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Convergence Theorems for Common Solutions of Nonlinear Problems and Applications

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ABSTRACT. In this work, two inertial algorithms for approximating common elements of the sets of solutions of three important problems are constructed. The first problem is a generalized mixed equilibrium one involving relaxed monotone mapping, the second is a zero problem of inverse strongly monotone mappings, while the third one is a fixed point problem of a family of relatively nonexpansive mappings. The first algorithm is a shrinking projection type for a common solution of all the three problems. The second is a generalized Alber projection free method for the second and the third problems. Each of the devised algorithms uses the conjugate gradient-like direction, which allows it to accelerate its iterates toward a solution of the problems. The strong convergence theorem for each of the algorithms is formulated and proved in a real 2 - uniformly convex and uniformly smooth Banach space. Additionally, the applications of our algorithms to convex optimization problems and image recovery problems are studied. The advantages and computational efficiency of our methods are analyzed based on their numerical performance in comparison to some of the existing and recently proposed methods using numerical example.

1. INTRODUCTION

Throughout this paper, $\mathcal{H}$ will be a real Hilbert space and $\mathcal{C}$ will be a nonempty convex and closed subset of $\mathcal{H}$. A mapping $\mathcal{B} : \mathcal{C} \to \mathcal{H}$ is said to be monotone, if

$$\langle \mathcal{B}u - \mathcal{B}v, u - v\rangle \geq 0, \quad \forall u, v \in \mathcal{C},$$

and μ - inverse strongly monotone, if there exists a positive real number μ such that

$$\langle \mathcal{B}u - \mathcal{B}v, u - v\rangle \geq \mu||\mathcal{B}u - \mathcal{B}v||^2, \quad \forall u, v \in \mathcal{C}. \tag{1.1}$$

The zero problem with respect to a μ - inverse strongly monotone mapping $\mathcal{B}$, is to find $z \in \mathcal{C}$, such that

$$\mathcal{B}z = 0, \tag{1.2}$$

and the set of all zeros of $\mathcal{B}$, is denoted by $\mathcal{B}^{-1}(0) = \{z \in \mathcal{C} : \mathcal{B}z = 0\}$.

In 2003, Fang and Huang [26], introduced a concept of relaxed $\eta - \alpha$ monotone mapping for solving mixed equilibrium problem. A mapping $\mathcal{B} : \mathcal{C} \to \mathcal{H}$ is said to be relaxed $\eta - \alpha$ monotone (see, [26]), if there exists a mapping $\eta : \mathcal{C} \times \mathcal{C} \to \mathcal{H}$ and a function $\alpha : \mathcal{H} \to \mathbb{R}$ with $\alpha(tz) = t^q\alpha(z),\ \forall z \in \mathcal{H}$ and a constant $q > 1$, such that

$$\langle \mathcal{B}u - \mathcal{B}v, \eta(u, v)\rangle \geq \alpha(u - v),\ \forall u, v \in \mathcal{C}. \tag{1.3}$$

It is noticed that when $\eta(u, v) = u - v,\ \forall u, v \in \mathcal{C}$, then, $\mathcal{B}$ becomes a relaxed α monotone. In a situation where $\eta(u, v) = u - v,\ \forall u, v \in \mathcal{C}$ and $\alpha(z) = r||z||^q$, where $q > 1$ and $r > 0$ are two constants, then, $\mathcal{B}$ is called q - monotone (see, [28, 50, 58]). In fact, if $q = 2$, then, $\mathcal{B}$ is r - strongly monotone. One also sees that the class of monotone mappings is contained

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in the class of $\eta - \alpha$ monotone mappings. It has been proved in [26] that under some mild assumptions, the variational inequality problem of finding $u \in \mathcal{C}$, satisfying

$$\langle \mathcal{B}u, \eta(v,u)\rangle + \Phi(v) - \Phi(u) \geq 0, \ \forall v \in \mathcal{C}, \tag{1.4}$$

is solvable, where $\Phi : \mathcal{C} \to \mathbb{R} \cup \{+\infty\}$ is a function which is convex and proper. The problem (1.4) is shown to be equivalent to the problem below, in which we find $u \in \mathcal{C}$, satisfying

$$\langle \mathcal{B}v, \eta(v,u)\rangle + \Phi(v) - \Phi(u) \geq \alpha(v-u), \ \forall v \in \mathcal{C}. \tag{1.5}$$

In this paper, the symbol $MVIP(\mathcal{B}, \Phi)$ is used to represent the set of solutions of the problem (1.4) or (1.5).

For a bimapping $\eta : \mathcal{C} \times \mathcal{C} \to \mathcal{H}$, a mapping $\mathcal{B} : \mathcal{C} \to \mathcal{H}$ is called η-hemicontinuous (see [26]), if for all $u, v \in \mathcal{C}$, a function $f : [0,1] \to (-\infty, +\infty)$ defined by

$$f(t) = \langle \mathcal{B}\big((1-t)x + ty\big), \eta(u,v)\rangle,$$

is continuous at 0^+. Since the inception of the monotone mapping described in (1.3), many methods were proposed and studied for solving problems involving this type of generalized monotone mapping. For example, Chen et al. [15] studied a new equilibrium problem of finding $z \in \mathcal{C}$, such that

$$\xi(u,v) + \langle \mathcal{B}u, \eta(v,u)\rangle + \Phi(v) - \Phi(u) \geq 0, \ \forall v \in \mathcal{C}, \tag{1.6}$$

where $\xi : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ is a bifunction, $\Phi : \mathcal{C} \to \mathbb{R} \cup \{+\infty\}$ is a convex proper and lower semicontinuous function and $\mathcal{B} : \mathcal{C} \to \mathcal{H}$ is an $\eta - \alpha$ relaxed monotone mapping. The solutions' set of problem (1.6) is represented by

$$EP(\xi, \mathcal{B}, \Phi) = \big\{u \in \mathcal{C} : \xi(u,v) + \langle \mathcal{B}u, \eta(v,u)\rangle + \Phi(v) - \Phi(u) \geq 0, \ \forall v \in \mathcal{C}\big\}.$$

If $\mathcal{B}$ is monotone, that is $\eta(u,v) = u - v, \ \forall u, v \in \mathcal{C}$ and $\alpha \equiv 0$, then, (1.6) reduces to the generalized mixed equilibrium problem of Peng and Yao [46], which is to find $u \in \mathcal{C}$ such that

$$\xi(u,v) + \langle \mathcal{B}u, v - u\rangle + \Phi(v) - \Phi(u) \geq 0, \ \forall v \in \mathcal{C}. \tag{1.7}$$

If $\mathcal{B}(u) = 0, \ \forall u \in \mathcal{C}$, then, (1.6) reduces to the mixed equilibrium problem, studied by Ceng and Yao [12], in which we find $u \in \mathcal{C}$, such that

$$\xi(u,v) + \Phi(v) - \Phi(u) \geq 0, \ \forall y \in \mathcal{C}. \tag{1.8}$$

The set of solutions of (1.8) is denoted by $MEP(\xi, \Phi)$. If $\mathcal{B}$ is monotone and $\Phi(u) = 0, \ \forall u \in \mathcal{C}$, then, (1.6) becomes the generalized equilibrium problem studied by Takahashi and Takahashi [51] of finding $u \in \mathcal{C}$, such that

$$\xi(u,v) + \langle \mathcal{B}u, v - u\rangle \geq 0, \ \forall v \in \mathcal{C}. \tag{1.9}$$

The set of solutions of (1.9) is denoted by $GEP(\xi, \mathcal{B})$. If $\mathcal{B}(u) = 0$ and $\Phi(u) = 0, \ \forall u \in \mathcal{C}$, then, (1.6) reduces to the classical equilibrium problem, studied by Blum and Oettli [9] of finding $u \in \mathcal{C}$, such that

$$\xi(u,v) \geq 0, \ \forall v \in \mathcal{C}. \tag{1.10}$$

The set of solutions of (1.10) is denoted by $EP(\xi)$. If $\xi(u,v) = 0, \ \forall u, v \in \mathcal{C}$, then, (1.6) also becomes the mixed variational inequality problem (1.4) or (1.5).

Therefore, the problem (1.6) seems to be more general in nature, since many important problems can be classified as its special cases, such as variational inequality problems, optimization problems and several problems of interest (see, [38, 59, 60]). However, various methods have been introduced for approximating common solutions of zero problems, fixed point problems and equilibrium problems, see for examples, [6, 13, 18, 20, 30, 33, 43, 45, 57] and the references therein.

Hybrid shrinking projection method proposed by Takahashi et al. [54], has been used to approximate a solution of a fixed point and an equilibrium problems in various spaces, see for examples, [15, 30, 55] and the references therein. The rationale behind the method is to obtain the strong convergence of the iterative schemes.

In most cases, (see, [22, 23]), the Mann algorithm [41], has very slow convergence speed. However, fast convergence is needed in many practical applications (see, [16, 25, 35, 45]). Consequently, various methods have been introduced to obtain the fast convergence. Some of these methods are called conjugate gradient-like methods contained in [44], widely seen as an efficient accelerated versions of most gradient methods.

Dong and Yuan [23], introduced accelerated Mann algorithm, by combining the Mann Algorithm in [41] and conjugate gradient method in [44]. They consequently obtained the algorithm;

$$\begin{cases} p_{n+1} = \frac{1}{\lambda}\big(T(v_n) - v_n\big) + \beta_n p_n \\ y_n = v_n + \lambda p_{n+1} \\ v_{n+1} = \mu\gamma_n v_n + (1-\mu\gamma_n) y_n, \qquad \forall n \geq 0, \end{cases} \tag{1.11}$$

where $\mu \in (0,1]$, $\lambda > 0$, $p_0 = \frac{Tv_0 - v_0}{\lambda}$, $\{\gamma_n\}$ and $\{\beta_n\}$ are nonnegative real sequences. They proved the weak convergence of the sequence $\{v_n\}$ to a fixed point of a nonexpansive mapping T, provided that the following conditions hold:
$(A1)$ $\sum_{n=0}^{\infty} \mu\gamma_n(1-\mu\gamma_n) = \infty$;
$(A2)$ $\sum_{n=0}^{\infty} \beta_n < \infty$;
$(A3)$ $\big\{T(v_n) - v_n\big\}$ is bounded.

Polyak [47] also introduced an inertial - type algorithm, which is mostly used in fastening the sequence of iterates generated by various algorithms towards the solution of a problem. Since then, many authors incorporated the inertial term with their algorithms for fast convergence (see, [3, 10, 22, 31, 32] and the references therein).

By combining the accelerated Mann algorithm (1.11) and an inertial term, Dong et al. [24] proposed the following modified inertial Mann algorithm in Hilbert space.

$$\begin{cases} v_0, v_1 \in \mathcal{H}, \\ t_n = v_n + \alpha_n(v_n - v_{n-1}), \\ p_{n+1} = \frac{1}{\lambda}\big(T(t_n) - t_n\big) + \beta_n p_n \\ y_n = t_n + \lambda p_{n+1} \\ v_{n+1} = \mu\gamma_n t_n + (1-\mu\gamma_n) y_n, \qquad \forall n \geq 1, \end{cases} \tag{1.12}$$

where $\alpha_n \in [0,\alpha]$ is nonincreasing with $\alpha_1 = 0$, $0 \leq \alpha < 1$, $0 < \mu \leq 1$, $\lambda > 0$, $p_1 = \frac{Tv_1 - v_1}{\lambda}$, $\{\gamma_n\}$ and $\{\beta_n\}$ satisfy some conditions. They proved a weak convergence theorem of (1.12) to a fixed point of a nonexpansive mapping T and numerically justified that algorithm (1.12) converges faster than algorithm (1.11), provided that for a sequence $\{t_n\}$, the following conditions hold:
(D1) $\{Tt_n - t_n\}$ is bounded;
(D2) $\{Tt_n - y\}$ is bounded for any $y \in Fix(T)$.

Recently, Tan et al. [56] introduce the following algorithm called modified inertial shrinking algorithm, by combining algorithm (1.12) and the method in [54] in Hilbert space.

$$\begin{cases} v_0, v_1 \in \mathcal{H}, \\ t_n = v_n + \alpha_n(v_n - v_{n-1}), \\ p_{n+1} = \frac{1}{\lambda}\big(T(t_n) - t_n\big) + \beta_n p_n, \\ y_n = t_n + \lambda p_{n+1}, \\ g_n = \theta_n t_n + (1-\theta_n) y_n, \\ \mathcal{C}_{n+1} = \big\{z \in \mathcal{C}_n : ||g_n - z||^2 \leq ||t_n - z||^2 - \theta_n(1-\theta_n)||t_n - y_n||^2 + \delta_n\big\}, \\ v_{n+1} = P_{\mathcal{C}_{n+1}} v_0, \quad n \geq 0, \end{cases} \tag{1.13}$$

where $\lambda > 0$, $\{\alpha_n\},\ \{\theta_n\},\ \{\beta_n\}$, $\delta_n = \lambda\beta_n M_2[\lambda\beta_n M_2 + 2M_1]$, $M_1 = diam\ \mathcal{C} = \underset{u,v\in\mathcal{C}}{sup}||u-v||$, $M_2 = max\{\underset{1\leq k\leq n_0}{max}||p_k||, \frac{2}{\lambda}M_1\}$, satisfy some conditions. The strong convergence of the scheme (1.13) to a fixed point of a nonexpansive mapping T is established, under the boundedness assumption on a nonempty closed and convex subset $\mathcal{C}$ of $\mathcal{H}$.

Remark 1.1. It is noted that the convergence results of all the methods (1.11), (1.12) and (1.13) with conjugate gradient-like directions hold under the assumptions (A3), (D1)-(D2) and the boundedness of $\mathcal{C}$, respectively. These assumptions appeared to be too restrictive and it would be interesting to consider dispensing them.

For zeros of a countable family of mappings defined in (1.1) that are fixed points of a countable family of relatively weak J-nonexpansive mappings in 2 - uniformly convex and uniformly smooth Banach space, Chidume et al. [18], formulated and proved the strong convergence theorem for the following algorithm.

$$\begin{cases} v_1 \in \mathcal{X} = \mathcal{C}_1, \\ t_n = J^{-1}\big(Jv_n - \lambda\big(\sum_{i=1}^{\infty}\beta_i\mathcal{B}_i\big)v_n\big), \\ y_n = J^{-1}\big(\sum_{i=1}^{\infty}\alpha_i T_i\big)t_n, \\ \mathcal{C}_{n+1} = \big\{z \in \mathcal{C}_n : \phi(z, y_n) \leq \phi(z, v_n)\big\}, \\ v_{n+1} = \Pi_{\mathcal{C}_{n+1}} v_1, \quad n \geq 1, \end{cases} \tag{1.14}$$

where $\beta_i,\ \alpha_i \in (0,1)$, such that $\sum_{i=1}^{\infty}\beta_i = 1,\ \sum_{i=1}^{\infty}\alpha_i = 1,\ \lambda \in (0, \frac{\mu}{2L}),\ L > 0$ is a Lipschitz constant of J^{-1}, $\mu \geq \underset{i\geq 1}{\inf}\mu_i > 0$ and for each $i \geq 1,\ \mu_i$ is the constant appearing in (1.1) for $\mathcal{B}_i$.

Very recently, motivated by the results in [19], Adamu et al. [2] proposed an inertial algorithm for zeros of a sum of two monotone mappings that are J - fixed points of a relatively J-nonexpansive mapping in 2 - uniformly convex and uniformly smooth Banach space.

Inspired and motivated by the results in [2, 15, 18, 19, 55, 56], we propose two inertial algorithms for common elements of the sets of solutions of some important nonlinear problems. The first is developed based on the hybrid projection technique for approximating common solutions of a generalized mixed equilibrium problem and a zero problem of a countable family of inverse strongly monotone mappings, that are fixed points of a countable family of relatively nonexpansive mappings. The second algorithm is a generalized Alber projection free method for zeros of a countable family of inverse strongly monotone mappings, that are fixed points of a countable family of relatively nonexpansive mappings. Each of the constructed algorithms uses the conjugate gradient-like direction, which allows it to accelerate its sequence of iterates towards a common solution of the problems. The special point of the two proposed conjugate gradient-like methods in this work, over some existing methods with conjugate gradient-like directions (see e.g., [23, 24, 56]) is that, the strong convergence theorem for each of the two algorithms is formulated and proved in a real 2 - uniformly convex and uniformly smooth Banach space, in which we dispensed with the strong assumptions used in [23, 24] and the boundedness assumption on $\mathcal{C}$ used in [56] as discussed in the Remark 1.1. To the best of our knowledge, this is the first study in a setting of a real 2 - uniformly convex and uniformly smooth Banach space, using the generalized Alber projection and Lyapunov distance function with conjugate gradient-like direction. The advantages and computational efficiency of our methods are analyzed based on their numerical performance, in comparison to some of the existing and recently proposed methods in solving a numerical example. Additionally, the applications of our algorithms to convex optimization problems are studied and our second algorithm is used to solve image recovery problems.

2. Preliminaries

Let $\mathcal{C}$ be a nonempty convex and closed set in a real Banach space $\mathcal{X}$. The generalized duality mapping $J_\psi : \mathcal{X} \to 2^{\mathcal{X}^*}$ ($\mathcal{X}^*$ is the dual space of $\mathcal{X}$) associated with a gauge function ψ is defined by

$$J_\psi(u) = \{u^* \in \mathcal{X}^* : \langle u, u^* \rangle = ||u|| ||u^*||, ||u^*|| = \psi(||u||)\},$$

where $\psi(t) = t^{p-1}$, for all $t \geq 0$ and $1 < p < \infty$. In particular, if $p = 2$, then,$J_\psi = J_2$ is called the normalized duality mapping defined by

$$J(u) = \{u^* \in \mathcal{X}^* : \langle u, u^* \rangle = ||u||^2, ||u^*|| = ||u||\}. \tag{2.15}$$

Remark 2.2. (see, [17, 21, 34]) The following are some of the well known properties of the map in (2.15).
(i) If $\mathcal{X}$ is reflexive, strictly convex and smooth, then, J is surjective, injective and single-valued respectively.
(ii) If $\mathcal{X}$ is uniformly smooth, then, J is norm - to - norm uniformly continuous on bounded subsets of $\mathcal{X}$.
(iii) If $\mathcal{X}$ is a real Hilbert space $\mathcal{H}$, the normalized duality map J reduces to identity map, i.e., $Ju = \{u\}$.

Throughout this paper, we use $\mathbb{R}$ and $\mathbb{N}$ to denote the sets of real and natural numbers, $\mathcal{S}(\mathcal{X})$ to denote a unit sphere in $\mathcal{X}$, i,e., $\mathcal{S}(\mathcal{X}) = \{u \in \mathcal{X} : ||u|| = 1\}$ and the set of ω - weak cluster limits of $\{v_n\}$ is represented by $\omega_w(v_n)$.

A Banach space $\mathcal{X}$ (see, [17]) is said to be uniformly convex, if for any $\epsilon \in (0, 2]$, there exists $\delta > 0$, such that $\forall u, v \in \mathcal{S}(\mathcal{X})$, with $||u - v|| \geq \epsilon$, we have $\frac{||u+v||}{2} \leq 1 - \delta$. $\mathcal{X}$ is strictly convex, if $\forall u, v \in \mathcal{S}(\mathcal{X})$, with $u \neq v$, we have $\frac{||u+v||}{2} < 1$. $\mathcal{X}$ is also said to be smooth, if $\forall u, v \in \mathcal{S}(\mathcal{X})$, the limit

$$\lim_{t \to 0} \frac{||u + tv|| - ||u||}{t} \quad exists. \tag{2.16}$$

$\mathcal{X}$ is also said to be uniformly smooth, if $\forall u, v \in \mathcal{S}(\mathcal{X})$, the limit in (2.16) exists uniformly. A space $\mathcal{X}$ is also said to have the Kadec-Klee property see [17], if for any sequence $\{v_n\} \subseteq \mathcal{X}$, such that $v_n \rightharpoonup u^* \in \mathcal{X}$ and $||v_n|| \to ||u^*||$ implies $v_n \to u^*$.

Remark 2.3. (see, [52, 53]) Every uniformly convex Banach space is known to be a reflexive with Kadec-Klee property.

Let $\phi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a Lyapunov functional defined by

$$\phi(u, v) = ||u||^2 - 2\langle u, Jv\rangle + ||v||^2,\ u, v \in \mathcal{X},$$

where J is the normalized duality mapping. It is easily seen that in $\mathcal{H}$, $\phi(u, v) = ||u - v||^2,\ \forall u, v \in \mathcal{H}$. In 1996, Alber [4], introduced a generalized projection $\Pi_{\mathcal{C}}$ in a Banach space $\mathcal{X}$ with a nonempty closed convex subset $\mathcal{C}$, which is an analogue of the metric projection in Hilbert spaces. The generalized projection $\Pi_{\mathcal{C}} : \mathcal{X} \to \mathcal{C}$ is defined by $\Pi_{\mathcal{C}} u = u^0 \in \mathcal{C}$ such that

$$\phi(u^0, u) = arg\min_{v \in \mathcal{C}} \phi(v, u).$$

The properties of ϕ and the strict monotonicity of J (see, [4]) implies that a unique point $\Pi_{\mathcal{C}}$ exists in $\mathcal{C}$. Let $\mathcal{X}$ be a smooth, strictly convex and reflexive Banach space, then, for all $u, v, z \in \mathcal{X}$ and $\varrho \in (0, 1)$, we have the following properties.
(B1) $\big(||u|| - ||v||\big)^2 \le \phi(u, v) \le \big(||u|| + ||v||\big)^2$,
(B2) $\phi(u, v) = \phi(u, z) + \phi(z, v) + 2\langle z - u, Jv - Jz\rangle$,
(B3) $\phi\big(u, J^{-1}(\varrho Jv + (1-\varrho)Jz)\big) \le \varrho\phi(u, v) + (1-\varrho)\phi(u, z)$,
(B4) $\phi(u, v) \le ||u||||Ju - Jv|| + ||v||||u - v||$.

Define a map $V : \mathcal{X} \times \mathcal{X}^* \to \mathbb{R}$ by

$$V(u, u^*) = ||u||^2 - 2\langle u, u^*\rangle + ||u^*||^2.$$

Then it is not difficult to see that $V(u, u^*) = \phi\big(u, J^{-1}(u^*)\big),\ \forall u \in \mathcal{X},\ u^* \in \mathcal{X}^*$.

A nonlinear map $T : \mathcal{C} \to \mathcal{C}$ is said to be Nonexpansive if

$$||Tu - Tv|| \le ||u - v||,\ \forall u, v \in \mathcal{C}, \tag{2.17}$$

and a fixed point problem with respect to T, is to find a point $u \in \mathcal{C}$, such that

$$Tu = u. \tag{2.18}$$

A fixed point $u^* \in \mathcal{C}$ (see, [14]), is said to be an asymptotic fixed point of T, if there exists a sequence $\{v_n\} \subseteq \mathcal{C}$, such that $v_n \rightharpoonup u^*$ and $\lim_{n\to\infty} ||v_n - Tv_n|| = 0$. $F(T)$ and $\widehat{F}(T)$ are used to denote the set of fixed point and the set of asymptotic fixed points of T respectively. T is said to be relatively nonexpansive (see, [42, 48]), if $\widehat{F}(T) = F(T) \neq \emptyset$, and

$$\phi(p, Tu) \le \phi(p, u),\ \forall u \in \mathcal{C},\ p \in F(T), \tag{2.19}$$

and T is said to be closed, if for any sequence $\{v_n\} \subset \mathcal{C}$, with $v_n \to u$ and $Tv_n \to u^*$, then $u^* = Tu$.

Remark 2.4. Observe that if $\mathcal{X} = \mathcal{H}$, the mapping in (2.19) reduces to quasi - nonexpansive mapping.

To solve the generalized mixed equilibrium problem (1.6), the function ξ is assumed to satisfy the following.
(C1) $\xi(u, u) = 0, \quad \forall u \in \mathcal{C}$;
(C2) ξ is monotone, i.e., $\xi(u, v) + \xi(v, u) \le 0, \quad \forall u, v \in \mathcal{C}$;
(C3) for each $u, v, z \in \mathcal{C}$,

$$\lim_{s\to 0} \xi\big(sz + (1-s)u, v\big) \le \xi(u, v);$$

(C4) for each $u \in \mathcal{C},\ v \longmapsto \xi(u, v)$ is convex and lower semi continuous.

The following Lemmas will be needed in the proof of the main results.

Lemma 2.1. *([4]) Let $\mathcal{C}$ be a nonempty closed and convex subset of a smooth, strictly convex and reflexive Banach space $\mathcal{X}$. Let $u \in \mathcal{X}$ and $z \in \mathcal{C}$. Then, $z = \Pi_{\mathcal{C}} u, \Leftrightarrow$*

$$\langle z - v, Jz - Ju \rangle \leq 0, \ \forall v \in \mathcal{C}, \ and \tag{2.20}$$

$$\phi(v, \Pi_{\mathcal{C}} u) + \phi(\Pi_{\mathcal{C}} u, u) \leq \phi(v, u), \ \forall v \in \mathcal{C}. \ u \in \mathcal{X}. \tag{2.21}$$

Lemma 2.2. ([4]) *Let $\mathcal{X}$ be a smooth, strictly convex and reflexive Banach space with $\mathcal{X}^*$ as its dual. Then*

$$V(u, u^*) + 2\left\langle J^{-1}u^* - u, v^* \right\rangle \leq V(u, u^* + v^*),$$

for all $u \in \mathcal{X}$ and all $u^*, v^* \in \mathcal{X}^*$.

Lemma 2.3. ([60]) *Let $\mathcal{X}$ be a 2 - uniformly convex real Banach space, then*

$$||u - v|| \leq \frac{2}{c}||Ju - Jv||, \tag{2.22}$$

for all $u, \ v \in \mathcal{X}$, where $0 < c \leq 1$ and c is the 2 - uniformly convex constant of $\mathcal{X}$.

Lemma 2.4. ([18]) *Let $\mathcal{X}$ be a uniformly convex and uniformly smooth Banach space with dual space $\mathcal{X}^*$. Let $\mathcal{B}_i : \mathcal{X} \to \mathcal{X}^*$, $i = 1, 2, \ldots$, be a countable family of mappings defined in* (1.1) *with $\mu_i > 0$ such that $\mu = \inf_{i \geq 1} \mu_i > 0$ and $\bigcap_{i=1}^{\infty} \mathcal{B}_i^{-1}(0) \neq \emptyset$. Let $\mathcal{B} : \mathcal{X} \to \mathcal{X}^*$ be defined by $\mathcal{B}u = \sum_{i=1}^{\infty} \upsilon_i \mathcal{B}_i u$, for each $u \in \mathcal{X}$, where $\{\upsilon_i\}_{i=1}^{\infty}$ is a sequence in $(0, 1)$ such that $\sum_{i=1}^{\infty} \upsilon_i = 1$. Then,*
(i) $\mathcal{B}$ is well defined,
(ii) $\mathcal{B}$ also satisfies (1.1) *with $\mu > 0$,*
(iii) $\mathcal{B}^{-1}(0) = \bigcap_{i=1}^{\infty} \mathcal{B}_i^{-1}(0)$.

Lemma 2.5. ([60]) *Let $\mathcal{X}$ be a real uniformly convex Banach space and $q > 0$ be a fixed number. Then there exists a continuous and strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$, such that;*

$$||\sigma u + (1 - \sigma)v||^2 \leq \sigma||u||^2 + (1 - \sigma)||v||^2 - \sigma(1 - \sigma)\varphi(||u - v||),$$

for all u, v in $B_q = \{u \in \mathcal{X} : ||u|| \leq q\}$ and $\sigma \in [0, 1]$.

Lemma 2.6. ([42]) *Let $\mathcal{C}$ be a nonempty convex and closed set in a smooth, strictly convex and reflexive Banach space $\mathcal{X}$, and $T : \mathcal{C} \to \mathcal{C}$ be a relatively nonexpansive map. Then, $F(T)$ is convex and closed.*

Lemma 2.7. *([62]) Let $\mathcal{X}$ be a real 2-uniformly smooth Banach space. Then there exists a constant $\varepsilon > 0$ such that for all $u, v \in \mathcal{X}$*

$$||u + v||^2 \leq ||u||^2 + 2\langle v, Ju \rangle + \varepsilon||v||^2,$$

where $\varepsilon = 1$ in a real Hilbert space.

Lemma 2.8. ([37]) *Let $\{u_n\}$ and $\{v_n\}$ be sequences in a smooth and uniformly convex Banach space $\mathcal{X}$ such that either $\{u_n\}$ or $\{v_n\}$ is bounded. Then, $\lim_{n\to\infty} \phi(u_n, v_n) = 0$ implies that $\lim_{n\to\infty} ||u_n - v_n|| = 0$.*

Remark 2.5. If both $\{u_n\}$ and $\{v_n\}$ are bounded in Lemma 2.8, then, the converse holds by the property $(B4)$.

Lemma 2.9. ([1]) *Let $\mathcal{X}$ be a real 2-uniformly convex and uniformly smooth Banach space and let $v_0, v_1, v \in \mathcal{X}$ and $\varrho_n \in (0,1)$. Let $\{q_n\}$ be a sequence in $\mathcal{X}$ defined by $q_n = J^{-1}(Jv_n + \varrho_n(Jv_n - Jv_{n-1}))$. Then*

$$\begin{aligned}\phi(v,q_n) \leq\ & \phi(v,v_n) + \varepsilon\varrho_n^2||Jv_n - Jv_{n-1}||^2 + \varrho_n\phi(v_n,v_{n-1}) \\ & + \varrho_n(\phi(v,v_n) - \phi(v,v_{n-1})),\end{aligned} \tag{2.23}$$

where ε is the constant appearing in Lemma 2.7.

Lemma 2.10. ([7]) *Let $\{\xi_n\},\ \{\Gamma_n\}$ and $\{\eta_n\}$ be sequences in $[0,\infty)$ such that $\forall n \geq 1$,*

$$\xi_{n+1} \leq \xi_n + \eta_n(\xi_n - \xi_{n-1}) + \Gamma_n,$$

If there exists an η satisfying $0 \leq \eta_n \leq \eta < 1,\ \forall n \in \mathbb{N}$ and $\sum_{n=1}^{\infty}\Gamma_n < \infty$. Then, the following hold:
(i) $\sum_{n=1}^{\infty}[\xi_n - \xi_{n-1}]_+ < \infty$, where $[t]_+ = \max\{t,0\}$;
(ii) there exists $\xi^* \in [0,\infty)$ such that $\lim_{n\to\infty}\xi_n = \xi^*$.

Lemma 2.11. ([61]) *Let $\{c_n\}$ be a sequence of nonnegative real numbers such that*

$$c_{n+1} \leq (1-\sigma_n)c_n + \sigma_n\gamma_n + b_n, \quad \forall n \in \mathbb{N},$$

where $\{\sigma_n\},\ \{\gamma_n\}$ and $\{b_n\}$ are sequences in $\mathbb{R}$ satisfying (i) $\{\sigma_n\} \subset [0,1]$ such that $\sum_{n=1}^{\infty}\sigma_n = \infty$; (ii) $\limsup_{n\to\infty}\gamma_n \leq 0$ and (iii) $b_n \geq 0,\ \sum_{n=1}^{\infty}b_n < \infty$. Then, $\lim_{n\to\infty}c_n = 0$.

Lemma 2.12. ([39]) *Let $\mathcal{C}$ be a nonempty closed convex subset of a real uniformly convex and uniformly smooth Banach space $\mathcal{X}$ and let $T_i : \mathcal{C} \to \mathcal{X},\ i = 1,2,3,\ldots$, be a countable family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty}F(T_i) \neq \emptyset$. Suppose that $\{\delta_i\}$ and $\{\tau_i\}$ are sequences in $(0,1)$ such that $\sum_{i=1}^{\infty}\delta_i = 1$ and $T : \mathcal{X} \to \mathcal{X}$ is defined by*

$$Tu = J^{-1}\Big(\sum_{i=1}^{\infty}\delta_i\big(\tau_iJu + (1-\tau_i)JT_iu\big)\Big),$$

for each $u \in \mathcal{C}$. Then, T is relatively nonexpansive and $F(T) = \bigcap_{i=1}^{\infty}F(T_i)$.

Lemma 2.13. *Let $\mathcal{C}$ be a nonempty closed convex set in $\mathcal{H}$ and let $T_i : \mathcal{C} \to \mathcal{H},\ i = 1,2,3,\ldots$, be a countable family of quasi - nonexpansive mappings such that $\bigcap_{i=1}^{\infty}F(T_i) \neq \emptyset$. Suppose that $\{\delta_i\}$ and $\{\tau_i\}$ are sequences in $(0,1)$ such that $\sum_{i=1}^{\infty}\delta_i = 1$ and $T : \mathcal{H} \to \mathcal{H}$ is defined by*

$$Tu = \sum_{i=1}^{\infty}\delta_i\big(\tau_iu + (1-\tau_i)T_iu\big),$$

for each $u \in \mathcal{C}$. Then, T is quasi - nonexpansive and $F(T) = \bigcap_{i=1}^{\infty}F(T_i)$.

Proof. Let T_i for each $i = 1,2\ldots$ be a quasi - nonexpansive mapping and $p \in \bigcap_{i=1}^{\infty}F(T_i)$.

$$\begin{aligned}||p - Tu|| &= \Big|\Big|p - \sum_{i=1}^{\infty}\delta_i\big(\tau_iu + (1-\tau_i)T_iu\big)\Big|\Big| \\ &\leq \sum_{i=1}^{\infty}\delta_i||p - \big(\tau_iu + (1-\tau_i)T_iu\big)|| \\ &\leq \sum_{i=1}^{\infty}\delta_i\big(\tau_i||p-u|| + (1-\tau_i)||p - T_iu||\big) \\ &\leq ||p-u||.\end{aligned}$$

Thus, T is a quasi - nonexpansive. To prove that $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$, one can follow the same lines of proof of Lemma 3 in [11], with quasi - nonexpansive mappings $T_i, \ \forall i \geq 1$. □

Lemma 2.14. ([40]) *Let $\{a_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{a_{n_r}\}$ of $\{a_n\}$ that satisfies $a_{n_r} < a_{n_r+1}, \ \forall r \geq 0$. Consider the sequence of integers $\{\beta(n)\}_{n \geq n_0}$ defined by*

$$\beta(n) = \max\{s \leq n : a_s < a_{s+1}\},$$

then, $\{\beta(n)\}_{n \geq n_0}$ is a nondecreasing sequence satisfying $\lim\limits_{n\to\infty} \beta(n) = \infty$ and for all $n \geq n_0$, it holds that

$$a_{\beta(n)} \leq a_{\beta(n)+1} \quad and \quad a_n \leq a_{\beta(n)+1}$$

.

Lemma 2.15. ([5]) *Let $\mathcal{X}$ be a strictly convex, smooth and reflexive Banach space with the dual $\mathcal{X}^*$ and let C be a nonempty, closed, convex and bounded subset of $\mathcal{X}$. Let $J : \mathcal{X} \to \mathcal{X}^*$ be a normalized duality mapping. Let $A : C \to \mathcal{X}^*$ be an η-hemicontinuous and relaxed $\eta - \alpha$ monotone mapping and $\xi : C \times C \to \mathbb{R}$ be a bifunction satisfying $(C1)$ and $(C4)$. Let $g : C \to \mathbb{R}$ be a proper, convex lower semicontinuous. For $r > 0$ and $u \in \mathcal{X}$. Suppose*
(i) $\eta(u, u) = 0$, for all $u \in C$;
(ii) $\eta(v, y) + \eta(y, v) = 0, \ \forall v, y \in C$;
(iii) $\langle Au, \eta(., v)\rangle$ is convex and lower semicontinuous for fixed $u, v \in C$;
(iv) $\alpha : \mathcal{X} \to \mathbb{R}$ is weakly lower semicontinuous.

Then there exists $z \in C$, such that

$$\xi(z, v) + \langle Az, \eta(v, z)\rangle + g(v) - g(z) + \frac{1}{r}\langle Jz - Ju, v - z\rangle \geq 0, \ \forall y \in C.$$

Lemma 2.16. ([15]) *Let $\mathcal{C}$ be a nonempty convex closed and bounded set in a uniformly smooth and strictly convex Banach space $\mathcal{X}$ with the dual space $\mathcal{X}^*$. Let $\mathcal{A} : \mathcal{C} \to \mathcal{X}^*$ be an η-hemicontinuous and relaxed $\eta - \alpha$ monotone mapping, $\xi : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ be a function which satisfies the conditions (C1) - (C4) and let $g : \mathcal{C} \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. For $r > 0$ and $u \in \mathcal{X}$, define a map $T_r : \mathcal{X} \to \mathcal{C}$ as follows*

$$T_r(u) = \left\{ z \in \mathcal{C} : \xi(z, v) + \langle \mathcal{A}z, \eta(v, z)\rangle + g(v) - g(z) + \frac{1}{r}\langle v - z, Jz - Ju\rangle \geq 0, \ \forall y \in \mathcal{C} \right\},$$

Assume that
(i) $\eta(u, v) + \eta(v, u) = 0, \ \forall u, v \in \mathcal{C}$;
(ii) for any fixed $a, b \in \mathcal{C}$, the mapping $u \mapsto \langle Ab, \eta(u, a)\rangle$ is convex and lower semicontinuous;
(iii) $\alpha : \mathcal{X} \to \mathbb{R}$ is weakly lower semicontinuous; that is for any net $\{u_\beta\}$, u_β converges to u in $\sigma(\mathcal{X}, \mathcal{X}^*)$ implying that $\alpha(u) \leq \liminf \alpha(u_\beta)$;
(iv) for any $u, v \in \mathcal{C}, \ \alpha(u - v) + \alpha(v - u) \geq 0$;
(v) $\big\langle A\big(tz_1 + (1 - t)z_2\big), \eta\big(v, tz_1 + (1 - t)z_2\big)\big\rangle \geq t\,\langle Az_1, \eta(v, z_1)\rangle + (1 - t)\,\langle Az_2, \eta(v, z_2)\rangle$, $\forall\, z_1, z_2, v \in \mathcal{C}$ and $t \in [0, 1]$.

Then, the following properties hold:
(1) T_r is single-valued,
(2) T_r is firmly nonexpansive, i.e., for $u, v \in \mathcal{X}$,

$$\langle T_r u - T_r v, JT_r u - JT_r v\rangle \leq \langle T_r u - T_r v, Ju - Jv\rangle\,;$$

(3) $F(T_r) = MEP(\xi, A)$,
(4) $MEP(\xi, A)$ *is closed and convex.*
(5) $\phi(p, T_r u) + \phi(T_r u, u) \leq \phi(p, u),\ p \in F(T_r),\ u \in \mathcal{X}$.

3. Main results

In the first, the following lemma is proved.

Lemma 3.17. *Let $\mathcal{X}$ be a real 2 - uniformly convex and uniformly smooth Banach space with its dual space $\mathcal{X}^*$. Let $\mathcal{B} : \mathcal{X} \to \mathcal{X}^*$ be a μ - inverse strongly monotone map, $\mathcal{A} : \mathcal{X} \to \mathcal{X}^*$ be a relaxed $\eta-\alpha$ monotone and η-hemicontinuous mapping, $\xi : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be a function which satisfies (C1) - (C4) and $g : \mathcal{X} \to \mathbb{R}$ be a convex, proper and lower semicontinuous function. Let $T : \mathcal{X} \to \mathcal{X}$ be a relatively nonexpansive map such that $\Omega = F(T) \cap GMEP(\xi, \mathcal{A}, g) \cap \mathcal{B}^{-1}(0) \neq \emptyset$. Assume that the conditions (i) - (v) of Lemma 2.16 and the following condition are satisfied:*

(vi) $\forall u, v, w, z \in \mathcal{X}$,

$$\limsup_{t \to 0} \left\langle \mathcal{A}z, \eta\big(u, tv + (1-t)w\big) \right\rangle \leq \langle \mathcal{A}z, \eta(u, w) \rangle . \tag{3.24}$$

Let $\{v_n\}$ be a sequence generated as follows

$$\begin{cases} v_0, v_1 \in \mathcal{X}, \\ C_0 = \mathcal{X}, \\ t_n = v_n + \theta_n(v_n - v_{n-1}), \\ g_n = J^{-1}\big(Jt_n - \delta B t_n\big), \\ p_{n+1} = J^{-1}\Big(\frac{1}{\lambda}\big(JT(g_n) - Jg_n\big) + \beta_n J p_n\Big), \\ y_n = J^{-1}\big(Jg_n + \lambda J p_{n+1}\big), \\ z_n = J^{-1}\big(\alpha_n J t_n + (1-\alpha_n) J y_n\big), \\ u_n \in \mathcal{X} \text{ such that } \xi(u_n, u) + \langle A u_n, \eta(u, u_n) \rangle \\ +g(u) - g(u_n) + \frac{1}{r_n} \langle u - u_n, J u_n - J z_n \rangle \geq 0,\ \forall u \in \mathcal{X}, \\ \mathcal{C}_{n+1} = \big\{ z \in \mathcal{C}_n : \phi(z, u_n) \leq \phi(z, t_n) + \gamma_n \big\}, \\ v_{n+1} = \Pi_{\mathcal{C}_{n+1}} v_0, \quad n \geq 0, \end{cases} \tag{3.25}$$

where J is the map defined in (2.15), $p_1 = \frac{Tv_1 - v_1}{\lambda}$, $\gamma_n = 2\lambda(1-\alpha_n)||Tg_n||||p_n||\beta_n + \lambda^2||p_n||^2(1-\alpha_n)\beta_n^2$, $\lambda > 0$, $\{\theta_n\}$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,\infty)$, $\{r_n\} \subset [b,\infty)$ for some $b > 0$ and $0 < \delta < \frac{c\mu}{2}$, with the 2 - uniformly convex constant of X as c. Then $\{v_n\}$ converges strongly to $v^* \in \Omega$, provided that the following conditions hold:
(D1) $\lim_{n \to \infty} \beta_n = 0$,
(D2) $\alpha_n \in (a, 1-a)$, for some $a \in (0,1)$.

Proof. Observe that

$$\begin{aligned} y_n &= J^{-1}\big(Jg_n + \lambda J p_{n+1}\big) \\ &= J^{-1}\Big(Jg_n + \lambda\big(\frac{1}{\lambda}(JTg_n - Jg_n) + \beta_n J p_n\big)\Big) \\ &= J^{-1}\big(JTg_n + \lambda\beta_n J p_n\big). \end{aligned} \tag{3.26}$$

The proof is divided into steps;

Step (1): We show that $\{v_n\}$ is well defined and $\Omega \subset \mathcal{C}_n,\ \forall n \geq 0$.

Observe that

$$\phi(z, u_n) \le \phi(z, t_n) + \gamma_n \;\Leftrightarrow 2\langle z, Jt_n - Ju_n\rangle \le ||t_n||^2 - ||u_n||^2 + \gamma_n. \tag{3.27}$$

Therefore, using (3.27), it is easily seen that $\mathcal{C}_n$ is closed and convex, $\forall n \ge 0$. Thus, $\{v_n\}$ is well defined.

We also observe that for $n = 0$, $\Omega \subset \mathcal{C}_0 = \mathcal{X}$. Assume that $\Omega \subset \mathcal{C}_n$ and let $p \in \Omega$. Then, from the scheme (3.25), Lemma 2.2, Lemma 2.3 and the definition of the map V, we obtain

$$\begin{aligned}
\phi(p, g_n) &= \phi\Big(p, J^{-1}\big(Jt_n - \delta Bt_n\big)\Big)\\
&= V\big(p, Jt_n - \delta Bt_n\big)\\
&\le V\Big(p, \big(Jt_n - \delta Bt_n\big) + \delta Bt_n\Big) - 2\left\langle J^{-1}\big(Jt_n - \delta Bt_n\big) - p, \delta Bt_n\right\rangle\\
&= V\big(p, Jt_n\big) - 2\delta\left\langle J^{-1}\big(Jt_n - \delta Bt_n\big) - p, Bt_n\right\rangle\\
&= \phi(p, t_n) - 2\delta\langle t_n - p, Bt_n\rangle - 2\delta\left\langle J^{-1}\big(Jt_n - \delta Bt_n\big) - t_n, Bt_n\right\rangle\\
&= \phi(p, t_n) - 2\delta\langle t_n - p, Bt_n - Bp\rangle - 2\delta\left\langle J^{-1}\big(Jt_n - \delta Bt_n\big) - t_n, Bt_n\right\rangle\\
&\le \phi(p, t_n) - 2\delta\mu||Bt_n||^2 + 2\delta||J^{-1}(Jt_n - \delta Bt_n) - J^{-1}(Jt_n)||||Bt_n||\\
&\le \phi(p, t_n) - 2\delta\mu||Bt_n||^2 + \frac{4\delta^2}{c}||Bt_n||^2\\
&= \phi(p, t_n) - 2\delta\big(\mu - \frac{2\delta}{c}\big)||Bt_n||^2.
\end{aligned} \tag{3.28}$$

Using the assumption that $0 < \delta < \frac{c\mu}{2}$, we have

$$\phi(p, g_n) \le \phi(p, t_n). \tag{3.29}$$

Using (3.26), (3.29) and the notion that T is relatively nonexpansive, we get

$$\begin{aligned}
\phi(p, y_n) &= \phi\big(p, J^{-1}(JTg_n + \lambda\beta_n Jp_n)\big)\\
&= ||p||^2 - 2\langle p, JTg_n + \lambda\beta_n Jp_n\rangle + ||J^{-1}(JTg_n + \lambda\beta_n Jp_n)||^2\\
&= ||p||^2 - 2\langle p, JTg_n\rangle - 2\lambda\beta_n\langle p, Jp_n\rangle + ||JTg_n + \lambda\beta_n Jp_n||^2\\
&\le ||p||^2 - 2\langle p, JTg_n\rangle - 2\lambda\beta_n\langle p, Jp_n\rangle + \big(||JTg_n|| + \lambda\beta_n||Jp_n||\big)^2\\
&= ||p||^2 - 2\langle p, JTg_n\rangle - 2\lambda\beta_n\langle p, Jp_n\rangle + ||Tg_n||^2 + 2\lambda\beta_n||p_n||||Tg_n||\\
&\quad + \lambda^2\beta_n^2||p_n||^2\\
&\le \phi(p, Tg_n) - 2\lambda\beta_n\langle p, Jp_n\rangle + 2\lambda||p_n||||Tg_n||\beta_n + \lambda^2||p_n||^2\beta_n^2\\
&\le \phi(p, g_n) - 2\lambda\beta_n\langle p, Jp_n\rangle + 2\lambda||p_n||||Tg_n||\beta_n + \lambda^2||p_n||^2\beta_n^2\\
&\le \phi(p, t_n) - 2\lambda\beta_n\langle p, Jp_n\rangle + 2\lambda||p_n||||Tg_n||\beta_n + \lambda^2||p_n||^2\beta_n^2.
\end{aligned} \tag{3.30}$$

Now, putting $u_n = T_{r_n} z_n$, using (3.30), (B3), convexity of $||.||^2$ and Lemma 2.16(5), we get

$$\begin{aligned}
\phi(p, u_n) &= \phi(p, T_{r_n} z_n) \\
&\leq \phi(p, z_n) \\
&= \phi\Big(p, J^{-1}\big(\alpha_n J t_n + (1-\alpha_n) J y_n\big)\Big) \\
&\leq \alpha_n \phi(p, t_n) + (1-\alpha_n)\big(\phi(p, t_n) - 2\lambda\beta_n \langle p, Jp_n\rangle \\
&\quad + 2\lambda ||p_n|| ||Tg_n|| \beta_n + \lambda^2 ||p_n||^2 \beta_n^2\big) \\
&\leq \phi(p, t_n) - 2\lambda\beta_n \langle p, Jp_n\rangle + 2\lambda\beta_n \langle p, Jp_n\rangle \\
&\quad + 2\lambda(1-\alpha_n)||p_n|| ||Tg_n|| \beta_n + \lambda^2 ||p_n||^2 (1-\alpha_n)\beta_n^2 \\
&\leq \phi(p, t_n) + 2\lambda(1-\alpha_n)||p_n|| ||Tg_n|| \beta_n + \lambda^2 ||p_n||^2 (1-\alpha_n)\beta_n^2 \\
&= \phi(p, t_n) + \gamma_n.
\end{aligned} \tag{3.31}$$

Hence, we have $p \in \mathcal{C}_{n+1}$. So, by induction, we have that $\Omega \subset \mathcal{C}_n, \ \forall n \geq 0$.

Step (2): We show that $v_n \to v^* \in \mathcal{X}$ as $n \to \infty$.

From the definition of v_n, we have $v_n = \Pi_{\mathcal{C}_n} v_0$. Using (2.21), we obtain

$$\begin{aligned}
\phi(v_n, v_0) &= \phi(\Pi_{\mathcal{C}_0} v_0, v_0) \\
&\leq \phi(p, x_o) - \phi(p, \Pi_{\mathcal{C}_n} v_0) \\
&\leq \phi(p, v_0).
\end{aligned} \tag{3.32}$$

$\forall p \in \Omega \subset \mathcal{X}$. Therefore, $\big\{\phi(v_n, v_0)\big\}$ is bounded. Consequently, $\{v_n\}, \ \{t_n\}, \ \{g_n\}, \ \{y_n\}, \ \{u_n\}$, $\{p_n\}$ and $\{Tg_n\}$ are bounded.

On the other hand, since $v_{n+1} \in \mathcal{C}_{n+1} \subset \mathcal{C}_n$, and $v_n = \Pi_{\mathcal{C}_n} v_0$, we have

$$\phi(v_n, v_0) \leq \phi(v_{n+1}, v_0), \ \forall n \geq 0. \tag{3.33}$$

Showing that, $\big\{\phi(v_n, v_0)\big\}$ is nondecreasing. From (3.32) and (3.33), we have that $\lim_{n\to\infty} \phi(v_n, v_0)$ exists. Hence for any positive integer $m > n$, by (2.21), we have

$$\begin{aligned}
\phi(v_m, v_n) &= \phi(v_m, \Pi_{\mathcal{C}_n} v_0) \\
&\leq \phi(v_m, v_0) - \phi(v_n, v_0).
\end{aligned} \tag{3.34}$$

Using the fact that $\lim_{n\to\infty} \phi(v_n, v_0)$ exists, we obtain that

$$\lim_{m,n\to\infty} \phi(v_m, v_n) = 0. \tag{3.35}$$

Consequently, by Lemma 2.8, we have

$$\lim_{m,n\to\infty} ||v_m - v_n|| = 0. \tag{3.36}$$

This shows that $\{v_n\}$ is Cauchy sequence in $\mathcal{X}$. Hence, there exists $v^* \in \mathcal{X}$ such that

$$\lim_{n\to\infty} v_n = v^*. \tag{3.37}$$

Step (3): We show that $v^* \in \mathcal{B}^{-1}(0)$.

Considering $m = n + 1$ in (3.36), we see that

$$\lim_{n\to\infty} ||v_{n+1} - v_n|| = 0. \tag{3.38}$$

From the definition of t_n and (3.38), we get

$$\begin{aligned} ||v_n - t_n|| &= \theta_n ||v_n - v_{n-1}|| \\ &\leq ||v_n - v_{n-1}|| \to 0 \ as \ n \to \infty. \end{aligned} \tag{3.39}$$

In view of Remark 2.5 and the boundedness $\{t_n\}$, one see that $\lim_{n\to\infty} \phi(v_n, t_n) = 0$. Also from (3.38) and (3.39), we have $||v_{n+1} - t_n|| \to 0$. Again, by Remark 2.5, we obtain

$$\lim_{n\to\infty} \phi(v_{n+1}, t_n) = 0. \tag{3.40}$$

Now, from $v_{n+1} = \Pi_{\mathcal{C}_{n+1}} v_0 \in \mathcal{C}_{n+1}$, we get

$$\phi(v_{n+1}, u_n) \leq \phi(v_{n+1}, t_n) + \gamma_n, \ \forall n \geq 0.$$

Since $\lim_{n\to\infty} \beta_n = 0$, we have from the boundedness of $\{p_n\}$ and $\{Tg_n\}$ that $\lim_{n\to\infty} \gamma_n = 0$. Thus, from (3.40) we get

$$\lim_{n\to\infty} \phi(v_{n+1}, u_n) = 0.$$

Again, from Lemma 2.8, we obtain

$$\lim_{n\to\infty} ||v_{n+1} - u_n|| = 0. \tag{3.41}$$

Using (3.38) and (3.41), we obtain

$$||v_n - u_n|| \leq ||v_n - v_{n+1}|| + ||v_{n+1} - u_n|| \to 0 \ as \ n \to \infty. \tag{3.42}$$

From (3.39) and (3.42), we have

$$\lim_{n\to\infty} ||t_n - u_n|| = 0. \tag{3.43}$$

Using the uniform continuity of J on a bounded set in $\mathcal{X}$, we get

$$\lim_{n\to\infty} ||Jt_n - Ju_n|| = 0. \tag{3.44}$$

The inequalities (3.28), (3.30) and (3.31) imply

$$\begin{aligned} \phi(p, u_n) &\leq (1-\alpha_n)\phi(p, y_n) + \alpha_n \phi(p, t_n) \\ &\leq \alpha_n \phi(p, t_n) + (1-\alpha_n)\phi(p, g_n) - 2\lambda(1-\alpha_n)\beta_n \langle p, Jp_n \rangle \\ &\quad + 2\lambda(1-\alpha_n)||p_n|| ||Tg_n||\beta_n + \lambda^2 ||p_n||^2 (1-\alpha_n)\beta_n^2 \\ &\leq \alpha_n \phi(p, t_n) + (1-\alpha_n)\big(\phi(p, t_n) - 2\delta\big(\mu - \frac{2\delta}{c}\big)||Bt_n||^2\big) + \gamma_n \\ &= \phi(p, t_n) + \gamma_n - 2\delta(1-\alpha_n)\big(\mu - \frac{2\delta}{c}\big)||Bt_n||^2. \end{aligned} \tag{3.45}$$

Therefore, we obtain from (3.45) and condition (D2) that

$$2\delta a\big(\mu - \frac{2\delta}{c}\big)||Bt_n||^2 < 2\delta(1-\alpha_n)\big(\mu - \frac{2\delta}{c}\big)||Bt_n||^2 \leq \phi(p, t_n) - \phi(p, u_n) + \gamma_n, \tag{3.46}$$

where

$$
\begin{aligned}
\phi(p,t_n)-\phi(p,u_n) &= ||t_n||^2 - 2\langle p, Jt_n\rangle + ||p||^2 \\
&\quad - \big(||p||^2 - 2\langle p, Ju_n\rangle + ||u_n||^2\big) \\
&= -2\langle p, Jt_n\rangle + ||t_n||^2 + 2\langle p, Ju_n\rangle - ||u_n||^2 \\
&= ||t_n||^2 - ||u_n||^2 - 2\langle p, Jt_n - Ju_n\rangle \\
&\le \Big|||t_n||^2 - ||u_n||^2\Big| + 2\big|\langle p, Jt_n - Ju_n\rangle\big| \\
&\le 2||p||||Jt_n - Ju_n|| + \Big|||t_n|| - ||u_n||\Big|\big(||t_n|| + ||u_n||\big) \\
&\le ||t_n - u_n||\big(||t_n|| + ||u_n||\big) + 2||p||||Jt_n - Ju_n||.
\end{aligned} \tag{3.47}
$$

Using (3.43) and (3.44), we obtain from (3.47) that

$$\lim_{n\to\infty}\big(\phi(p,t_n)-\phi(p,u_n)\big)=0. \tag{3.48}$$

Now, from (3.46), (3.48), the fact that $0<\delta<\frac{c\mu}{2}$, the condition (D1) and the boundedness of $\{p_n\}$ and $\{Tg_n\}$, we get

$$\lim_{n\to\infty}||\mathcal{B}t_n||=0. \tag{3.49}$$

Since $\mathcal{B}$ is μ - inverse strongly monotone, it is $\frac{1}{\mu}$ - Lipschitz continuous. It therefore follows from (3.37), (3.39) and (3.49) that $v^*\in\mathcal{B}^{-1}(0)$.

Step (4): We show that $v^*\in F(T)$.

Let $r=\sup_{n\in\mathbb{N}}\big\{||t_n||,||Tg_n||\big\}$. Since $\mathcal{X}$ is uniformly smooth Banach space, then, $\mathcal{X}^*$ is uniformly convex Banach space. So, for $p\in\Omega$, putting $u_n=T_{r_n}z_n$, using (3.26), (3.29), Lemma 2.16(5) and the notion that T is relatively nonexpansive mapping, we obtain

$$
\begin{aligned}
\phi(p,u_n) &= \phi(p,T_{r_n}z_n)\\
&\le \phi\Big(p,J^{-1}\big(\alpha_n Jt_n+(1-\alpha_n)Jy_n\big)\Big)\\
&\le ||p||^2-2\alpha_n\langle p,Jt_n\rangle-2(1-\alpha_n)\langle p,JTg_n+\lambda\beta_n Jp_n\rangle\\
&\quad +||\alpha_n Jt_n+(1-\alpha_n)(JTg_n+\lambda\beta_n Jp_n)||^2\\
&\le ||p||^2-2\alpha_n\langle p,Jt_n\rangle-2(1-\alpha_n)\langle p,JTg_n\rangle\\
&\quad +\alpha_n||t_n||^2+(1-\alpha_n)||Tg_n||^2-\alpha_n(1-\alpha_n)g\big(||Jt_n-JTg_n||\big)\\
&\quad +2||p_n||\lambda(1-\alpha_n)\big(\alpha_n||t_n||+(1-\alpha_n)||Tg_n||\big)\beta_n\\
&\quad -2(1-\alpha_n)\langle p,\lambda\beta_n Jp_n\rangle+\lambda^2||p_n||^2(1-\alpha_n)^2\beta_n^2\\
&\le \alpha_n\phi(p,t_n)+(1-\alpha_n)\phi(p,Tg_n)-\alpha_n(1-\alpha_n)g\big(||Jt_n-JTg_n||\big) \qquad (3.50)\\
&\quad +2||p_n||\lambda(1-\alpha_n)\big(\alpha_n||t_n||+(1-\alpha_n)||Tg_n||\big)\beta_n\\
&\quad -2(1-\alpha_n)\langle p,\lambda\beta_n Jp_n\rangle+\lambda^2||p_n||^2(1-\alpha_n)^2\beta_n^2\\
&\le \alpha_n\phi(p,t_n)+(1-\alpha_n)\phi(p,g_n)-\alpha_n(1-\alpha_n)g\big(||Jt_n-JTg_n||\big)\\
&\quad +2||p_n||\lambda(1-\alpha_n)\big(\alpha_n||t_n||+(1-\alpha_n)||Tg_n||\big)\beta_n\\
&\quad +\lambda^2||p_n||^2(1-\alpha_n)^2\beta_n^2\\
&\le \alpha_n\phi(p,t_n)+(1-\alpha_n)\phi(p,t_n)-\alpha_n(1-\alpha_n)g\big(||Jt_n-JTg_n||\big)\\
&\quad +2||p_n||\lambda(1-\alpha_n)\big(\alpha_n||t_n||+(1-\alpha_n)||Tg_n||\big)\beta_n\\
&\quad +\lambda^2||p_n||^2(1-\alpha_n)^2\beta_n^2\\
&= \phi(p,t_n)+2||p_n||\lambda(1-\alpha_n)\big(\alpha_n||t_n||+(1-\alpha_n)||Tg_n||\big)\beta_n\\
&\quad +\lambda^2||p_n||^2(1-\alpha_n)^2\beta_n^2-\alpha_n(1-\alpha_n)g\big(||Jt_n-JTg_n||\big).
\end{aligned}
$$

The condition (D2) implies

$$\phi(p,u_n)\le\phi(p,t_n)+\lambda^2||p_n||^2(1-a)^2\beta_n^2+2||p_n||\lambda\beta_n(1-a)^2\big(||t_n||+||Tg_n||\big)-a^2g\big(||Jt_n-JTg_n||\big).$$

Thus,

$$a^2g\big(||Jt_n-JTg_n||\big)\le\phi(p,t_n)-\phi(p,u_n)+\lambda^2||p_n||^2(1-a)^2\beta_n^2+2||p_n||\lambda\beta_n(1-a)^2\big(||t_n||+||Tg_n||\big).$$

Using (3.48), the conditions (D1), (D2) and the boundedness of $\{p_n\},\ \{t_n\}$ and $\{Tg_n\}$, we have

$$\lim_{n\to\infty} g\big(||Jt_n-JTg_n||\big)=0.$$

The property of g gives

$$\lim_{n\to\infty}||Jt_n-JTg_n||=0.$$

Uniform continuity of J^{-1} on bounded set in $\mathcal{X}^*$ leads to

$$\lim_{n\to\infty}||t_n-Tg_n||=0. \tag{3.51}$$

Also, from (3.25), we obtain

$$||Jg_n-Jt_n||=\delta||Bt_n||.$$

Using (3.49), we have

$$\lim_{n\to\infty}||Jg_n-Jt_n||=0. \tag{3.52}$$

Similar argument as in obtaining (3.51) also leads to

$$\lim_{n\to\infty}||g_n-t_n||=0. \tag{3.53}$$

Combining (3.51) and (3.53), we get

$$\begin{aligned} ||g_n - Tg_n|| &= ||g_n - t_n + t_n - Tg_n|| \\ &\le ||g_n - t_n|| + ||t_n - Tg_n|| \to 0 \ as \ n \to \infty. \end{aligned} \tag{3.54}$$

Since $v_{n_q} \rightharpoonup v^*$, then from (3.39), we get $t_{n_q} \rightharpoonup v^*$. Combining with $g_{n_q} - w_{n_q} \to 0$, it follows that $g_{n_q} \rightharpoonup v^*$. Together with (3.54) and the notion of the relative nonexpansiveness of T, we obtain $v^* \in F(T) = \widehat{F}(T)$.

Step (5): We show that $v^* \in GMEP(\xi, A, g)$.

Putting $u_n = T_{r_n} z_n$ and using Lemma 2.16(5), we have

$$\begin{aligned} \phi(u_n, z_n) &= \phi(T_{r_n} z_n, z_n) \\ &\le \phi(p, z_n) - \phi(p, T_{r_n} z_n) \\ &\le \phi(p, t_n) + \gamma_n - \phi(p, u_n) \\ &= \phi(p, t_n) - \phi(p, u_n) + \gamma_n. \end{aligned}$$

From (3.48) and condition $(D1)$, we get

$$\lim_{n\to\infty} \phi(u_n, z_n) = 0.$$

It follows from Lemma 2.8 that

$$\lim_{n\to\infty} ||u_n - z_n|| = 0. \tag{3.55}$$

Now, from $v_{n_q} \rightharpoonup v^*$, (3.42) and (3.55), we have $z_{n_q} \rightharpoonup v^*$ and $u_{n_q} \rightharpoonup v^*$ as $q \to \infty$. Applying the uniform continuity of J on a bounded set in $\mathcal{X}$ and (3.55), we have

$$\lim_{n\to\infty} ||Ju_n - Jz_n|| = 0. \tag{3.56}$$

From $r_n \ge b$, we have

$$\lim_{n\to\infty} \left|\left| \frac{Ju_n - Jz_n}{r_n} \right|\right| \le \lim_{n\to\infty} \frac{1}{b} ||Ju_n - Jz_n|| = 0. \tag{3.57}$$

By $u_n = T_{r_n} z_n$, we see that

$$\xi(u_n, z) + \langle \mathcal{A}u_n, \eta(z, u_n)\rangle + g(z) - g(u_n) + \frac{1}{r_n} \langle z - u_n, Ju_n - Jz_n\rangle \ge 0, \ \forall z \in \mathcal{X}.$$

Replacing n by n_q, we have from the conditions (C4) and (i) that

$$\begin{aligned} \frac{1}{r_{n_q}} ||z - u_{n_q}|| ||Ju_{n_q} - Jz_{n_q}|| &\ge \left\langle \mathcal{A}u_{n_q}, \eta(u_{n_q}, z)\right\rangle \\ &\quad + g(u_{n_q}) - g(z) - \xi(u_{n_q}, z) \\ &\ge \left\langle \mathcal{A}u_{n_q}, \eta(u_{n_q}, z)\right\rangle + g(u_{n_q}) - g(z) + \xi(z, u_{n_q}) \end{aligned} \tag{3.58}$$

Letting $q \to \infty$ in (3.58), using (3.57), (C4) and (ii), we get

$$\langle \mathcal{A}v^*, \eta(v^*, z)\rangle + g(v^*) - g(z) + \xi(z, v^*) \le 0, \quad \forall z \in \mathcal{X}. \tag{3.59}$$

Assume that $s \in (0, 1]$. For $z \in \mathcal{X}$, let $z_s = sz + (1 - s)v^*$. It is clear that, $z_s \in \mathcal{X}$ and therefore,

$$\langle \mathcal{A}v^*, \eta(v^*, z_s)\rangle + g(v^*) - g(z_s) + \xi(z_s, v^*) \le 0. \tag{3.60}$$

Now using (C1), (C4), (i), (ii), the convexity of g and (3.60), we have

$$\begin{aligned} 0 &= \xi(z_s,z_s)+\langle \mathcal{A}v^*,\eta(z_s,z_s)\rangle+g(z_s)-g(z_s) \\ &= \xi\big(z_s,sz+(1-s)v^*\big)+\langle \mathcal{A}v^*,\eta(z_s,z_s)\rangle \\ &\leq s\big(\xi(z_s,z)+\langle \mathcal{A}v^*,\eta(z,z_s)\rangle+g(z)-g(z_s)\big) \\ &\quad +(1-s)\big(\xi(z_s,v^*)+\langle \mathcal{A}v^*,\eta(v^*,z_s)\rangle+g(v^*)-g(z_s)\big) \\ &\leq s\big(\xi(z_s,z)+\langle \mathcal{A}v^*,\eta(z,z_s)\rangle+g(z)-g(z_s)\big) \end{aligned}$$

and dividing by s, we have

$$\xi(z_s,z)+\langle Av^*,\eta(z,z_s)\rangle+g(z)-g(z_s)\geq 0, \quad \forall z\in\mathcal{X}.$$

By taking limit as $s\to 0$, using (C3), (vi) together with the lower semicontinuity of g, we obtain that

$$\xi(v^*,z)+\langle \mathcal{A}v^*,\eta(z,v^*)\rangle+g(z)-g(v^*)\geq 0, \quad \forall z\in\mathcal{X}.$$

Showing that $v^*\in GMEP(\xi,\mathcal{A},g)$ and so $v^*\in\Omega$. Hence the proof is complete.

□

In view of Lemma 3.17, the following is proved as our first main theorem.

Theorem 3.1. *Let $\mathcal{X}$ be a real 2 - uniformly convex and uniformly smooth Banach space with its dual space $\mathcal{X}^*$. Let $\mathcal{B}_i:\mathcal{X}\to\mathcal{X}^*$, $i=1,2,\ldots$, be a countable family of mappings defined in* (1.1) *with $\mu_i>0$ and $\mu=\inf_{i\geq 1}\mu_i>0$, $\mathcal{A}:\mathcal{X}\to\mathcal{X}^*$ be a relaxed $\eta-\alpha$ monotone and η-hemicontinuous mapping, $\xi:\mathcal{X}\times\mathcal{X}\to\mathbb{R}$ be a bifunction which satisfies the conditions (C1) - (C4) and $g:\mathcal{X}\to\mathbb{R}$ be a convex proper and lower semicontinuous function. Let $T_i:\mathcal{X}\to\mathcal{X}$, $i=1,2,3,\ldots$, be a countable family of mappings in* (2.19) *such that $\bigcap_{i=1}^{\infty}F(T_i)\cap\bigcap_{i=1}^{\infty}\mathcal{B}_i^{-1}(0)\cap GMEP(\xi,\mathcal{A},g)\neq\emptyset$. Suppose that $\{\delta_i\}$, $\{\upsilon_i\}$ and $\{\tau_i\}$ are sequences in $(0,1)$ which satisfy $\sum_{i=1}^{\infty}\delta_i=1$ and $\sum_{i=1}^{\infty}\upsilon_i=1$. Let $\mathcal{B}:\mathcal{X}\to\mathcal{X}^*$ be defined by $\mathcal{B}u=\sum_{i=1}^{\infty}\upsilon_i\mathcal{B}_iu$ and $T:\mathcal{X}\to\mathcal{X}$ be defined by $Tu=J^{-1}\Big(\sum_{i=1}^{\infty}\delta_i\big(\tau_iJu+(1-\tau_i)JT_iu\big)\Big)$ for all $u\in\mathcal{X}$. Assume that the conditions (i) - (v) of Lemma 2.16 and the following condition are satisfied:*

(vi) $\forall u,v,w,z\in\mathcal{X}$,

$$\limsup_{t\to 0}\big\langle \mathcal{A}z,\eta\big(u,tv+(1-t)w\big)\big\rangle\leq\langle \mathcal{A}z,\eta(u,w)\rangle\,. \tag{3.61}$$

Let $\{v_n\}$ be a sequence generated as follows

$$\begin{cases} v_0,u_1\in\mathcal{X}, \\ \mathcal{C}_0=\mathcal{X}, \\ t_n=v_n+\theta_n(v_n-v_{n-1}), \\ g_n=J^{-1}\big(Jt_n-\delta\mathcal{B}t_n\big), \\ p_{n+1}=J^{-1}\Big(\frac{1}{\lambda}\big(JT(g_n)-Jg_n\big)+\beta_nJp_n\Big), \\ y_n=J^{-1}\big(Jg_n+\lambda Jp_{n+1}\big), \\ z_n=J^{-1}\big(\alpha_nJt_n+(1-\alpha_n)Jy_n\big), \\ u_n\in\mathcal{X} \text{ such that } \xi(u_n,u)+\langle \mathcal{A}u_n,\eta(u,u_n)\rangle \\ +g(u)-g(u_n)+\frac{1}{r_n}\langle u-u_n,Ju_n-Jz_n\rangle\geq 0,\ \forall u\in\mathcal{X}, \\ \mathcal{C}_{n+1}=\big\{z\in\mathcal{C}_n:\phi(z,u_n)\leq\phi(z,t_n)+\gamma_n\big\}, \\ v_{n+1}=\Pi_{\mathcal{C}_{n+1}}v_0, \quad n\geq 0, \end{cases} \tag{3.62}$$

where J is the map defined in (2.15), *$p_1=\frac{Tv_1-v_1}{\lambda}$, $\gamma_n=2\lambda(1-\alpha_n)||p_n||||Tg_n||\beta_n+\lambda^2||p_n||^2(1-\alpha_n)\beta_n^2$, $\lambda>0$, $\{\theta_n\}$, $\{\alpha_n\}\subset(0,1)$, $\{\beta_n\}\subset[0,\infty)$, $\{r_n\}\subset[b,\infty)$ for some $b>0$, $0<\delta<\frac{c\mu}{2}$,* *with the 2 - uniformly convex constant of $\mathcal{X}$ as c. Then, the sequence $\{v_n\}$ converges strongly to $v^* \in \Omega = F(T) \cap \mathcal{B}^{-1}(0) \cap GMEP(\xi, \mathcal{A}, g)$, provided that the conditions (D1) and (D2) hold.*

Proof. It follows from Lemma 2.4 that $\mathcal{B}$ is μ - inverse strongly monotone mapping and $\mathcal{B}^{-1}(0) = \bigcap_{i=1}^{\infty} \mathcal{B}_i^{-1}(0)$. It also follows from Lemma 2.12 that T satisfies (2.19) and $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$. Thus, the conclusion follows from Lemma 3.17. This completes the proof of the first main theorem. □

For the convergence analysis of the second algorithm, we make the following assumption;

Assumption 1: Let $\mathcal{X}$ be a real 2-uniformly convex and uniformly smooth Banach space with its dual $\mathcal{X}^*$ and $v_0, v_1 \in \mathcal{X}$ be arbitrary points. For the iterates v_{n-1} and v_n $\forall n \geq 1$, choose $\varsigma_n \in (0,1)$ such that $\sum_{n=1}^{\infty} \varsigma_n < \infty$, $\theta_n \in [0, \bar{\theta_n}]$ and any $\eta \geq 0$ such that

$$\bar{\theta}_n := \begin{cases} \min\left\{\frac{n-1}{n+\eta-1}, \frac{\varsigma_n}{||Jv_n - Jv_{n-1}||}, \frac{\varsigma_n}{\phi(v_n, v_{n-1})}\right\} & if\ v_n \neq v_{n-1}, \\ \frac{n-1}{n+\eta-1}. \end{cases}$$

We obtained this idea based on the recent inertial extrapolation steps in [2, 8].

Remark 3.6. It is immediately seen from Assumption 1 that for every $n \geq 1$, we have

$$\theta_n ||Jv_n - Jv_{n-1}||^2 \leq \varsigma_n \quad \text{and} \quad \theta_n \phi(v_n, v_{n-1}) \leq \varsigma_n.$$

Together with $\sum_{n=1}^{\infty} \varsigma_n < \infty$ and the constant ε appearing in Lemma 2.7, we respectively obtain

$$\sum_{n=1}^{\infty} \varepsilon\theta_n ||Jv_n - Jv_{n-1}||^2 < \infty \tag{3.63}$$

and

$$\sum_{n=1}^{\infty} \theta_n \phi(v_n, v_{n-1}) < \infty. \tag{3.64}$$

Next, we consider the following lemma;

Lemma 3.18. *Let $\mathcal{X}$ be a real 2 - uniformly convex and uniformly smooth Banach space with its dual space $\mathcal{X}^*$. Let $\mathcal{B} : \mathcal{X} \to \mathcal{X}^*$ and $T : \mathcal{X} \to \mathcal{X}$ be mappings which satisfy* (1.1) *with $\mu > 0$ and* (2.19) *respectively, such that $\Omega = F(T) \cap \mathcal{B}^{-1}(0) \neq \emptyset$. Let $\{v_n\}$ be a sequence generated as follows;*

$$\begin{cases} v_0, v_1 \in \mathcal{X}, \\ t_n = v_n + \theta_n(v_n - v_{n-1}), \\ g_n = J^{-1}\big(Jt_n - \delta Bt_n\big), \\ p_{n+1} = J^{-1}\Big(\frac{1}{\lambda}\big(JT(g_n) - Jg_n\big) + \beta_n Jp_n\Big), \\ y_n = J^{-1}\big(Jg_n + \lambda Jp_{n+1}\big), \\ z_n = J^{-1}\big(\alpha_n Jt_n + (1-\alpha_n)Jy_n\big), \\ v_{n+1} = J^{-1}\big(\sigma_n Ju + (1-\sigma_n)Jz_n\big), \quad n \geq 0, \end{cases} \tag{3.65}$$

where J is the map defined in (2.15), $\lambda > 0$, $0 < \delta < \frac{c\mu}{2}$, where c is the 2 - uniformly convex constant of $\mathcal{X}$, $p_1 = \frac{Tv_1 - v_1}{\lambda}$ and $\{\beta_n\} \subset [0, \bar{\beta}_n]$, $\forall n \geq 1$, such that

$$\bar{\beta}_n = \frac{\vartheta_n}{\max\{||p_n|| ||Tg_n||, ||p_n||, \beta\}}, \quad \text{for any } \beta > 0. \tag{3.66}$$

Then, the sequence $\{v_n\}$ converges strongly to $v^* \in \Omega$, provided that Assumption 1 and the following conditions hold;
(E1) $\vartheta_n \in [0,\infty)$, such that $\sum_{n=0}^{\infty} \vartheta_n < \infty$,
(E2) $\alpha_n \in (a, 1-a)$, for some $a \in (0,1)$ and $\sigma_n \in (b, 1-b)$, for some $b \in (0,1)$ such that $\lim\limits_{n\to\infty} \sigma_n = 0$ and $\sum_{n=1}^{\infty} \sigma_n = \infty$.

Proof. We begin our proof with the following remark.

Remark 3.7. It is immediately seen from the definition of β_n in (3.66) that for all $n \geq 1$, we have

$$\beta_n||p_n||||Tg_n|| \leq \vartheta_n \quad \text{and} \quad \beta_n||p_n|| \leq \vartheta_n.$$

Together with condition (E1), we respectively obtain

$$\sum_{n=1}^{\infty} \beta_n||p_n||||Tg_n|| < \infty \tag{3.67}$$

and

$$\sum_{n=1}^{\infty} \beta_n||p_n|| < \infty. \tag{3.68}$$

In a similar fashion, the following steps are followed for the proof.

Step (I): We start by showing that $\{v_n\}$ is bounded.

Let $p \in \Omega$. In view of the inequality (3.31), one sees that

$$\phi(p, z_n) \leq \phi(p, t_n) + \gamma_n, \tag{3.69}$$

where $\gamma_n = 2\lambda(1-\alpha_n)||p_n||||Tg_n||\beta_n + \lambda^2||p_n||^2(1-\alpha_n)\beta_n^2$.

Now, using $(B3)$, inequality (3.69), Lemma 2.9 and the fact that $\theta_n \in (0,1)$, one sees that

$$\begin{aligned} \phi(p, v_{n+1}) &= \phi\big(p, J^{-1}\big(\sigma_n Ju + (1-\sigma_n)Jz_n\big)\big) \\ &\leq \sigma_n\phi(p,u) + (1-\sigma_n)\big(\phi(p,t_n) + \gamma_n\big) \\ &\leq \sigma_n\phi(p,u) + (1-\sigma_n)\big(\phi(p,v_n) + \theta_n\big(\phi(p,v_n) - \phi(p,v_{n-1})\big) \\ &\quad + \varepsilon\theta_n^2||Jv_n - Jv_{n-1}||^2 + \theta_n\phi(v_n, v_{n-1}) + \gamma_n\big) \\ &\leq \max\big\{\phi(p,u), \phi(p,v_n) + \theta_n\big(\phi(p,v_n) - \phi(p,v_{n-1})\big) \\ &\quad + \varepsilon\theta_n||Jv_n - Jv_{n-1}||^2 + \theta_n\phi(v_n, v_{n-1}) + \gamma_n\big\} \end{aligned} \tag{3.70}$$

If $\phi(p,u)$ is the maximum, then we obtained the desired result. Otherwise, there exists $n_0 \in \mathbb{N}$, such that $\forall n \geq n_0$, we have

$$\begin{aligned} \phi(p, v_{n+1}) &\leq \phi(p,v_n) + \theta_n\big(\phi(p,v_n) - \phi(p,v_{n-1})\big) \\ &\quad + \varepsilon\theta_n||Jv_n - Jv_{n-1}||^2 + \theta_n\phi(v_n, v_{n-1}) + \gamma_n. \end{aligned}$$

Hence, by Assumption 1, condition $(E1)$, Lemma 2.10 and (3.66) , we obtain that for every $p \in \Omega$, the sequence $\{\phi(p, v_n)\}$ converges and thus, bounded. Furthermore, by $(B1)$, $\{v_n\}$ is bounded. Consequently, $\{t_n\}$, $\{g_n\}$, $\{z_n\}$, $\{y_n\}$, $\{p_n\}$ and $\{Tg_n\}$ are bounded.

Step (II): Next is to show that $\lim\limits_{n\to\infty} ||v_{n+1} - v_n|| = 0$.

Let $p \in \Omega$. Following the same lines of proof as in (3.50), we first estimate $\phi(p, z_n)$ as follows;

$$\begin{aligned} \phi(p, z_n) &\leq 2||p_n||\lambda(1-\alpha_n)\big(\alpha_n||t_n|| + (1-\alpha_n)||Tg_n||\big)\beta_n + \phi(p, t_n) \\ &\quad + \lambda^2||p_n||^2(1-\alpha_n)^2\beta_n^2 - \alpha_n(1-\alpha_n)g\big(||Jt_n - JTg_n||\big). \end{aligned} \tag{3.71}$$

Using the inequality (3.71), conditions (E2), Lemma 2.9 and the fact that $\sigma_n, \ \theta_n \in (0,1)$, we estimate $\phi(p, v_{n+1})$ as follows;

$$\begin{aligned} \phi(p, v_{n+1}) &\leq (1-\sigma_n)\phi(p, z_n) + \sigma_n\phi(p, u) \\ &\leq \sigma_n\phi(p, u) + \phi(p, t_n) + 2||p_n||\lambda(1-\sigma_n)(1-\alpha_n)\big(\alpha_n||t_n|| \\ &\quad + (1-\alpha_n)||Tg_n||\big)\beta_n + \lambda^2||p_n||^2(1-\sigma_n)(1-\alpha_n)^2\beta_n^2 \\ &\quad - \alpha_n(1-\alpha_n)(1-\sigma_n)g\big(||Jt_n - JTg_n||\big) \\ &\leq \phi(p, v_n) + \sigma_n\phi(p, u) + \theta_n\big(\phi(p, v_n) - \phi(p, v_{n-1})\big) \\ &\quad + \varepsilon\theta_n||Jv_n - Jv_{n-1}||^2 + \theta_n\phi(v_n, v_{n-1}) + \Gamma_n \\ &\quad - a^2bg\big(||Jt_n - JTg_n||\big), \end{aligned} \tag{3.72}$$

where $\Gamma_n = ||p_n||\lambda(1-\sigma_n)(1-\alpha_n)\Big(2\big(\alpha_n||t_n|| + (1-\alpha_n)||Tg_n||\big) + ||p_n||\lambda(1-\alpha_n)\beta_n\Big)\beta_n.$

Rearranging the terms in (3.72), we get

$$\begin{aligned} a^2bg\big(||Jt_n - JTg_n||\big) &\leq \sigma_n\phi(p, u) - \phi(p, v_{n+1}) + \phi(p, v_n) \\ &\quad + \varepsilon\theta_n||Jv_{n-1} - Jv_n||^2 + \theta_n\phi(v_n, v_{n-1}) \\ &\quad + \theta_n\big(\phi(p, v_n) - \phi(p, v_{n-1})\big) + \Gamma_n. \end{aligned} \tag{3.73}$$

In the remaining part of the proof, the following two cases are considered.

Case 1. Assume that $n_0 \in \mathbb{N}$ exists, such that for any $n \geq n_0$,

$$\phi(p, v_{n+1}) \leq \phi(p, v_n).$$

Then, $\{\phi(p, v_n)\}$ converges. Therefore, it follows from the inequality (3.73), conditions $(E1), \ (E2)$, the boundedness of $\{t_n\}$, the definition of $\bar{\beta}_n$ in (3.66) and the facts that $\lim_{n\to\infty}\phi(p, v_n)$ exists, $\lim_{n\to\infty}\theta_n\phi(v_n, v_{n-1}) = 0$ and $\lim_{n\to\infty}\varepsilon\theta||Jv_n - Jv_{n-1}||^2 = 0$, we obtain that

$$\lim_{n\to\infty} g\big(||Jt_n - JTg_n||\big) = 0.$$

The property of g also implies

$$\lim_{n\to\infty}||Jt_n - JTg_n|| = 0. \tag{3.74}$$

Similar argument as in (3.51) also leads to

$$\lim_{n\to\infty}||t_n - Tg_n|| = 0.$$

Observe that since $\lim_{n\to\infty}\theta_n\phi(v_n, v_{n-1}) = 0$, Lemma 2.8 provides that $\lim_{n\to\infty}\theta_n||v_n - v_{n-1}|| = 0$. The uniform continuity of J on bounded sets implies that $\lim_{n\to\infty}\theta_n||Jv_n - Jv_{n-1}|| = 0$. Furthermore, since $||Jt_n - Jv_n|| = \theta_n||v_n - v_{n-1}|$, then

$$\lim_{n\to\infty}||Jt_n - Jv_n|| = 0. \tag{3.75}$$

Considering the equation (3.75) and the uniform continuity of J^{-1} on bounded sets ensure that

$$\lim_{n\to\infty}||t_n - v_n|| = 0. \tag{3.76}$$

Combining (3.26), (3.74), (3.75), the condition $(E1)$ and equation (3.66), one sees that

$$\begin{aligned} ||Jv_n - Jy_n|| &\leq ||Jv_n - JTg_n|| + \lambda||p_n||\beta_n \\ &\leq ||Jv_n - Jt_n|| + ||Jt_n - JTg_n|| + \lambda||p_n||\beta_n \to 0 \; as \; n \to \infty. \end{aligned} \tag{3.77}$$

Similarly, from (3.75), (3.77) and $\alpha_n \in (0,1)$, we obtain

$$||Jz_n - Jv_n|| \leq \alpha_n||Jt_n - Jv_n|| + (1-\alpha_n)||Jv_n - Jy_n|| \to 0 \; as \; n \to \infty. \tag{3.78}$$

Combining (3.75) and (3.78), we have

$$||Jt_n - Jz_n|| \leq ||Jt_n - Jv_n|| + ||Jv_n - Jz_n|| \to 0 \; as \; n \to \infty. \tag{3.79}$$

By the uniform continuity of J^{-1} and equation (3.79), we get

$$\lim_{n\to\infty} ||t_n - z_n|| = 0. \tag{3.80}$$

Using (3.78), the boundedness of $\{v_n\}$ and $\lim_{n\to\infty} \sigma_n = 0$, we get

$$||Jv_{n+1} - Jv_n|| \leq \sigma_n||Ju - Jv_n|| + (1-\sigma_n)||Jz_n - Jv_n|| \to 0 \; as \; n \to \infty. \tag{3.81}$$

By the uniform continuity of J^{-1} on a bounded set, we see that

$$\lim_{n\to\infty} ||v_{n+1} - v_n|| = 0. \tag{3.82}$$

Step (III): We prove that $\omega_w(v_n) \subset \Omega$.

Observe that the boundedness of $\{v_n\}$ implies that $\omega_w(v_n) \neq \emptyset$. Let $v^* \in \omega_w(v_n)$. Then, a subsequence $\{v_{n_q}\}$ of $\{v_n\}$ such that $v_{n_q} \rightharpoonup v^*$ exists. From (3.76) and (3.80), we have $t_{n_q} \rightharpoonup v^*$ and $z_{n_q} \rightharpoonup v^*$ respectively.

We first show that $v^* \in \mathcal{B}^{-1}(0)$. Following the same lines of proof as in (3.47), one sees that

$$\phi(p, t_n) - \phi(p, z_n) \leq ||t_n - z_n||\big(||t_n|| + ||z_n||\big) + 2||p||||Jt_n - Jz_n||. \tag{3.83}$$

Combining (3.79), (3.80), (3.83) and the boundedness of $\{t_n\}$ and $\{z_n\}$, we have

$$\lim_{n\to\infty} \big(\phi(p, t_n) - \phi(p, z_n)\big) = 0. \tag{3.84}$$

In view of (3.45), (3.66), (3.84), the boundedness of $\{Tg_n\}$, the conditions $(E1),\ (E2)$ and the definition of z_n in (3.65), we obtain

$$2\delta a\big(\mu - \frac{2\delta}{c}\big)||Bt_n||^2 \leq \phi(p, t_n) - \phi(p, z_n) + \gamma_n \to 0 \; as \; n \to \infty. \tag{3.85}$$

It follows from (3.85) and the fact that $0 < \delta < \frac{c\mu}{2}$ that

$$\lim_{n\to\infty} ||\mathcal{B}t_n|| = 0. \tag{3.86}$$

Continuity of $\mathcal{B}$, together with the equations (3.76), (3.86) and $v_{n_q} \rightharpoonup v^*$, we see that $v^* \in \mathcal{B}^{-1}(0)$.

Next, we show that $v^* \in F(T)$. In connection with (3.86) and the definition of g_n in (3.65), we equivalently obtain similar result as in (3.52). Uniform continuity of J^{-1} on bounded sets also leads to the same result as in (3.53). Combining (3.52) and (3.74), we have

$$\lim_{n\to\infty} ||Jg_n - JTg_n|| = 0. \tag{3.87}$$

Uniform continuity of J^{-1} on bounded sets implies

$$\lim_{n\to\infty} ||g_n - Tg_n|| = 0. \tag{3.88}$$

Since $v_{n_q} \rightharpoonup v^*$, then we obtain from (3.53) and (3.76) that $g_{n_q} \rightharpoonup v^*$. Using (3.88) and the fact that T satisfies (2.19), we get $v^* \in F(T) = \widehat{F}(T)$. So that $v^* \in \Omega$.

Finally, we show that $v_n \to p = \Pi_\Omega u \ as \ n \to \infty$. One sees that if $p = v^*$, then, we obtain what is needed. Suppose that $p \neq v^*$, then, the inequality (2.20), the boundedness of $\{v_n\}$ and the fact that Ω is closed and convex imply the existence of a subsequence $\{v_{n_q}\} \subset \{v_n\}$ such that

$$\limsup_{n\to\infty} \langle v_n - v, Ju - Jv\rangle = \lim_{q\to\infty} \left\langle v_{n_q} - v, Ju - Jv\right\rangle = \langle v^* - v, Ju - Jv\rangle \leq 0.$$

In connection with (3.81) and the uniform boundedness of J^{-1}, we see that

$$\limsup_{n\to\infty} \langle v_{n+1} - v, Ju - Jv\rangle \leq 0. \tag{3.89}$$

Using (3.69), Lemma 2.2, Lemma (2.15),

$$\begin{aligned}
\phi(p, v_{n+1}) &= \phi\big(p, J^{-1}(\sigma_n Ju + (1-\sigma_n)Jz_n)\big)\\
&= V(p, \sigma_n Ju + (1-\sigma_n)Jz_n)\\
&\leq V\big(p, \sigma_n Ju + (1-\sigma_n)Jz_n - \sigma_n(Ju - Jv)\big)\\
&\quad + 2\sigma_n \langle v_{n+1} - v, Ju - Jv\rangle\\
&= V\big(p, \sigma_n Jp + (1-\sigma_n)Jz_n\big) + 2\sigma_n \langle v_{n+1} - v, Ju - Jv\rangle\\
&= \phi(p, J^{-1}(\sigma_n Jp + (1-\sigma_n)Jz_n)\big) + 2\sigma_n \langle v_{n+1} - v, Ju - Jv\rangle\\
&\leq \sigma_n\phi(p,p) + (1-\sigma_n)\phi(p, z_n) + 2\sigma_n \langle v_{n+1} - v, Ju - Jv\rangle\\
&\leq (1-\sigma_n)\big(\phi(p, v_n) + \varepsilon\theta_n ||Jv_n - Jv_{n-1}||^2 + \theta_n\phi(v_n, v_{n-1})\\
&\quad + \theta_n(\phi(p, v_n) - \phi(p, v_{n-1}))\big) + (1-\sigma_n)\gamma_n + 2\sigma_n \langle v_{n+1} - v, Ju - Jv\rangle\\
&\leq (1-\sigma_n)\phi(p, v_n) + \varepsilon\theta_n ||Jv_n - Jv_{n-1}||^2 + \theta_n\phi(v_n, v_{n-1})\\
&\quad + (1-\sigma_n)\gamma_n + 2\sigma_n \langle v_{n+1} - v, Ju - Jv\rangle .
\end{aligned} \tag{3.90}$$

By Lemma 2.11, the inequalities (3.89), (3.90), the conditions (E1), (E2), the Assumption 1 and the definition of $\bar{\beta}_n$ in (3.66) respectively, we see that $\lim_{n\to\infty} \phi(p, v_n) = 0$. Together with Lemma 2.8, one obtains that $\lim_{n\to\infty} v_n = p$.

Case 2: If the assumption in case 1 does not hold, then, we ensured the existence of a subsequence $\{v_{m_r}\} \subset \{v_n\}$ such that

$$\phi(p, v_{m_r}) < \phi(p, v_{m_r+1}), \ \forall r \in \mathbb{N}.$$

By Lemma 2.14, a nondecreasing sequence $\{m_s\}$ exists, such that $\lim_{s\to\infty} m_s = \infty$ and the following hold

$$\phi(p, v_{m_s+1}) \geq \phi(p, v_{m_s}), \quad and \quad \phi(p, v_{m_s}) \geq \phi(p, v_s), \ \forall s \in \mathbb{N}.$$

From the Inequality (3.73), we have

$$\begin{aligned}
a^2 bg\big(||Jt_{m_s} - JTg_{m_s}||\big) &\leq \sigma_{m_s}\phi(p, u) + \phi(p, v_{m_s}) - \phi(p, v_{m_s+1})\\
&\quad + \varepsilon\theta_{m_s}^2 ||Jv_{m_s} - Jv_{m_s-1}||^2 + \theta_{m_s}\phi(v_{m_s}, v_{m_s-1})\\
&\quad + \theta_{m_s}\big(\phi(p, v_{m_s}) - \phi(p, v_{m_s-1})\big) + \Gamma_{m_s}\\
&\leq \sigma_{m_s}\phi(p, u) + \varepsilon\theta_{m_s} ||Jv_{m_s} - Jv_{m_s-1}||^2\\
&\quad + \theta_{m_s}\phi(v_{m_s}, v_{m_s-1}) + \Gamma_{m_s}\\
&\quad + \theta_{m_s}\big(\phi(p, v_{m_s}) - \phi(p, v_{m_s-1})\big)
\end{aligned} \tag{3.91}$$

Similar arguments as in the Case 1, lead to

$$\lim_{s\to\infty} ||g_{m_s} - Tg_{m_s}|| = 0, \quad \lim_{s\to\infty} ||v_{m_s+1} - v_{m_s}|| = 0$$

and

$$\limsup_{s\to\infty} \langle v_{m_s+1} - v, Ju - Jv\rangle \leq 0.$$

Equivalently, one easily sees from (3.90) and the fact that $\theta_{m_s} \in (0,1)$ that

$$\begin{aligned}\phi(p, v_{m_s+1}) \;\leq\;& (1-\sigma_{m_s})\phi(p, v_{m_s}) + \varepsilon\theta_{m_s}||Jv_{m_s} - Jv_{m_s-1}||^2 \\ &+ \theta_{m_s}\phi(v_{m_s}, v_{m_s-1}) + (\phi(p, v_{m_s}) - \phi(p, v_{m_s-1})) \\ &+ (1-\sigma_{m_s})\gamma_{m_s} + 2\sigma_{m_s}\langle v_{m_s+1} - v, Ju - Jv\rangle. \end{aligned} \tag{3.92}$$

Lemma 2.11 and the inequality (3.92) lead to conclude that $\lim_{s\to\infty} \phi(p, v_{m_s}) = 0$, which implies that

$$\limsup_{s\to\infty} \phi(p, v_s) \leq \lim_{s\to\infty} \phi(p, v_{m_s}) = 0.$$

Thus, $\limsup_{s\to\infty} \phi(p, v_s) = 0$. Hence, by Lemma 2.8 we get that $\lim_{s\to\infty} v_s = p$. This completes the proof. □

Using Lemma 3.18, we prove the following as our second main Theorem.

Theorem 3.2. *Let $\mathcal{X}$ be a real 2 - uniformly convex and uniformly smooth Banach space with its dual space $\mathcal{X}^*$. Let $\mathcal{B}_i : \mathcal{X} \to \mathcal{X}^*,\ i = 1, 2, \ldots,$ be a countable family of mappings which satisfies* (1.1) *with $\mu_i > 0$ and $\mu = \inf_{i\geq 1}\mu_i > 0$. Let $T_i : \mathcal{X} \to \mathcal{X},\ i = 1, 2, \ldots,$ be a countable family of mappings defined in* (2.19) *such that $\bigcap_{i=1}^{\infty} F(T_i) \cap \bigcap_{i=1}^{\infty} \mathcal{B}_i^{-1}(0) \neq \emptyset$. Suppose that $\{\delta_i\}$, $\{\upsilon_i\}$ and $\{\tau_i\}$ are sequences in $(0,1)$ such that $\sum_{i=1}^{\infty}\delta_i = 1$, $\sum_{i=1}^{\infty}\upsilon_i = 1$. Let $\mathcal{B} : \mathcal{X} \to \mathcal{X}^*$ and $T : \mathcal{X} \to \mathcal{X}$ be defined by $\mathcal{B}u = \sum_{i=1}^{\infty}\upsilon_i\mathcal{B}_i u$ and $Tu = J^{-1}\Big(\sum_{i=1}^{\infty}\delta_i\big(\tau_i Ju + (1-\tau_i)JT_i u\big)\Big)$ for every $u \in \mathcal{X}$.*

Let $\{v_n\}$ be a sequence generated as follows

$$\begin{cases} v_0, v_1 \in \mathcal{X}, \\ t_n = J^{-1}(Jv_n + \theta_n(Jv_n - Jv_{n-1})), \\ g_n = J^{-1}\big(Jt_n - \delta\mathcal{B}t_n\big), \\ p_{n+1} = J^{-1}\Big(\frac{1}{\lambda}\big(JT(g_n) - Jg_n\big) + \beta_n Jp_n\Big), \\ y_n = J^{-1}\big(Jg_n + \lambda Jp_{n+1}\big), \\ z_n = J^{-1}\big(\alpha_n Jt_n + (1-\alpha_n)Jy_n\big), \\ v_{n+1} = J^{-1}\big(\sigma_n Ju + (1-\sigma_n)Jz_n\big), \quad n \geq 0, \end{cases} \tag{3.93}$$

where J is the map defined in (2.15)*, $\lambda > 0$, $0 < \delta < \frac{c\mu}{2}$, where c is the 2 - uniformly convex constant of $\mathcal{X}$, $p_1 = \frac{Tv_1 - v_1}{\lambda}$ and $\{\beta_n\} \subset [0, \bar{\beta}_n]$, $\forall n \geq 1$, such that $\bar{\beta}_n$ is obtained by* (3.66)*. Then, the sequence $\{v_n\}$ converges strongly to $v^* \in \Omega = F(T) \cap \mathcal{B}^{-1}(0)$, provided that the Assumption 1 and the conditions (E1) and (E2) hold.*

Proof. It follows from Lemma 2.4 that $\mathcal{B}$ is μ - inverse strongly monotone map and $\mathcal{B}^{-1}(0) = \bigcap_{i=1}^{\infty} \mathcal{B}_i^{-1}(0)$. It also follows from Lemma 2.12 that T satisfies (2.19) and $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$. Hence, the conclusion follows from Lemma 3.18. □

If in the Theorems 3.1 and 3.2, $\mathcal{X} = \mathcal{H}$, then we respectively obtain the following from Remark 2.4.

Corollary 3.1. *Let $\mathcal{H}$ be a real Hilbert space. Let $\mathcal{B}_i : \mathcal{H} \to \mathcal{H},\ i = 1, 2, \ldots,$ be a countable family of μ_i - inverse strongly monotone maps with $\mu = \inf_{i\geq 1} \mu_i > 0$, $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ be an η-hemicontinuous and relaxed $\eta - \alpha$ monotone mapping, $\xi : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bifunction which satisfies the conditions (C1) - (C4) and $g : \mathcal{H} \to \mathbb{R}$ be a proper, convex and lower semicontinuous function. Let $T_i : \mathcal{H} \to \mathcal{H},\ i = 1, 2, 3, \ldots,$ be a countable family of quasi - nonexpansive mappings such that $\bigcap_{i=1}^{\infty} F(T_i) \cap \bigcap_{i=1}^{\infty} \mathcal{B}_i^{-1}(0) \cap GMEP(\xi, \mathcal{A}, g) \neq \emptyset$. Suppose that $\{\delta_i\}$, $\{\upsilon_i\}$ and $\{\tau_i\}$ are sequences in $(0, 1)$ such that $\sum_{i=1}^{\infty} \delta_i = 1$, $\sum_{i=1}^{\infty} \upsilon_i = 1$. Let $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ and $T : \mathcal{H} \to \mathcal{H}$ be defined by $\mathcal{B}u = \sum_{i=1}^{\infty} \upsilon_i \mathcal{B}_i u$ and $Tu = \sum_{i=1}^{\infty} \delta_i \big(\tau_i u + (1 - \tau_i) T_i u\big)$ for any $u \in \mathcal{H}$. Assume that the conditions (i) - (v) of Lemma 2.16 and the following condition are satisfied:*

(vi) $\forall u, v, w, z \in \mathcal{H}$,

$$\limsup_{t\to 0} \big\langle \mathcal{A}z, \eta\big(u, tv + (1 - t)w\big)\big\rangle \leq \langle \mathcal{A}z, \eta(u, w)\rangle . \tag{3.94}$$

Let $\{v_n\}$ be a sequence generated as follows

$$\begin{cases} v_0, v_1 \in \mathcal{H}, \\ \mathcal{C}_0 = \mathcal{H}, \\ t_n = v_n + \theta_n(v_n - v_{n-1}), \\ g_n = t_n - \delta \mathcal{B} t_n, \\ p_{n+1} = \frac{1}{\lambda}\big(T(g_n) - g_n\big) + \beta_n p_n, \\ y_n = g_n + \lambda p_{n+1}, \\ z_n = \alpha_n t_n + (1 - \alpha_n) y_n, \\ u_n \in \mathcal{H} \text{ such that } \xi(u_n, u) + \langle \mathcal{A}u_n, \eta(u, u_n)\rangle \\ +g(u) - g(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - z_n\rangle \geq 0,\ \forall u \in \mathcal{H}, \\ \mathcal{C}_{n+1} = \big\{z \in \mathcal{C}_n : ||u_n - z||^2 \leq ||t_n - z||^2 + \gamma_n\big\}, \\ v_{n+1} = P_{\mathcal{C}_{n+1}} v_0, \quad n \geq 0, \end{cases} \tag{3.95}$$

where $\gamma_n = 2\lambda(1 - \alpha_n)||p_n||||Tg_n||\beta_n + \lambda^2||p_n||^2(1 - \alpha_n)\beta_n^2$, $\lambda > 0$, $\{\theta_n\}$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, \infty)$, $\{r_n\} \subset [b, \infty)$ for some $b > 0$, $0 < \delta < \frac{\mu}{2}$ and $p_1 = \frac{Tv_1 - v_1}{\lambda}$. Then, the sequence $\{v_n\}$ converges strongly to $v^* \in \Omega = F(T) \cap \mathcal{B}^{-1}(0) \cap GMEP(\xi, \mathcal{A}, g)$, provided that the following the conditions (D1) and (D2) hold.

Corollary 3.2. *Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{B}_i : \mathcal{H} \to \mathcal{H},\ i = 1, 2, \ldots,$ be a countable family of mappings which satisfies* (1.1) *with $\mu_i > 0$ and $\mu = \inf_{i\geq 1} \mu_i > 0$. Let $T_i : \mathcal{H} \to \mathcal{H},\ i = 1, 2, \ldots,$ be a countable family of mappings defined in* (2.19) *such that $\bigcap_{i=1}^{\infty} F(T_i) \cap \bigcap_{i=1}^{\infty} \mathcal{B}_i^{-1}(0) \neq \emptyset$. Suppose that $\{\delta_i\}$, $\{\upsilon_i\}$ and $\{\tau_i\}$ are sequences in $(0, 1)$ such that $\sum_{i=1}^{\infty} \delta_i = 1$, $\sum_{i=1}^{\infty} \upsilon_i = 1$. Let $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ and $T : \mathcal{H} \to \mathcal{H}$ be defined by $\mathcal{B}u = \sum_{i=1}^{\infty} \upsilon_i \mathcal{B}_i u$ and $Tu = J^{-1}\Big(\sum_{i=1}^{\infty} \delta_i \big(\tau_i Ju + (1 - \tau_i) JT_i u\big)\Big)$ for every $u \in \mathcal{H}$. Let $\{v_n\}$ be a sequence generated as follows*

$$\begin{cases} v_0, v_1 \in \mathcal{H}, \\ t_n = v_n + \theta_n(v_n - v_{n-1}), \\ g_n = t_n - \delta \mathcal{B} t_n, \\ p_{n+1} = \frac{1}{\lambda}\big(T(g_n) - g_n\big) + \beta_n p_n, \\ y_n = g_n + \lambda p_{n+1}, \\ z_n = \alpha_n t_n + (1 - \alpha_n) y_n, \\ v_{n+1} = \sigma_n u + (1 - \sigma_n) z_n, \quad n \geq 0, \end{cases} \tag{3.96}$$

where $\lambda > 0,\ \{\alpha_n\} \subset (0,1), 0 < \delta < \frac{\mu}{2}, p_1 = \frac{Tv_1 - v_1}{\lambda}$ *and* $\{\beta_n\} \subset [0, \bar{\beta}_n],\ \forall n \geq 1$, *such that* $\bar{\beta}_n$ *is obtained by* (3.66). *Then, the sequence* $\{v_n\}$ *converges strongly to* $v^* \in \Omega = F(T) \cap \mathcal{B}^{-1}(0)$, *provided that Assumption* 1 *and the following the conditions (E*1*) and (E*2*) hold.*

4. Numerical example and applications

4.1. **Numerical example.**

Example 4.1. Consider $\mathcal{X} = \mathbb{R}$. Let $\mathcal{A} : \mathcal{X} \to \mathcal{X},\ \eta : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $\alpha : \mathcal{X} \to \mathbb{R}$ be defined by

$$\mathcal{A}(v) = -t^2 v,\ \eta(v, y) = -c(v - y),\ \alpha(v) = v^2,\ \forall v, y \in \mathcal{X},$$

then, one sees that $\mathcal{A}$ is η - hemicontinuous and relaxed $\eta - \alpha$ monotone with $t,\ c > 0$ and satisfies the conditions $(i) - (v)$ of Lemma 2.16 and condition (vi) of the Theorem 3.1. Define a bifunction $\xi : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ and a function $g : \mathcal{X} \to \mathbb{R}$ by $\xi(v, y) = 8y^2 + 2vy - 10v^2$ and $g(v) = v^2,\ \forall v, y \in \mathcal{X}$. Then, we observe that ξ and g satisfy all the stated conditions in the Theorem 3.1 and $0 \in GMEP(\xi, \mathcal{A}, g)$. Therefore, by Lemma 2.16, we obtain that for any $v \in \mathcal{X},\ r_n > 0,\ T_{r_n} v$ is nonempty and single-valued. Following similar arguments as in the example in [30], we get that

$$T_{r_n} v = \frac{2v}{2 + 40r_n + 2t^2 c r_n}.$$

For each $i = 1, 2, \ldots$, let $B_i : \mathcal{X} \to \mathcal{X}$ and $T_i : \mathcal{X} \to \mathcal{X}$ be defined by $B_i(v) = 3v$ and $T_i(v) = \frac{5}{8}v$ and $v, \forall v \in \mathcal{X}$. Then, it respectively follows from the Lemma 2.4 and Lemma 2.12 that $B(v) = 3v$ is $\frac{1}{3}$-inverse strongly monotone and $T(v) = \sum_{i=1}^{\infty} \delta_i(\tau_i v + (1 - \tau_i) T_i v)$ is relatively nonexpansive with $0 \in F(T) \cap \mathcal{B}^{-1}(0)$. The performance of our new devised algorithms (3.62) and (3.93) are compared with the algorithms of Adamu et al. [2] and Cholamjiak et al. [19] with $A(v) = 2v,\ \forall v \in \mathcal{X}$. For the implementation of all the algorithms, the initial values v_0 and v_1 are randomly chosen in four different cases; Case 1: $v_0 = v_1 = rand(1,1)$, Case 2: $v_0 = v_1 = 10rand(1,1)$, Case 3: $v_0 = 10rand(1,1),\ v_1 = rand(1,1)$ and Case 4: $v_0 = rand(1,1),\ v_1 = 10rand(1,1)$ and applied $||v_{n+1} - v_n|| \leq 10^{-6}$ as the stopping criteria with 300 as the maximum number of iterations. We represent the execution time in seconds by "Time", the number of iterations by "Iter." and set $p_1 = \frac{Tv_1 - v_1}{\lambda}$ and the following for the parameters;

• In the Algorithm (3.62), we set $\lambda = 5,\ \delta = 0.02,\ c = 10,\ t = 80,\ \beta_n = \frac{1}{(n+1)^{20}}$, $\alpha_n = \frac{1}{10n^2+1},\ r_n = \frac{2n}{3n+1}$ and $\theta_n = \frac{3n}{n^3+10}$.

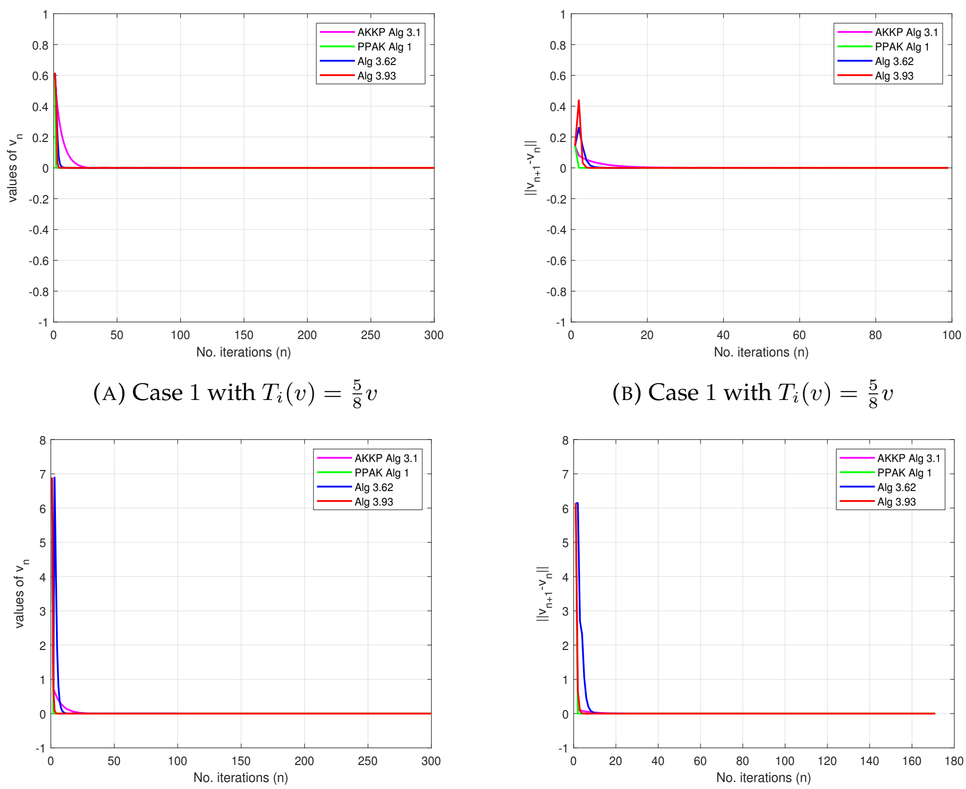
• In the Algorithm (3.93), we set $\lambda = 5,\ \delta = 0.3,\ \eta = 2,\ \beta = 0.01,\ \alpha_n = \frac{1}{10n^2+1}$, $\vartheta_n = \frac{1}{(n+1)^{100}},\ \varsigma_n = \frac{1}{10n^2+1}\ \sigma_n = \frac{1}{50000n+1}$ and $u = 0.5v_1$.

• In the Algorithm of Adamu et al. [2] (Abbreviated as AKKP Alg 3.1), the chosen parameters were adapted from [2].

• In the Algorithm of Cholamjiak et al. [19] (Abbreviated as PPAK Alg 1), we set $\lambda_n = 0.02,\ \beta_n = 0.999,\ \sigma_n = \frac{1}{50000n+1}$ and $u = 0.5v_1$.

TABLE 1. Numerical results of all algorithms under different initial values and mappings in Example 4.1

$T_i(v)$	Algorithms	Case 1		Case 2		Case 3		Case 4	
		Iter.	Time	Iter.	Time	Iter.	Time	Iter	Time
$T_i(v) = \frac{5}{8}v$	Alg 3.62	17	0.3525	21	0.4745	23	0.2481	2	0.0497
	Alg 3.93	6	0.0872	7	0.1822	6	0.0663	7	0.1220
	AKKP Alg 3.1	98	0.1847	106	0.2161	94	0.0942	80	0.1806
	PPAK Alg 1	58	0.1103	124	0.2607	138	0.1521	63	0.1383
$T_i(v) = v$	Our Alg 3.62	21	0.0181	24	0.0278	23	0.0253	2	0.0227
	Our Alg 3.93	7	1.24e-04	9	0.0068	9	0.0081	9	1.76e-04
	AKKP Alg 3.1	98	7.44e-04	98	0.0121	89	0.0085	83	4.24e-04
	PPAK Alg 1	69	1.62e-04	87	0.0078	170	0.0088	55	1.82e-04



(A) Case 1 with $T_i(v) = \frac{5}{8}v$

(B) Case 1 with $T_i(v) = \frac{5}{8}v$

(C) Case 3 with $T_i(v) = v$

(D) Case 3 with $T_i(v) = v$

FIGURE 1. Computational results of all algorithms for $T_i(v) = \frac{5}{8}v$ and $T_i(v) = v$ in Example 4.1.

Remark 4.8. Based on the numerical results reported in Table 1 and Figure 1, we observe that Algorithms 3.62 and 3.93 require fewer iterations to reach the stopping condition than the algorithms of Adamu et al. [2] and that of Cholamjiak et al. [19]. Moreover, Algorithm 3.93 provides smaller execution time than all the algorithms.

4.2. **Applications.**

4.2.1. *Approximating a common minimizer of continuously Fréchet differentiable convex functionals.*

Lemma 4.19. (see, [36]) *Let $f : \mathcal{X} \to \mathbb{R}$ be a continuously Fréchet differentiable, convex functional on a Banach space $\mathcal{X}$ and ∇f represents the gradient of f. If ∇f is $\frac{1}{\mu}$ - Lipschitz continuous, then ∇f is μ - inverse strongly monotone.*

To approximate a minimizer of f, the following conditions are assumed to be satisfied;

(1) f is a continuously Fréchet differentiable, convex functional on $\mathcal{X}$ and ∇f is $\frac{1}{\mu}$ - Lipschitz continuous;

(2) $\Phi = \underset{x \in \mathcal{X}}{argmin} f(u) = \{u^* \in S : f(u^*) = \min_{x \in X} f(u)\} \neq \emptyset.$

Suppose that for each $i = 1, 2, \dots,$ $f_i : \mathcal{X} \to \mathbb{R}$ satisfies the conditions $(1) - (2)$. Then, setting $\mathcal{B}_i = \nabla f_i$ for each $i = 1, 2, \dots$ in Algorithms 3.62 and 3.93 and assuming that $\bigcap_{i=1}^{\infty} F(T_i) \cap \bigcap_{i=1}^{\infty} \Phi_i \cap GMEP(\xi, \mathcal{A}, g) \neq \emptyset$. Suppose that $\{v_i\}$ is a sequence in $(0, 1)$ such that $\sum_{i=1}^{\infty} v_i = 1$ and define $\mathcal{B} : \mathcal{X} \to \mathbb{R}$ by $\mathcal{B}u = \sum_{i=1}^{\infty} v_i \mathcal{B}_i u$ for each $u \in \mathcal{X}$, then it follows from the Theorems 3.1 and 3.2 that $\{v_n\}$ converges strongly to $v^* \in \Omega_1 = F(T) \cap \mathcal{B}^{-1}(0) \cap GMEP(\xi, \mathcal{A}, g)$ and $v^* \in \Omega_2 = F(T) \cap B^{-1}(0)$ respectively.

4.2.2. *Image restoration.*

Example 4.2. In this example, we apply our Algorithm 3.93 to solve image restoration problem, which is concern with the reconstruction of an image degraded by blur and additive noise. Generally, the problem can be modeled as the following linear equation

$$b = Fv + \sigma, \tag{4.97}$$

where $v \in \mathbb{R}^N$ is the original image, $b \in \mathbb{R}^M$ is the degraded image with noise σ and $F : \mathbb{R}^N \to \mathbb{R}^M$; $(M < N)$ is a bounded linear operator known as the blurring operator. We use ℓ_1 - regularization problem to solve (4.97), which can be modeled and viewed as solving the following LASSO problem [27]

$$\min_{v \in \mathbb{R}^N} \{\tau ||v||_1 + \frac{1}{2} ||b - Fv||_2^2\}, \tag{4.98}$$

where $\tau > 0$ is the balancing parameter. In view of the results in [29], we have $\nabla \Big(\frac{1}{2} ||Fv - b||_2^2 \Big) = F^T(Fv - b)$.

In the experiments, we compare our Algorithm 3.93 and some relevant existing algorithms in [2, 19, 49], in solving problem (4.97) and the test images of Abdul (344×258), Abubakar (258×258), Lena (320×320), Barbara (320×320), MATLAB blur function "$fspecial('motion', 20,\ 30)$" and added random noise are used. The stopping criteria of

10^{-5} and the maximum number of iterations $N = 3000$ are used for all the algorithms. The number of iterations is denoted by ”Iter.” and the execution time is represented in seconds by ”Time”. We set the following for the implementations.

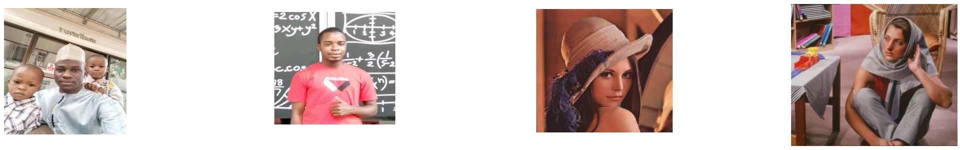
• In our Algorithm (3.93), we set $\mathcal{B}v = \nabla\Big(\frac{1}{2}||Fv - b||_2^2\Big),\ Tv = \frac{nv}{n+1},\ \theta_n = 0.95,\ \sigma_n = \frac{1}{10^3 n+1},\ \alpha_n = \frac{1}{n^7},\ \vartheta_n = \frac{1}{(n+1)^{10}},\ \beta_n = \frac{1}{(10n+1)^{100}},\ \varsigma_n = \frac{1}{(n+1)^2},\ \lambda = 100$ and $\eta = 5$.

• In the Algorithms of Adamu et al. [2] (Abbreviated as AKKP Alg 3.1), Cholamjiak et al. [19] (Abbreviated as PPAK Alg 1), and Shehu [49] (Abbreviated as Shehu Alg 3.3), we set $Av = \nabla\Big(\frac{1}{2}||Fv - b||_2^2\Big),\ Tv = \frac{nv}{n+1},\ Bv = \partial(\tau||v||_1)$ and the chosen parameters were adapted from [2, 19]. In particular, we select $\lambda_n = 0.11$ for the Shehu Alg 3.3.

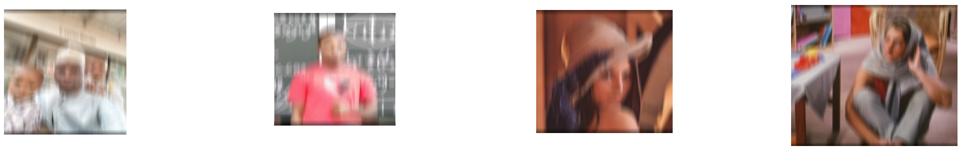
In Figure 2, the original, degraded and restored images by all the agorithms are presented.

TABLE 2. Computational results of all the algorithms in example 4.2

Figure	Algorithms	Iter.	Time	SNR	PSNR
Abdul	Alg 3.93	1779	43.7959	52.1880	28.3388
	Shehu Alg 3.3	2740	97.0222	49.0335	26.1896
	PPAK Alg 1	3000	104.6158	42.4158	24.0568
	AKKP Alg 3.1	2794	90.4897	41.1098	23.4103
Abubakar	Alg 3.93	2282	40.0348	47.5674	26.8269
	Shehu Alg 3.3	3000	61.0223	43.6486	25.0520
	PPAK Alg 1	3000	58.2490	36.6962	21.9048
	AKKP Alg 3.1	3000	60.9556	35.5236	21.3199
Lena	Alg 3.93	1720	46.9639	53.1598	32.4825
	Shehu Alg 3.3	2346	91.6060	51.5655	32.5016
	PPAK Alg 1	2965	95.5873	43.7120	29.3881
	AKKP Alg 3.1	2720	85.7591	42.7090	28.9011
Barbara	Alg 3.93	1840	127.8623	52.8909	31.4123
	Shehu Alg 3.3	2639	263.8614	50.0878	30.9593
	PPAK Alg 1	3000	403.2466	43.6449	28.1606
	AKKP Alg 3.1	2885	317.0611	42.4174	27.5585



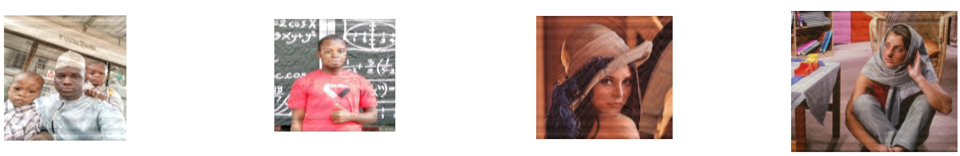
(A) Original Images



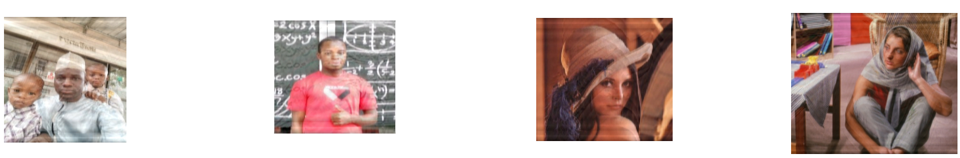
(B) Test images degraded by motion blur and random noise

(C) Degraded images in (B) restored with Alg 3.93

(D) Degraded images in (B) restored with Shehu Alg 3.3



(E) Degraded images in (B) restored with PPAK Alg 1



(F) Degraded images in (B) restored with AKKP Alg 3.1

FIGURE 2. Degradation of the test images and their restorations using Alg 3.93, Shehu Alg 3.3, PPAK Alg 1 and AKKP Alg 3.1

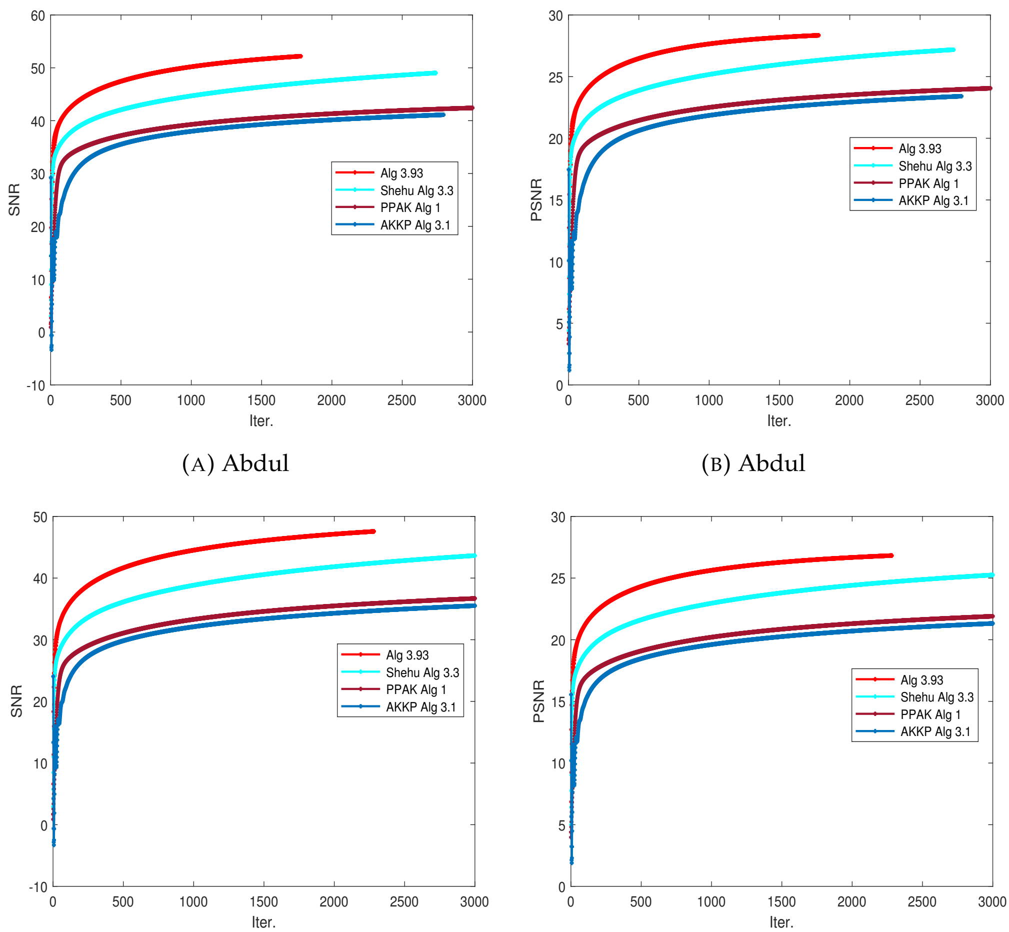
One observes that it is not easy to identify which of the algorithms has better performance in the restoration process from the Figure 2. For that purpose, to measure the quality of restored image by each algorithm, we apply the two different tools, SNR known as signal-to-noise ratio and $PSNR$ called peak signal-to-noise ratio, respectively defined by

$$SNR = 20 \times \log_{10}\left(\frac{||v||_2^2}{||v - v^*||_2^2}\right) \quad \text{and} \quad PSNR = 20 \times \log_{10}\left(\frac{\text{MAX}_I}{\sqrt{MSE}}\right), \tag{4.99}$$

where MAX_I denotes the image's possible maximum pixel value and MSE represents the mean square error, which is computed by

$$MSE = \frac{1}{N}\sum\sum (v - v^*)^2, \tag{4.100}$$

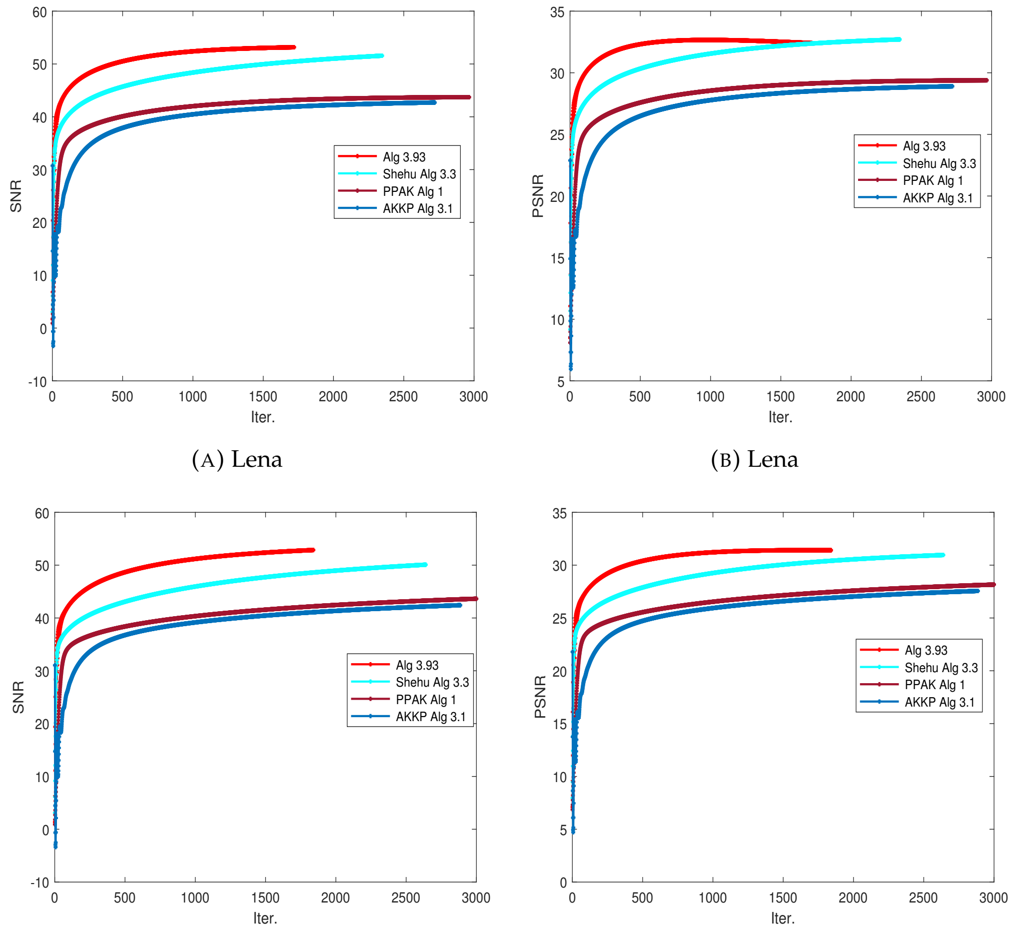
where N is the image size and v and v^* are respectively the original and the restored images. The larger the SNR or $PSNR$, the better the quality of the restored image. The results are reported in Figure 3, 4 and Table 2.



(A) Abdul (B) Abdul

(C) Abubakar (D) Abubakar

FIGURE 3. Graphs of SNR and PSNR for Abdul and Abubakar images restored via all the algorithms .



(A) Lena (B) Lena

(C) Barbara (D) Barbara

FIGURE 4. SNR and PSNR graphs for Lena and Barbara images restored via all the algorithms .

Remark 4.9. In view of the numerical results of Example 4.2 in Figure 3, Figure 4 and Table 2 for the restored images in Figure 2, the advantages and computational efficiency of our proposed Algorithm 3.93 in solving the problem in Example 4.2, over the existing methods considered in the experiments, which include the Shehu Alg 3.3 in [49], PPAK Alg 1 in [19] and AKKP Alg 3.1 in [2] are shown. More specifically, in its fewer iterations and execution time requirements to reach the stipulated tolerance with the largest value of SNR and $PSNR$ for all the test images than the methods in [2, 19, 49]. In particular, PPAK Alg 1 was only abled to restore the Lena Image in the whole experiments before the exhaustion of the maximum number of iterations. Similarly, Abubakar Image was also not abled to be restored by Shehu Alg 3.3 and AKKP Alg 3.1 before reaching the maximum number of iterations.

5. Conclusions

This paper presents efficient and accelerated inertial algorithms with conjugate gradient - like direction for solutions of a generalized mixed equilibrium problem with relaxed monotone mapping and zeros of a countable family of inverse strongly monotone mappings, that are fixed points of a family of relatively nonexpansive mappings. Applications of the theorems for a common minimizer of a countable family of smooth and convex functions are considered. Moreover, the numerical performance of our proposed methods in solving image restoration problems and numerical example are compared with that of algorithms in [2, 19, 49]. The numerical results justify the advantages and computational efficiency of our proposed algorithms over those in [2, 19, 49].

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