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Modified General Inertial Mann and General Inertial Viscosity Algorithms for Fixed Point and Common Fixed Point Problems with Applications

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ABSTRACT. In this paper, we propose a modified general inertial Mann algorithm and prove that it generates a sequence which converges weakly to a fixed point of a nonexpansive mapping in Hilbert spaces. Moreover, by using the viscosity method, we introduce a general inertial viscosity algorithm and prove that it generates a sequence which converges strongly to a common fixed point of a countable family of nonexpansive operators. We also derive schemes for solving constrained convex optimization, monotone inclusion, and nonsmooth convex optimization problems. Finally, we apply one of our proposed algorithms to solve image restoration problem.

1. INTRODUCTION

Let $\mathcal{H}$ be a real Hilbert space and D be a nonempty closed convex subset of $\mathcal{H}$. A self mapping S on $\mathcal{H}$ is said to be nonexpansive if

$$\|Sy - Sz\| \leq \|y - z\|,$$

for all $y, z \in D$. The set of fixed points of the mapping $S : D \to D$ is defined by $F(S) = \{f \in D : Sf = f\}$. This paper considers fixed point problem for a nonexpansive operator and common fixed point problem for a countable family of nonexpansive operators.

One of the most extensively studied iterative algorithm for approximating fixed points of nonexpansive mappings is Mann algorithm [23] which is formulated as follows:

$$t_{n+1} = \theta_n t_n + (1 - \theta_n) S t_n, \tag{1.1}$$

where $\{\theta_n\} \subset [0, 1]$, $\lim_{n\to\infty} \theta_n = 0$, and $\sum_{n=1}^{\infty} \theta_n = \infty$.

Due to the fact that fast convergence is needed in many practical applications and Mann algorithm is slow in general (see, [11, 18, 19, 24]), many researchers modified the Mann algorithm and incorporated inertial extrapolation methods to speed up its convergence (see [1, 6, 10, 12, 21, 22, 29–31]). One of such methods is the general inertial Mann algorithm [14] which is of the form:

$$\begin{cases} w_n = t_n + \gamma_n (t_n - t_{n-1}) \\ z_n = t_n + \theta_n (t_n - t_{n-1}) \\ t_{n+1} = (1 - \zeta_n) w_n + \zeta_n S z_n, \end{cases} \tag{1.2}$$

for each $n \geq 1$, where $\{\gamma_n\}$, $\{\theta_n\}$, and $\{\zeta_n\}$ satisfy:
(D1) $\{\gamma_n\} \subset [0, \gamma]$ and $\{\theta_n\} \subset [0, \theta]$ are nondecreasing with $\gamma_1 = \theta_1 = 0$ and $\gamma, \theta \in [0, 1)$;
(D2) for any $\zeta, \alpha, \beta > 0$, $\beta > \dfrac{\gamma\tau(1+\tau) + \gamma\alpha}{1 - \gamma^2}$ and $0 < \zeta \leq \zeta_n \leq \dfrac{\beta - \gamma[\tau(1+\tau) + \gamma\beta + \alpha]}{\beta[1 + \tau(1+\tau) + \gamma\beta + \alpha]}$ where $\tau = \max\{\gamma, \theta\}$.

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Since the iterative sequence $\{t_n\}$ defined by (1.1) has only weak convergence and strong convergence is often much more desirable than the weak convergence in many practical problems, Moudafi [25] introduced viscosity approximation method which is shown below:

$$t_{n+1} = \delta_n v(t_n) + (1 - \delta_n) S t_n, \tag{1.3}$$

$n \in \mathbb{N}, t_1 \in \mathcal{H}$, $\delta_n \subset (0, 1)$, and v is a contraction operator. After that, several algorithms for fixed points of nonexpansive operators and common fixed points of a countable family of nonexpansive operators were developed (see [3,16,17,28,32–34]).

It is the purpose of this paper to introduce a modified general inertial Mann algorithm which generalizes the work of Dong et al. [14] by using inertial extrapolation algorithms mixed with the convex combination of three iterated vectors. Moreover, we prove strong convergence theorem by combining the general inertial Mann algorithm with the viscosity method for a countable family of nonexpansive mappings.

The paper is organized as follows: Section 2 presents some lemmas and definitions which are necessary in the proofs of our theorems. Section 3 establishes weak and strong convergence theorems. Section 4 shows applications of the proposed algorithms. Finally, section 5 gives some concluding remarks.

2. Preliminaries

Now, we review some definitions and lemmas which will be used in the sequel.

Lemma 2.1. *Let $\mathcal{H}$ be a real Hilbert space. Then for all $r, q, p \in \mathcal{H}$, we have:*

(1) $\|\theta q + (1 - \theta)\|^2 = \theta\|q\|^2 + (1 - \theta)\|p\|^2 - \theta(1 - \theta)\|q - p\|^2, \theta \in [0, 1]$;
(2) $\|q \pm p\|^2 = \|q\|^2 \pm 2\langle q, p\rangle + \|p\|^2$;
(3) $\|q + p\|^2 \leq \|q|^2 + 2\langle p, q + p\rangle$;
(4) $\|\beta r + \eta q + \zeta p\|^2 = \beta\|r\|^2 + \eta\|q\|^2 + \zeta\|p\|^2 - \beta\eta\|r - q\|^2 - \beta\zeta\|r - p\|^2 - \eta\zeta\|q - p\|^2$, $\beta, \eta, \zeta \in [0, 1)$ *such that* $\beta + \eta + \zeta = 1$.

Lemma 2.2. *[2] Let $\{\tau_n\}, \{\theta_n\}, \{\zeta_n\} \subset [0, \infty)$. If $\sum_{n=1}^{\infty} \zeta_n < \infty$, there exists a real number θ with $0 \leq \theta_n \leq \theta < 1$ for all $n \in \mathbb{N}$, and $\tau_{n+1} \leq \tau_n + \theta_n(\tau_n - \tau_{n-1}) + \zeta_n$ for each $n \geq 1$, then*

(1) $\sum_{n \geq 1}[\tau_n - \tau_{n-1}]_+ < \infty$, *where* $[t]_+ = max\{t, 0\}$;
(2) *there exists* $\tau^* \in [0, \infty)$ *such that* $\lim_{n\to\infty} \tau_n = \tau^*$.

Lemma 2.3. *[5] Let D be a nonempty closed convex subset of $\mathcal{H}$ and $S : D \to \mathcal{H}$ be a nonexpansive mapping. Let $\{y_n\}$ be a sequence in D such that $y_n \rightharpoonup y \in \mathcal{H}$ and $Sy_n - y_n \to 0$ as $n \to \infty$. Then $y \in F(S)$.*

Lemma 2.4. *[5] Let E be a nonempty subset of $\mathcal{H}$ and $\{z_n\}$ be a sequence in $\mathcal{H}$. If for all $z \in E$, $\lim_{n\to\infty} \|z_n - z\|$ exists and every sequential weak cluster point of $\{z_n\}$ is in E, then the sequence $\{z_n\}$ converges weakly to a point in E.*

Lemma 2.5. *[27] Let $\{s_k\} \subset [0, \infty)$, $\{t_k\} \subset (-\infty, \infty)$, and $\{u_k\} \subset (0, 1)$ satisfying $\sum_{n=1}^{\infty} u_k = \infty$ and $s_{k+1} \leq (1 - u_k)s_k + u_k t_k$, $k \in \mathbb{N}$. If $\limsup_{l\to\infty} t_{k_l} \leq 0$ and for every subsequence $\{k_l\}$ of $\{k\}$, $\liminf_{l\to\infty}(s_{k_l+1} - s_{k_l}) \geq 0$, then $\lim_{k\to\infty} s_k = 0$.*

Definition 2.1. [4] Let D be a nonempty closed convex subset of $\mathcal{H}$ and $\{S_m\}$ be a sequence of nonexpnsive operators such that $S_m : D \to D$ for $m \geq 1$. Suppose that for every bounded sequence $\{t_m\}$ in D, $\lim_{m\to\infty} \|t_m - S_m t_m\| = 0$ implies that every cluster point of $\{t_m\}$ belongs to $\Omega := \cap_{m=1}^{\infty} F(S_m)$, then $\{S_m\}$ is said to satisfy condition (Z).

Definition 2.2. Let D be a nonempty closed convex subset of $\mathcal{H}$. The projection from $\mathcal{H}$ onto D, denoted by P_D, is defined in such a way that, for every $b \in \mathcal{H}$, $P_D b$ is the unique point in D such that

$$\|b - P_D b\| = \min\{\|b - c\| : c \in D\}.$$

Lemma 2.6. *Let D be a nonempty closed convex subset of $\mathcal{H}$. Then*

$$\langle b - P_D b, c - P_D b\rangle \leq 0, \tag{2.4}$$

for all $b \in \mathcal{H}$ and $c \in D$.

3. Main results

3.1. **Modified General Inertial Mann Algorithm for Nonexpansive Mappings.** In this section, we study the weak convergence of the Modified General Inertial Mann Algorithm (MGIM, for short) for nonexpansive mappings under the conditions $(E1)$ and $(E2)$ stated below.

Algorithm 1: Modified General Inertial Mann Algorithm (MGIM)

Initialization:Take $t_0, t_1 \in \mathcal{H}$ arbitrarily and the followig conditions hold:

$(E1)$ $\{\theta_n\} \subset [0, \theta]$, $\{\phi_n\} \subset [0, \phi]$, and $\{\gamma_n\} \subset [0, \gamma]$ are nondecreasing with $\theta_1 = \phi_1 = \gamma_1 = 0$ and $\theta, \phi, \gamma \in [0, 1)$ and $\{a_n\}, \{b_n\}, \{c_n\} \subset [0, 1)$ such that $a_n + b_n + c_n = 1$;

$(E2)$ for any $c, \xi, \lambda > 0$,

$$\lambda > \frac{\max\{\theta,\phi\} - \min\{\theta,\phi\} + \max\{\theta,\phi\}[\kappa(1+\kappa) + \xi]}{1 - (\max\{\theta,\phi\})^2},$$

$$0 < c \leq c_n \leq \frac{\min\{\theta,\phi\} - \max\{\theta,\phi\} + \lambda - \max\{\theta,\phi\}[\kappa(1+\kappa) + \lambda\max\{\theta,\phi\} + \xi]}{\lambda[1 + \kappa(1+\kappa) + \lambda\max\{\theta,\phi\} + \xi]},$$

where $\kappa = \max\{\theta, \phi, \gamma\}$.

Iterative Steps: Calculate t_{n+1} as follows:

$$\begin{cases} w_n = t_n + \theta_n(t_n - t_{n-1}) \\ y_n = t_n + \phi_n(t_n - t_{n-1}) \\ z_n = t_n + \gamma_n(t_n - t_{n-1}) \\ t_{n+1} = a_n w_n + b_n y_n + c_n S z_n, \end{cases} \tag{3.5}$$

Theorem 3.1. *Let $S : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping and assume that $F(S) \neq \emptyset$. Then the sequence $\{t_n\}$ generated by* ***Algorithm 1*** *converges weakly to a point of $F(S)$.*

Proof. Pick $f \in F(S)$.
From (3.5), it follows that

$$\begin{aligned} \|t_{n+1} - f\|^2 =& \|a_n(w_n - f) + b_n(y_n - f) + c_n(Sz_n - f)\|^2 \\ =& a_n\|w_n - f\|^2 + b_n\|y_n - f\|^2 + c_n\|Sz_n - f\|^2 - a_nb_n\|w_n - y_n\|^2 - \\ & a_nc_n\|Sz_n - w_n\|^2 - b_nc_n\|Sz_n - y_n\|^2 \\ \leq& a_n\|w_n - f\|^2 + b_n\|y_n - f\|^2 + c_n\|z_n - f\|^2 - \\ & a_nb_n(\theta_n - \phi_n)^2\|t_n - t_{n-1}\|^2 - a_nc_n\|Sz_n - w_n\|^2 - \\ & b_nc_n\|Sz_n - y_n\|^2 \end{aligned} \tag{3.6}$$

Again using (3.5), we get

$$\begin{aligned} \|w_n - f\|^2 =& \|(1+\theta_n)t_n - \theta_n(t_{n-1} - f)\|^2 \\ =& (1+\theta_n)\|t_n - f\|^2 - \theta_n\|t_{n-1} - f\|^2 + \theta_n(1+\theta_n)\|t_n - t_{n-1}\|^2. \end{aligned} \tag{3.7}$$

Similarly, we have

$$\|y_n - f\|^2 = (1+\phi_n)\|t_n - f\|^2 - \phi_n\|t_{n-1} - f\|^2 + \phi_n(1+\phi_n)\|t_n - t_{n-1}\|^2 \tag{3.8}$$

and

$$\|z_n - f\|^2 = (1+\gamma_n)\|t_n - f\|^2 - \gamma_n\|t_{n-1} - f\|^2 + \gamma_n(1+\gamma_n)\|t_n - t_{n-1}\|^2 \tag{3.9}$$

Substituting (3.7), (3.8), and (3.9) into (3.6), we get

$$\begin{aligned}\|t_{n+1} - f\|^2 - (1+\Omega_n)\|t_n - f\|^2 + \Omega_n\|t_{n-1} - f\|^2 \leq & - a_n c_n\|Sz_n - w_n\|^2 - \\ & b_n c_n\|Sz_n - y_n\|^2 + \\ & (\Omega_n + \Psi_n)\|t_n - t_{n-1}\|^2,\end{aligned} \tag{3.10}$$

where $\Omega_n = a_n\theta_n + b_n\phi_n + c_n\gamma_n$ and $\Psi_n = a_n\theta_n^2 + b_n\phi_n^2 + c_n\gamma_n^2 - a_n b_n(\theta_n - \phi_n)^2$. From $(E1)$, $(E2)$, and $\kappa = \max\{\theta, \phi, \gamma\}$, it follows that $\Omega_n \subset [0, \kappa]$ is nondecreasing with $\Omega_1 = 0$.

Again from (3.5), we get

$$\begin{aligned}\|Sz_n - y_n\|^2 =& \Big\| \frac{1}{c_n}\Big(t_{n+1} - t_n\Big) + \frac{d_n}{c_n}\Big(t_{n-1} - t_n\Big)\Big\|^2 \\ =& \frac{1}{c_n^2}\|t_{n+1} - t_n\|^2 + \frac{d_n^2}{c_n^2}\|t_{n-1} - t_n\|^2 + \\ & 2\frac{d_n}{c_n^2}\langle t_{n+1} - t_n, t_{n-1} - t_n\rangle \\ \geq & \frac{1}{c_n^2}\|t_{n+1} - t_n\|^2 + \frac{d_n^2}{c_n^2}\|t_{n-1} - t_n\|^2 + \\ & \frac{d_n}{c_n^2}\Big(-\nu_n\|t_{n+1} - t_n\|^2 - \frac{1}{\nu_n}\|t_{n-1} - t_n\|^2\Big),\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\|Sz_n - w_n\|^2 =& \Big\| \frac{1}{c_n}\Big(t_{n+1} - t_n\Big) + \frac{e_n}{c_n}\Big(t_{n-1} - t_n\Big)\Big\|^2 \\ \geq & \frac{1}{c_n^2}\|t_{n+1} - t_n\|^2 + \frac{e_n^2}{\psi_n^2}\|t_{n-1} - t_n\|^2 + \\ & \frac{e_n}{c_n^2}\Big(-\nu_n\|t_{n+1} - t_n\|^2 - \frac{1}{\nu_n}\|t_{n-1} - t_n\|^2\Big),\end{aligned} \tag{3.12}$$

where $d_n = a_n(\theta_n - \phi_n) + \phi_n, e_n = b_n(\phi_n - \theta_n) + \theta_n$, and $\nu_n = \dfrac{1}{\min\{d_n, e_n\} + \lambda c_n}$.

Now, substituting (3.11) and (3.12) into (3.10), we get

$$\begin{aligned}\|t_{n+1} - f\|^2 - (1+\Omega_n)\|t_n - f\|^2 + \Omega_n\|t_{n-1} - f\|^2 \leq & \zeta_n\|t_{n-1} - t_n\|^2 + \\ & \frac{\eta_n}{c_n}\|t_{n+1} - t_n\|^2,\end{aligned} \tag{3.13}$$

where

$$\zeta_n = \Omega_n + \Psi_n + \frac{b_n d_n(1 - \nu_n d_n)}{\nu_n c_n} + \frac{a_n e_n(1 - \nu_n e_n)}{\nu_n c_n} \geq 0$$

and

$$\eta_n = a_n(\nu_n e_n - 1) + b_n(\nu_n d_n - 1).$$

Considering two cases for $\min\{d_n, e_n\}$, we can verify that for all $n \geq 1$

$$\zeta_n \leq \Omega_n + \Psi_n + \max\{d_n, e_n\}(1 - c_n)\lambda, \tag{3.14}$$

where $\lambda = \dfrac{1-\nu_n \min\{d_n, e_n\}}{\nu_n c_n}$.
Similarly, we can show that $\eta_n \leq 0$ for all $n \geq 1$ by taking into account the condition for λ.
Let

$$\zeta_n' := \Omega_n + \Psi_n + \max\{d_n, e_n\}(1-c_n)\lambda.$$

In view of (3.14), (3.13) becomes

$$\|t_{n+1} - f\|^2 - (1+\Omega_n)\|t_n - f\|^2 + \Omega_n \|t_{n-1} - f\|^2 \leq \zeta_n' \|t_n - t_{n-1}\|^2 + \frac{\eta_n}{c_n}\|t_{n+1} - t_n\|^2. \tag{3.15}$$

Moreover, we have

$$\zeta_n' \leq \kappa(1+\kappa) + \lambda \max\{\theta, \phi\}. \tag{3.16}$$

Next, we show that

$$\sum_{n=1}^{\infty} \|t_{n+1} - t_n\|^2 < \infty,$$

by adapting some techniques from [1, 7]. To do so, first, we let $\sigma_n = \|t_n - f\|^2$ and $\tau_n := \sigma_n - \Omega_n \sigma_{n-1} + \zeta_n' \|t_n - t_{n-1}\|^2$, for all $n \geq 1$.
Now, using the fact that $\{\Omega_n\}$ is monotone and $\sigma_n \geq 0$ for all $n \in \mathbb{N}$, we obtain

$$\tau_{n+1} - \tau_n = \sigma_{n+1} - (1+\Omega_n)\sigma_n + \Omega_n \sigma_{n-1} + \zeta_{n+1}' \|t_{n+1} - t_n\|^2 - \zeta_n' \|t_n - t_{n-1}\|^2. \tag{3.17}$$

By (3.15), we have

$$\sigma_{n+1} - (1+\Omega_n)\sigma_n + \Omega_n \sigma_{n-1} - \zeta_n' \|t_n - t_{n-1}\|^2 \leq \frac{\eta_n}{c_n}\|t_{n+1} - t_n\|^2. \tag{3.18}$$

Combining (3.17) and (3.18), we get

$$\tau_{n+1} - \tau_n \leq \left(\frac{\eta_n}{c_n} + \zeta_{n+1}'\right)\|t_{n+1} - t_n\|^2. \tag{3.19}$$

Now, we claim that

$$\frac{\eta_n}{c_n} + \zeta_{n+1}' \leq -\xi, \tag{3.20}$$

for each $n \in \mathbb{N}$.
After some manipulations, the bove claim is the same as:

$$\max\{\theta, \phi\} - \min\{\theta, \phi\} - \lambda + \max\{\theta, \phi\}(\zeta_{n+1}' + \xi) + \lambda c_n (1 + \zeta_{n+1}' + \xi) \leq 0.$$

This claim can be verified easily by using the upper bounds of ζ_{n+1}' and c_n.
Now, (3.19) becomes

$$\tau_{n+1} - \tau_n \leq -\xi \|t_{n+1} - t_n\|^2. \tag{3.21}$$

Taking in to account that $\{\tau_n\}_{n\geq 1}$ is nonincreasing and $\Omega_n \in [0, \kappa]$, we get

$$-\kappa\sigma_{n-1} \leq \sigma_n - \kappa\sigma_{n-1} \leq \tau_n \leq \tau_1, \tag{3.22}$$

for each $n \geq 1$.
From (3.22), we get

$$\sigma_n \leq \kappa^n \sigma_0 + \tau_1 \sum_{m=1}^{n-1} \kappa^m \leq \kappa^n \sigma_0 + \frac{\tau_1}{1-\kappa}, \tag{3.23}$$

for each $n \geq 1$.
Now, using (3.21), (3.22),and (3.23), we get

$$\xi \sum_{k=1}^{\infty} \|t_{n+1} - t_n\|^2 \leq \tau_1 - \tau_{n+1} \leq \tau_1 + \kappa\sigma_n \leq \kappa^{n+1}\sigma_0 + \frac{\kappa\tau_1}{1-\kappa}, \tag{3.24}$$

which implies

$$\sum_{k=1}^{\infty} \|t_{n+1} - t_n\|^2 < \infty. \tag{3.25}$$

Thus, we have

$$\lim_{n\to\infty} \|t_{n+1} - t_n\| = 0. \tag{3.26}$$

From (3.5) and (3.26), we see that

$$\|y_n - t_{n+1}\| \leq \|t_n - t_{n+1}\| + \theta_n \|t_n - t_{n-1}\|,$$

which in turn implies that

$$\lim_{n\to\infty} \|y_n - t_{n+1}\| = 0. \tag{3.27}$$

Similarly, we have

$$\lim_{n\to\infty} \|z_n - t_{n+1}\| = 0 \tag{3.28}$$

and

$$\lim_{n\to\infty} \|w_n - t_{n+1}\| = 0. \tag{3.29}$$

Now, using (3.5), (3.26) and (3.27), we get

$$\begin{aligned} \|Sz_n - y_n\| \leq& \frac{a_n|\theta_n - \gamma_n|}{c_n}\|t_n - t_{n-1}\| + \frac{1}{c_n}\|t_{n+1} - y_n\| \\ \leq& \frac{1}{c}\left(\|t_n - t_{n-1}\| + \|t_{n+1} - y_n\|\right), \end{aligned} \tag{3.30}$$

which implies that

$$\lim_{n\to\infty} \|Sz_n - y_n\| = 0. \tag{3.31}$$

Using (3.5), (3.26) and (3.31), we get

$$\|Sz_n - z_n\| \leq \|Sz_n - y_n\| + \|y_n - z_n\| \leq \|Sz_n - y_n\| + |\phi_n - \gamma_n|\|t_n - t_{n-1}\|, \tag{3.32}$$

which implies that

$$\lim_{n\to\infty} \|Sz_n - z_n\| = 0. \tag{3.33}$$

Using (3.15), (3.16), and (3.25), we can see that all the conditions of Lemma 2.2 are satisfied. Hence $\lim_{n\to\infty} \|t_n - f\|$ exists for an arbitrary $f \in F(S)$ which implies that $\{t_n\}$ is bounded. Now, let t be a sequential weak cluster point of $\{t_n\}$. It follows that $\{t_n\}$ has a subsequence $\{t_{n_k}\}$ such that $t_{n_k} \rightharpoonup t$ as $k \to \infty$. Since $\lim_{n\to\infty} \|z_n - t_n\| = 0$, it follows that $z_{n_k} \rightharpoonup t$ as $k \to \infty$. This together with (3.33) and Lemma 2.3 show that $t \in F(S)$. We can see that all the conditions of Lemma 2.4 are satisfied. Therefore, the sequence $\{t_n\}$ converges weakly to a point in $F(S)$. □

Remark:

(1) If we put $\theta_n = \phi_n$, then MGIM algorithm becomes general inertial Mann algorithm [14].

(2) If we put $\theta_n = \phi_n = 0$, then MGIM algorithm becomes Reflected Mann Algorithm [14].
(3) If we put $\theta_n = \phi_n = \gamma_n$, then Theorem 3.1 becomes Theorem 5 of Bot et al. [8].
(4) If we put $\theta_n = \phi_n$ and $\gamma_n = 0$, then MGIM algorithm becomes accelerated Mann algorithm [13].
(5) If we put $\gamma_n = 0$ in Theorem 3.1, then we get a new algorithm which we call it General Accelerated Mann Algorithm. It is formulated as follows:

$$\begin{cases} w_n = t_n + \theta_n(t_n - t_{n-1}) \\ y_n = t_n + \phi_n(t_n - t_{n-1}) \\ t_{n+1} = a_n w_n + b_n y_n + c_n S t_n, \end{cases} \tag{3.34}$$

3.2. **General Inertial Viscosity Algorithm for Nonexpansive Mappings.** Now, we present a viscosity method for solving a common fixed point of a countable family of nonexpansive operators in real Hilbert spaces which we call it General Inertial Mann Viscosity (GIMV, for short) algorithm. Let $v : \mathcal{H} \to \mathcal{H}$ be a η-contraction mapping where $\eta \in [0, 1)$ and $\{S_k\}$ be a sequence of nonexpansive mappings $S_k : \mathcal{H} \to \mathcal{H}$ for $k \geq 1$.
We take the following assumptions to prove the strong convergence of the sequence generated by GIMV algorithm:

(1) $\{S_k\}$ satisfies condition (Z) (See Definition 2.1).
(2) $\Gamma := \cap_{k=1}^{\infty} F(S_k) \neq \emptyset$.

Algorithm 3: General Inertial Mann Viscosity Algorithm (GIMV)

Initialization:Take $t_0, t_1 \in \mathcal{H}$ arbitrarily and positive sequences $\{\alpha_k\}$, $\{\beta_k\}$, $\{\lambda_k\}$, $\{\gamma_k\}$ which satisfy the following conditions:

$$\{\alpha_k\}, \{\beta_k\}, \{\lambda_k\}, \{\gamma_k\} \subset (0, 1),$$
$$\lim_{k\to\infty} \gamma_k = 0, \text{ and } \sum_{k=1}^{\infty} \gamma_k = \infty.$$

Step 1: Choose $\{\mu_k\}, \{\theta_k\} \subset [0, \infty)$ and bounded.
For $k \geq 1$, set

$$\alpha_k = \begin{cases} \min\left\{\mu_k, \dfrac{\tau_k}{\|t_k - t_{k-1}\|}\right\} & \text{if } t_k \neq t_{k-1}, \\ \mu_k & \text{otherwise}, \end{cases}$$

$$\beta_k = \begin{cases} \min\left\{\theta_k, \dfrac{\zeta_k}{\|t_k - t_{k-1}\|}\right\} & \text{if } t_k \neq t_{k-1}, \\ \theta_k & \text{otherwise}, \end{cases}$$

where $\{\tau_k\}, \{\zeta_k\} \subset (0, \infty)$; $\lim_{k\to\infty} \frac{\tau_k}{\gamma_k} = 0$, and $\lim_{k\to\infty} \frac{\zeta_k}{\gamma_k} = 0$

Step 2: Compute t_{k+1}.

$$\begin{cases} y_k = t_k + \alpha_k(t_k - t_{k-1}) \\ z_k = t_k + \beta_k(t_k - t_{k-1}) \\ w_k = (1 - \lambda_k) y_k + \lambda_k S_k z_k, \\ t_{k+1} = \gamma_k v(w_k) + (1 - \gamma_k) S_k z_k \end{cases} \tag{3.35}$$

Update $k = k + 1$ and return to Step 1.

Theorem 3.2. *The sequence $\{t_k\}$ generated by* ***Algorithm 3*** *converges strongly to an element $f \in \Gamma$, where $f = P_\Gamma v(f)$.*

Proof. Pick $f \in \Gamma$.
By using (3.35), we have

$$
\begin{aligned}
\|t_{k+1} - f\| =& \|\gamma_k(v(w_k) - f) + (1-\gamma_k)(S_k z_k - f)\| \\
\leq& \gamma_k\|v(w_k) - f\| + (1-\gamma_k)\|S_k z_k - f\| \\
\leq& \gamma_k\|v(w_k) - v(f)\| + \gamma_k\|v(f) - f\| + (1-\gamma_k)\|z_k - f\| \\
\leq& \eta\gamma_k\|w_k - f\| + \gamma_k\|v(f) - f\| + (1-\gamma_k)\|z_k - f\| \\
=& \eta\gamma_k\|(1-\lambda_k)(y_k - f) + \lambda_k(S_k z_k - f)\| + \gamma_k\|v(f) - f\| + \\
& (1-\gamma_k)\|z_k - f\| \\
\leq& \eta\gamma_k(1-\lambda_k)\|y_k - f\| + \eta\gamma_k\lambda_k\|z_k - f\| + \gamma_k\|v(f) - f\| + \\
& (1-\gamma_k)\|z_k - f\| \\
=& \eta\gamma_k(1-\lambda_k)\|y_k - f\| + [\eta\gamma_k\lambda_k + 1 - \gamma_k]\|z_k - f\| + \gamma_k\|v(f) - f\|.
\end{aligned} \tag{3.36}
$$

Again from (3.35), we have

$$\|y_k - f\| \leq \|t_k - f\| + \alpha_k\|t_k - t_{k-1}\| \tag{3.37}$$

and

$$\|z_k - f\| \leq \|t_k - f\| + \beta_k\|t_k - t_{k-1}\|. \tag{3.38}$$

Substituting (3.37) and (3.38) into (3.36), we get

$$\|t_{k+1} - f\| \leq [1-(1-\eta)\gamma_k]\|t_k - f\| + \gamma_k\left[\frac{\alpha_k}{\gamma_k}\|t_k - t_{k-1}\| + \frac{\beta_k}{\gamma_k}\|t_k - t_{k-1}\| + \|v(f) - f\|\right]. \tag{3.39}$$

By the conditions of α_k and β_k, we have $\lim_{k\to\infty} \frac{\alpha_k}{\gamma_k}\|t_k - t_{k-1}\| = 0$ and $\lim_{k\to\infty} \frac{\beta_k}{\gamma_k}\|t_k - t_{k-1}\| = 0$, respectively. Hence, we can find constants M, $N \geq 0$ such that

$$\frac{\alpha_k}{\gamma_k}\|t_k - t_{k-1}\| \leq M \text{ and } \frac{\beta_k}{\gamma_k}\|t_k - t_{k-1}\| \leq N,$$

for all $k \geq 1$.
Now, (3.39) becomes

$$
\begin{aligned}
\|t_{k+1} - f\| \leq& [1-(1-\eta)\gamma_k]\|t_k - f\| + \gamma_k\left[M + N + \|v(f) - f\|\right] \\
=& [1-(1-\eta)\gamma_k]\|t_k - f\| + \gamma_k(1-\eta)\left[\frac{M + N + \|v(f) - f\|}{1-\eta}\right].
\end{aligned}
$$

Proceeding inductively, we arrive at

$$\|t_{k+1} - f\| \leq \max\left\{\|t_1 - f\|, \frac{M + N + \|v(f) - f\|}{1-\eta}\right\},$$

for all $k \geq 1$ which proves the boundness of $\{t_k\}$. The boundness of $\{t_k\}$ again implies that $\{y_k\}, \{z_k\}, \{S_k z_k\}, \{w_k\}$, and $v(w_k)$ are all bounded.
From (3.35), we get

$$
\begin{aligned}
\|w_k - f\|^2 =& \|(1-\lambda_k)(y_k - f) + \lambda_k(S_k z_k - f)\|^2 \\
=& (1-\lambda_k)\|y_k - f\|^2 + \lambda_k\|S_k z_k - f\|^2 - \lambda_k(1-\lambda_k)\|S_k z_k - y_k\|\|^2 \\
\leq& (1-\lambda_k)\|y_k - f\|^2 + \lambda_k\|z_k - f\|^2 - \lambda_k(1-\lambda_k)\|S_k z_k - y_k\|^2,
\end{aligned} \tag{3.40}
$$

and

$$\begin{aligned}\|t_{k+1}-f\|^2 =&\|\gamma_k(v(w_k)-f)+(1-\gamma_k)(S_kz_k-f)\|^2\\ =&\|\gamma_k(v(w_k)-v(f)+v(f)-f)+(1-\gamma_k)(S_kz_k-f)\|^2\\ =&\|[\gamma_k(v(w_k)-v(f))+(1-\gamma_k)(S_kz_k-f)]+\gamma_k(v(f)-f)\|^2\\ \le&\|\gamma_k(v(w_k)-v(f))+(1-\gamma_k)(S_kz_k-f)\|^2+\\ &2\gamma_k\langle v(f)-f,t_{k+1}-f\rangle\\ \le&\gamma_k\|v(w_k)-v(f)\|^2+(1-\gamma_k)\|S_kz_k-f)\|^2+\\ &2\gamma_k\langle v(f)-f,t_{k+1}-f\rangle\\ \le&\eta\gamma_k\|w_k-f\|^2+(1-\gamma_k)\|z_k-f)\|^2+\\ &2\gamma_k\langle v(f)-f,t_{k+1}-f\rangle.\end{aligned} \tag{3.41}$$

Substituting (3.40) into (3.41), we get

$$\begin{aligned}\|t_{k+1}-f\|^2 \le&\eta\gamma_k(1-\lambda_k)\|y_k-f\|^2+(\eta\gamma_k\lambda_k+1-\gamma_k)\|z_k-f\|^2-\\ &\eta\gamma_k\lambda_k(1-\lambda_k)\|S_kz_k-y_k\|^2+2\gamma_k\langle v(f)-f,t_{k+1}-f\rangle.\end{aligned} \tag{3.42}$$

From (3.35), we get

$$\begin{aligned}\|y_k-f\|^2 =&\|(t_k-f)+\alpha_k(t_k-t_{k-1})\|^2\\ =&\|t_k-f\|^2+\alpha_k^2\|t_k-t_{k-1}\|^2+2\alpha_k\langle t_k-f,t_k-t_{k-1}\rangle\\ \le&\|t_k-f\|^2+\alpha_k^2\|t_k-t_{k-1}\|^2+2\alpha_k\|t_k-f\|\|t_k-t_{k-1}\|.\end{aligned} \tag{3.43}$$

Similarly,

$$\|z_k-f\|^2 \le\|t_k-f\|^2+\beta_k^2\|t_k-t_{k-1}\|^2+2\beta_k\|t_k-f\|\|t_k-t_{k-1}\|. \tag{3.44}$$

Substituting (3.43) and (3.44) into (3.42), we get

$$\begin{aligned}\|t_{k+1}-f\|^2 \le&[1-(1-\eta)\gamma_k]\|t_k-f\|^2+\gamma_k\Bigg[\left(\frac{\alpha_k}{\gamma_k}\|t_k-t_{k-1}\|\right)^2+\\ &\left(\frac{\beta_k}{\gamma_k}\|t_k-t_{k-1}\|\right)^2+2\|t_k-f\|\left(\frac{\alpha_k}{\gamma_k}\|t_k-t_{k-1}\|\right)+\\ &2\|t_k-f\|\left(\frac{\beta_k}{\gamma_k}\|t_k-t_{k-1}\|\right)+2\langle v(f)-f,t_{k+1}-f\rangle\Bigg]-\\ &\eta\gamma_k\lambda_k(1-\lambda_k)\|S_kz_k-y_k\|^2,\end{aligned} \tag{3.45}$$

which implies

$$\eta\gamma_k\lambda_k(1-\lambda_k)\|S_kz_k-y_k\|^2\le\|t_k-f\|^2-\|t_{k+1}-f\|^2+\gamma_kM_1, \tag{3.46}$$

where

$$\begin{aligned}M_1 =&\sup_{k\ge1}\Bigg\{\left(\frac{\alpha_k}{\gamma_k}\|t_k-t_{k-1}\|\right)^2+\left(\frac{\beta_k}{\gamma_k}\|t_k-t_{k-1}\|\right)^2+2\|t_k-f\|\left(\frac{\alpha_k}{\gamma_k}\|t_k-t_{k-1}\|\right)+\\ &2\|t_k-f\|\left(\frac{\beta_k}{\gamma_k}\|t_k-t_{k-1}\|\right)+2\langle v(f)-f,t_{k+1}-f\rangle\Bigg\}.\end{aligned}$$

Next, we prove the strong convergence of $\{t_k\}$ to f.
Suppose $a_k:=\|t_k-f\|^2$ has a subsequence $\{a_{k_i}\}$ such that $\liminf_{i\to\infty}(a_{k_i+1}-a_{k_i})\ge0$.

Using (3.46) and applying the conditions of $\{\gamma_k\}$ and $\{\lambda_k\}$, we obtain

$$\begin{aligned}\limsup_{i\to\infty} \eta\gamma_{k_i}\lambda_{k_i}(1-\lambda_{k_i})\|S_{k_i}z_{k_i} - y_{k_i}\|^2 &\leq \limsup_{i\to\infty}(a_{k_i} - a_{k_i+1} + \gamma_{k_i}M_1) \\ &\leq \limsup_{i\to\infty}(a_{k_i} - a_{k_i+1}) + \limsup_{i\to\infty}\gamma_{k_i}M_1 \\ &\leq 0.\end{aligned}$$

This implies that:

$$\lim_{i\to\infty}\|S_{k_i}z_{k_i} - y_{k_i}\| = 0. \tag{3.47}$$

Now, we are in a position to prove that $\limsup_{i\to\infty}\langle v(f) - f, t_{k_i+1} - f\rangle \leq 0$.
Choose a subsequence $\{t_{k_{i_j}}\}$ of $\{t_{k_i}\}$ such that

$$\limsup_{i\to\infty}\langle v(f) - f, t_{k_i} - f\rangle = \lim_{j\to\infty}\langle v(f) - f, t_{k_{i_j}} - f\rangle.$$

The boundness of $\{t_{k_{i_j}}\}$ guarantees the existence of a subsequence $\{t_{k_{i_{j_p}}}\}$ of $t_{k_{i_j}}$ such that $t_{k_{i_{j_p}}} \rightharpoonup u \in \mathcal{H}$. With out loss of generality, we may assume that $t_{k_{i_j}} \rightharpoonup u \in \mathcal{H}$. Since by assumption condition (Z) is satisfied by $\{S_k\}$, it follows that $u \in \Gamma$. As $\lim_{i\to\infty}\|t_{k_i+1} - t_{k_i}\| = 0$ and $f = P_\Gamma v(f)$, and using (2.4), we obtain

$$\limsup_{i\to\infty}\langle v(f) - f, t_{k_i} - f\rangle = \langle v(f) - f, u - f\rangle \leq 0. \tag{3.48}$$

Combining (3.45) and (3.48), and using the hypothesis of Theorem 3.2 that is $\lim_{k\to\infty}\frac{\alpha_k}{\gamma_k}\|t_k - t_{k-1}\| = 0$, $\lim_{k\to\infty}\frac{\beta_k}{\gamma_k}\|t_k - t_{k-1}\| = 0$, and $\sum_{k=0}^{\infty}\gamma_k = \infty$, we can see that all the conditions of Lemma 2.5 are satisfied. Hence $\lim_{k\to\infty}\|t_k - f\| = 0$; that is, $\{t_k\}$ converges strongly to $f = P_\Gamma v(f)$. □

4. Applications

4.1. **Constrained Convex Minimization Problem.** Let D be a nonempty closed convex subset of $\mathcal{H}$ and $g : D \to \mathbb{R}$ be a real valued convex function. Then the constrained convex minimization problem:

$$\min_{t\in D} g(t), \tag{4.49}$$

where g is a differentiable function can be expressed as a fixed point problem as follows:

$$t = P_D(t - \beta\nabla g(t)), \tag{4.50}$$

where $\beta > 0$.
From (4.50), we can formulate the gradient-projection algorithm as:

$$t_{n+1} = P_D\big(I - \beta\nabla g\big)(t_n). \tag{4.51}$$

It is proved that the composite $P_D(I - \beta\nabla)$ is $((2+\beta)/4)$-averaged for $0 < \beta < 2/L$ if ∇g is L-Lipschitz continuous [35]. So, the operator $P_D(I - \beta\nabla)$ is nonexpansive.
Setting $S = P_D(I - \beta\nabla)$ in Algorithm 1, we come up with a new algorithm for (4.49) which we call it Modified General Inertial Gradient Projection (MGIGP, for short) algorithm. It is shown in Algorithm 4.

Algorithm 4: Modified General Inertial Gradient Projection Algorithm (MGIGP)

Initialization:Take $t_0, t_1 \in \mathcal{H}$ arbitrarily, and $(E1)$ and $(E2)$ hold.
Iterative Step: Calculate t_{k+1} via the manner

$$\begin{cases} w_n = t_n + \gamma_n(t_n - t_{n-1}) \\ y_n = t_n + \phi_n(t_n - t_{n-1}) \\ z_n = t_n + \theta_n(t_n - t_{n-1}) \\ t_{n+1} = a_n w_n + b_n y_n + c_n P_D(z_n - \beta \nabla g(z_n)), \end{cases} \tag{4.52}$$

Theorem 4.3. *Assume that problem* (4.49) *is consistent, ∇g is L-Lipschitz continuous, and the number $\beta \in (0, 2/L)$. Then the sequence $\{t_n\}$ generated by Algorithm 4 converges weakly to a minimizer of problem* (4.49).

4.2. **The Douglas-Rachford Splitting Method.** Let γ be a fixed parameter, and A and B be maximal monotone operators. The resolvents of A and B are defined as:

$$J_\gamma^A := (I + \gamma A)^{-1} \text{ and } J_\gamma^B := (I + \gamma B)^{-1},$$

respectively, which are firmly nonexpansive. The corresponding reflection operators are also defined as follows:

$$R_\gamma^A := 2J_\gamma^A - I \text{ and } R_\gamma^B := 2J_\gamma^B - I,$$

which are nonexpansive operators.
We know that $0 \in Tx$ for $T = A + B$ if and only if $x = J_\gamma^B(t)$, where $t = R_\gamma^A R_\gamma^B t$. To find a zero of $T = A + B$, we can apply the Mann iteration to $R_\gamma^A R_\gamma^B$.
As a result,we obtain the following iteration:

$$t_{n+1} := (1 - \lambda_n)t_n + \lambda_n R_\gamma^A R_\gamma^B t_n, \tag{4.53}$$

for $n \geq 1$, see [5] for more details. This algorithm provides us with the approximation of the orginal variable by setting $x_n := J_\gamma^B t_n$.
Using (4.53) and the definitions of reflection operators, we get

$$t_{n+1} := t_n + 2\lambda_n\big(J_\gamma^A(2J_\gamma^B t_n - t_n) - J_\gamma^B t_n\big). \tag{4.54}$$

Now, we are in a position to introduce our algorithm which we call it Modified General Inertial Douglas-Rachford Splitting (MGIDRS, for short) algorithm which is formulated as follows.

Algorithm 5: Modified General Inertial Douglas-Rachford Splitting Algorithm (MGIDRS)

Initialization:Take $t_0, t_1 \in \mathcal{H}$ arbitrarily, and $(E1)$ and $(E2)$ hold.
Iterative Step: Calculate t_{n+1} via the manner

$$\begin{cases} w_n = t_n + \theta_n(t_n - t_{n-1}) \\ y_n = t_n + \phi_n(t_n - t_{n-1}) \\ z_n = t_n + \gamma_n(t_n - t_{n-1}) \\ t_{n+1} = a_n w_n + b_n y_n + 2c_n\big(J_\gamma^A(2J_\gamma^B t_n - z_n)\big) - 2c_n J_\gamma^B t_n + c_n z_n, \end{cases} \tag{4.55}$$

Theorem 4.4. *The sequence $\{t_n\}$ generated by Algorithm 5 converges weakly to an element $t \in \mathcal{H}$ such that $J_\gamma^B t \in (A+B)^{-1}(0)$, that is, $x := J_\gamma^B t$ is a solution of the monotone inclusion problem for the operator $T := A + B$.*

4.3. **Nonsmooth Convex Optimization Model.** Finally, we apply Algorithm 3 for solving the nonsmooth convex optimization problem which has the following form:

$$\min_{t\in\mathcal{H}} \psi_1(t) + \psi_2(t), \tag{4.56}$$

where $\psi_1(t) : \mathcal{H} \to \mathbb{R}$ is smooth and convex with a Lipschitz continuous gradient constant $L > 0$ and $\psi_2(t) : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ is a proper convex and lower-semicontinuous function. The solution set of problem (4.56) is given by $\Omega := \text{Argmin}(\psi_1 + \psi_2)$.
Furthermore, t is a solution of (4.56) if it satisfies the following fixed point equation.

$$t = \text{prox}_{c\psi_2}(I - c\nabla\psi_1)(t), \tag{4.57}$$

where $c > 0$, $\text{prox}_{\psi_2} = (I + \partial\psi_2)^{-1}$, and $\partial\psi_2$ is the subdifferential of ψ_2 (see [5] for more details).
For solving (4.56), we can use the forward-backward splitting algorithm [20] which takes the following form:

$$t_{k+1} = \underbrace{\text{prox}_{c_k\psi_2}}_{\text{backward step}} \underbrace{(I - c_k\nabla\psi_1)(t_k)}_{\text{forward step}}, \tag{4.58}$$

$k \in \mathbb{N}$ where $t_1 \in \mathcal{H}$ and $0 < c_k < 2/L$.
Now, we can apply Algorithm 3 for solving the nonsmooth convex optimization problem (4.56) by assuming $\Omega := \text{Argmin}(\psi_1 + \psi_2) \neq \emptyset$ and setting $S_k = \text{prox}_{c_k\psi_2}(I - c_k\nabla\psi_1)$ which is a nonexpansive mapping where $c_k \in (0, 2/L)$. As a result, we come up with a new algorithm which we call it General Inertial Viscosity Forward-Backward Splitting (GIVFBS, for short) algorithm.

Algorithm 6: General Inertial Viscosity Forward-Backward Splitting (GIVFBS) Algorithm

---

Initialization:We follow the same initialization process as Algorithm 3.
Step 1: The same as Algorithm 3.
Step 2: Compute t_{k+1} via the manner:

$$\begin{cases} y_k = t_k + \alpha_k(t_k - t_{k-1}) \\ z_k = t_k + \beta_k(t_k - t_{k-1}) \\ w_k = (1-\lambda_k)y_k + \lambda_k \text{prox}_{c_k\psi_2}(1 - c_k\nabla\psi_1)z_k \\ t_{k+1} = \gamma_k v(w_k) + (1-\gamma_k)\text{prox}_{c_k\psi_2}(1 - c_k\nabla\psi_1)z_k \end{cases} \tag{4.59}$$

Update $k = k + 1$ and return to Step 1.

Theorem 4.5. *The sequence generated by Algorithm 6 converges strongly to an element $f \in \Omega$, where $f = p_\Omega v(f)$.*

4.4. **Image Restoration problem.** The general image restoration problem can be formulated by the inversion of the observation model given by

$$y = Ax + \epsilon, \tag{4.60}$$

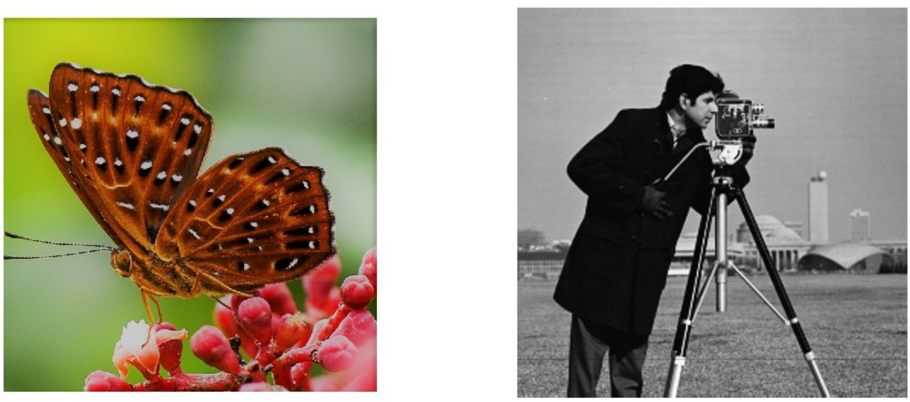
where x is the original image, y is the observed image, and ϵ is the additive noise. The kernel function A models the blurring operation.

A regularization method should be used in the image restoration process. The notion of regularization method has the goal of constraining, in the solution, the effect of the growth of the error coming from the data, by means of modifying the problem condition. The ℓ_1 regularization can remove noise in the restoration process and it is the problem of finding

$$\min_{y} \frac{1}{2}\|Ax - y\|_2^2 + \mu\|x\|_1, \tag{4.61}$$

where $\|.\|_2$ is the ℓ_2-norm, μ is a positive regularization parameter which measures the trade-off between a good fit and a regularized solution, and $\|.\|_1$ is the ℓ_1-norm. Finding the solutions of (4.60) can be seen as finding a solution to the least-square problem (4.61).

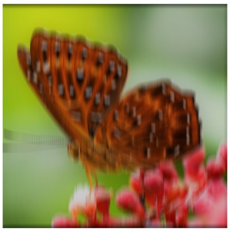
We apply Algorithm 6 (GIVFBS) for solving an image restoration problem (4.60) and compare the performance of our method in restoring blurred images with FBS [20] algorithm with 500 iterations. For comparison, we consider the standard test images of Butterfly (256×256) and Camera Man (512×512) (see Figure 1).



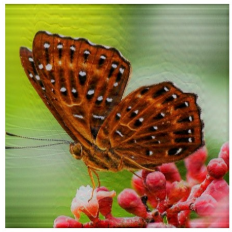
(A) Butterfly (B) Cameraman

FIGURE 1. Original test images: (A) Butterfly and (B) Cameraman

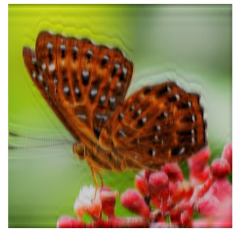
To measure the quality of recovered images, we calculate the improved signal-to-noise ratio (ISNR) and structural similarity index measure (SSIM). For the control parameters, we take $\tau_k = \zeta_k = \frac{10^{15}}{k^2}, \mu_k = \frac{9.5k}{6k+1}, \theta_k = \frac{9.5k}{10k+1}, \lambda_k = 0.95, \gamma_k = \frac{10^{-4}}{6k}$, and $c_k = 0.05$. Moreover, the contraction mapping is defined by $v(x) = 0.95x$. Now, by taking $\psi_1(x) = \|Ax - y\|_2^2$ and $\psi_2(x) = \mu\|x\|_1$, we can solve image restoration problems using our algorithm (GIVFBS) and the FBS algorithm.



(A) Blurred



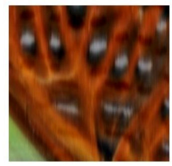
(B) GIVFBS



(C) FBS

(A) Blurred

(B) GIVFBS



(C) FBS

FIGURE 3. Degraded and restored Butterfly images.

(A) Blurred

(B) GIVFBS

(C) FBS

(A) Blurred

(B) GIVFBS

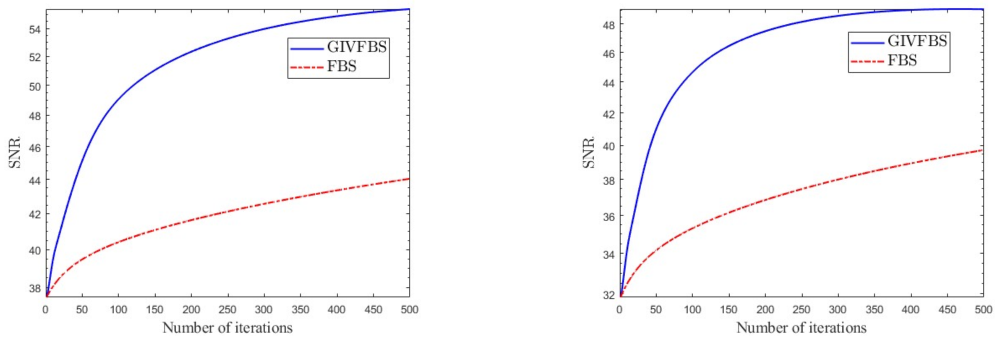
(C) FBS

FIGURE 5. Degraded and restored Cameraman images.

It can be seen from Figure 3 and Figure 5 that the recovered images by GIVFBS algorithm are better than that of FBS algorithm.

TABLE 1. ISNR and SSIM values for Butterfly and Cameraman images using GIVFBS and FBS algorithms

	GIVFBS		FBS	
Images	ISNR	SSIM	ISNR	SSIM
Butterfly	9.009122	0.999997	3.309512	0.999989
Cameraman	9.202300	0.999994	4.103025	0.999983



(A) Butterfly (B) Cameraman

FIGURE 6. Comparisons of SNR values for (A) Butterfly and (B) Cameraman images using GIVFBS and FBS algorithms

We can observe from Figure 6 and Table 1 that GIVFBS algorithm has higher SNR, ISNR, and SSIM values which shows that the quality of the images recovered by GIVFBS algorithm is better than that of FBS algorithm.

5. CONCLUSIONS

We introduce the modified general inertial Mann algorithm for finding the fixed point of nonexpansive mappings and prove weak convergence of the sequence generated by this algorithm. This new algorithm generalizes the algorithm which was developed by Dong et al. [14]. Moreover, we introduce the general inertial viscosity algorithm and prove strong convergence of the sequence generated by this algorithm to a common fixed point of a countable family of nonexpansive operators. Finally, we show an applications of Algorithm 6 to solve image restoration problem and the numerical results show that the proposed algorithm outperforms that of FBS algorithm in recovering the original image (See Figure 6 and Fable 1).

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