* \in \mathbb{R}^N$ is generated from a uniform distribution in $[-2, 2]$ with m nonzero components. The measured values y and $\tilde{y}$ are generated by white Gaussian noise with the signal-to-noise ratio (SNR) as 40 and 50 decibles, respectively. Set $t = m$. We test three cases as follows:

Case 1: $m = 10$, *Case 2:* $m = 50$, *Case 3:* $m = 100$.

We compare the signal recovery performance of Algorithm 2 with that of the following algorithm depending on $\|A\|$ and $\|B\|$.

Algorithm 5: Let $\{x_k\}$ be a sequence generated by (4.35) where

$$\lambda_k := \lambda \in \left(0, \frac{2}{\max\{\|A\|^2, \|B\|^2\}}\right).$$

Let $\beta_k = \frac{1}{k+1}$ and $\mu_k = \frac{2k}{k+1}$ for Algorithm 2 and $\beta_k = \frac{1}{k+1}$ and $\lambda = \frac{1}{\max\{\|A\|^2, \|B\|^2\}}$ for Algorithm 5. The process is started with the initial signal $x_1 = 0$. The restoration accuracy is measured by the mean squared error (MSE), i.e.,

$$\text{MSE}(k) = \frac{1}{N}\|x^* - x_k\|^2 < 10^{-4},$$

where x^* is the original signal and x_k is an estimated signal of x^*. Now, the numerical results of recovering the signal x^* are reported as Figures 2–7.

Remark 5.4. By the simple experiments as shown in Figures 2–7, we note that

(1) The original signals x^* can be recovered by Algorithms 2 and 5.
(2) If the number of spikes of x^* increases, then both methods also require an increase in the number of iterations and the CPU time. However, the number of iterations and the CPU time of using Algorithm 2 are less than those of using Algorithm 5.

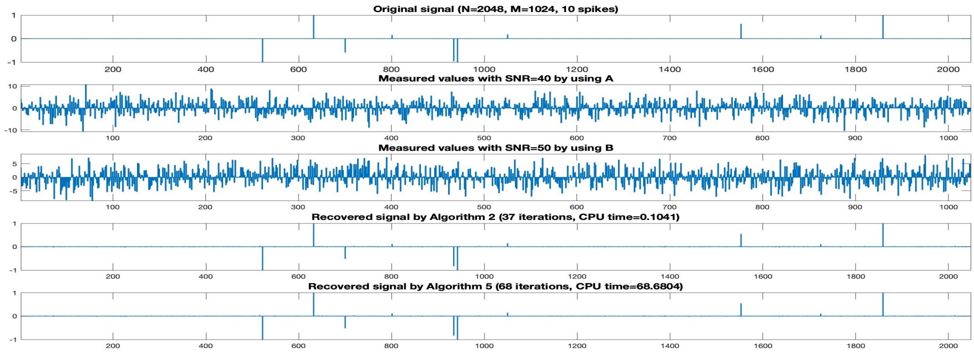

FIGURE 2. Signal recovery experiment in Case 1.
From top to bottom: original signal; observation data using A; observation data using B; recovered signal by Algorithm 2; recovered signal by Algorithm 5

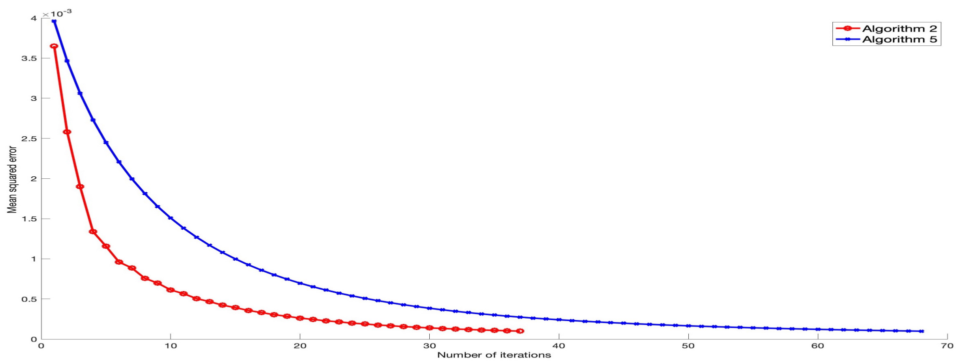

FIGURE 3. The mean squared error versus the number of iterations in Case 1

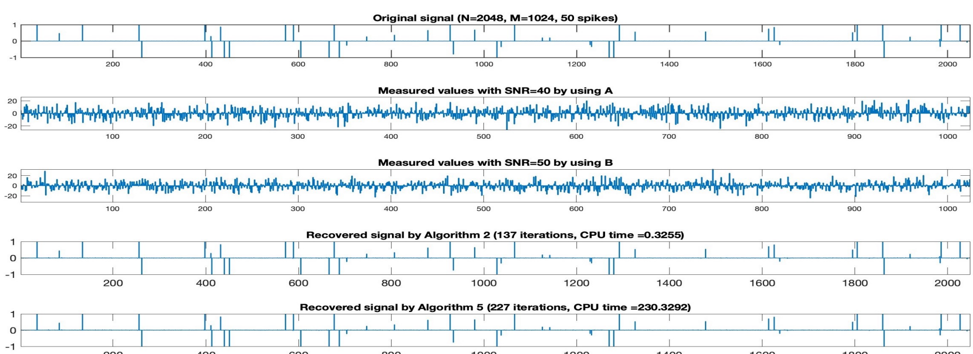

FIGURE 4. Signal recovery experiment in Case 2.
From top to bottom: original signal; observation data using A; observation data using B; recovered signal by Algorithm 2; recovered signal by Algorithm 5

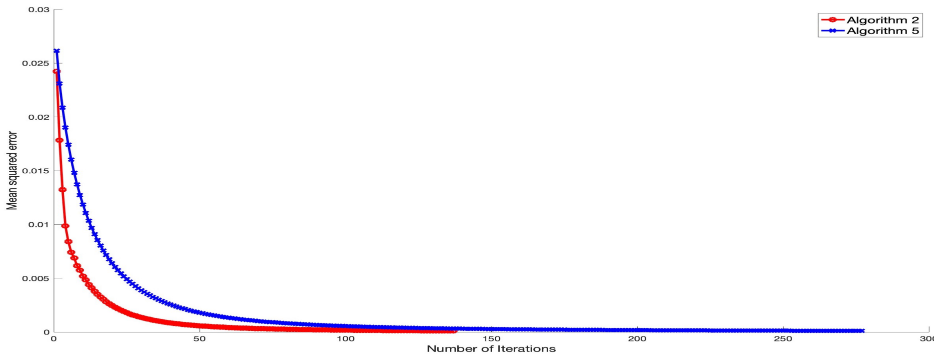

FIGURE 5. The mean squared error versus the number of iterations in Case 2

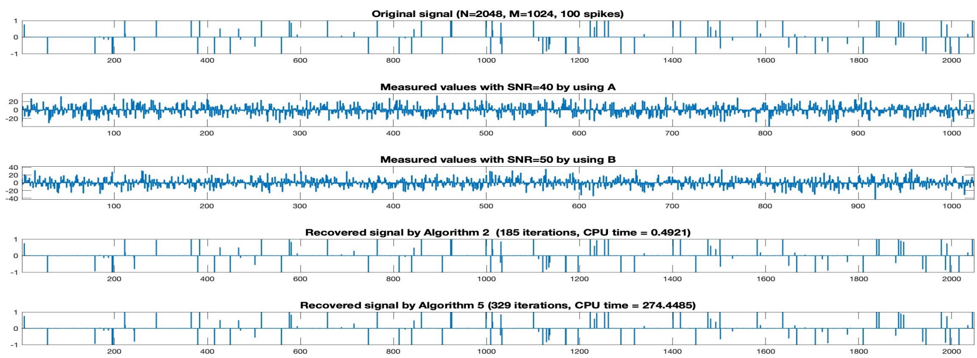

FIGURE 6. Signal recovery experiment in Case 3.
From top to bottom: original signal; observation data using A; observation data using B; recovered signal by Algorithm 2; recovered signal by Algorithm 5

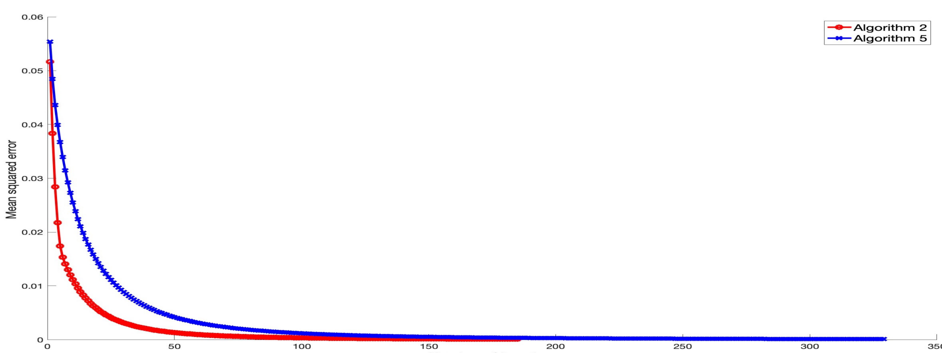

FIGURE 7. The mean squared error versus the number of iterations in Case 3

For the closed forms of some metric projections onto simple closed convex subsets in Hilbert spaces, the reader is referred to [6, Chapter 4]. There are also some examples for the split feasibility problem and related problems in the infinite-dimensional Hilbert spaces, see [23, 33, 35].

Conclusion

This paper discusses and analyzes the convergence results on the two-operator split feasibility problem (two-operator SFP) in Hilbert spaces, namely finding a point of a closed convex subset of a Hilbert space such that each of its images under two given bounded linear operators belongs to a closed convex subset of another Hilbert space. We introduce a self-adaptive CQ-type algorithm where the stepsize does not depend on such bounded linear operator norms. Under some mild conditions, we then prove that the sequence generated by the proposed algorithm converges strongly to the minimum-norm solution of the two-operator SFP. A relaxation version of our proposed algorithm is also introduced for solving the problem constrained by sub-level sets of convex functions. Our main results improve the result of Kangtunyakarn [21, Theorems 3.1] in terms of selecting the stepsize in the algorithm and generalize the results of Vinh et al. [35, Theorems 3.1 and 4.1] for the split feasibility problem (also improve the results of Xu [40], Wang and Xu [37], Yao et al. [43] and Chuang [7]). In addition, it is observed from our numerical experiments that our self-adaptive CQ-type algorithms (without any operator norms) are more efficient than the CQ-type algorithms based on the operator norms.

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