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An intermixed algorithm for solving fixed point problems of proximal operators in Hilbert Spaces

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ABSTRACT. The aim of this paper is to modify proximal operators in Hilbert spaces. We introduce an intermixed algorithm with viscosity technique to find the solution of fixed point problem of two proximal operators in a real Hilbert space, utilizing the modified proximal operators. Under some mild conditions, a strong convergence theorem is established for the proposed algorithm. We also apply our main result to the split feasibility problem. Finally we provide numerical examples for supporting the main result.

1. INTRODUCTION

Let H be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and an induced norm $\| \cdot \|$ and let $\Gamma_0(H)$ be a class of convex, lower semicontinuous, and proper functions from a Hilbert space H to $(-\infty, +\infty]$. Let C be closed convex subset of H. Let $S : C \to C$ be a nonlinear mapping, a point $x \in C$ is called a fixed point of S if $Sx = x$. We denote by $Fix(S)$, the set of all fixed points of S, i.e. $Fix(S) = \{x \in C : S(x) = x\}$. Consider the following convex minimization problem

$$\min_{x \in H} (f(x) + g(x)), \tag{1.1}$$

where $f \in \Gamma_0(H)$, $g : H \to \mathbb{R}$ is convex and differentiable with the Lipschitz continuous gradient denoted by ∇g. The solution set of (1.1) will be denoted by $\operatorname{Argmin}(f + g)$. In 2014, Xu [29] presented an important mathematical tool to demonstrate that the solution set of (1.1) is equivalent to the fixed point equation as follows:

$$\tilde{x} = \operatorname{Prox}_{\gamma f}(\tilde{x} - \gamma \nabla g(\tilde{x})), \tag{1.2}$$

where $\gamma > 0$ and $\operatorname{Prox}_{\gamma f} x := \operatorname{argmin}_{u \in H}\{f(u) + \frac{1}{2\gamma}\|u - x\|^2\}$ is the proximal mapping of f (see [2] for more informations on the proximal mapping). The most widely used algorithm for solving the convex minimization problem (1.1) is the so-called proximal-gradient algorithm. This proximal-gradient algorithm is given by: $x_1 \in H$ and

$$x_{n+1} = \operatorname{Prox}_{\gamma f}(I - \gamma \nabla g)(x_n), \quad \forall n \geq 1. \tag{1.3}$$

where Prox_f is the proximal operator of f, $\gamma \in (0, 2/L)$ and L is the Lipschitz constant of ∇g, then the sequence $\{x_n\}$ generated by algorithm (1.3) converges weakly to an element of $\operatorname{Argmin}(f + g)$ [see [2], Corollary 28.9]. This method is sometimes called the forward-backward algorithm. The proximal-gradient algorithm can be used in real-world applications, for example, in signal recovery, in image deblurring, and in machine learning (regression on highdimensional datasets) (see, [4], [20], [14], [22]). Recently there are extensive works in studying proximal gradient algorithm, see [19], [13], [28], [27], [10], [1], [25] and the references therein. For a set C, we denote by δ_C the indicator function of the set, that is, $\delta_C(x) = 0$ if $x \in C$ and ∞ otherwise. We denote the metric projection

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onto C as P_C. Clearly, by definition, $P_C x = \operatorname{Prox}_{\delta_C} x$. When $f = \delta_C$, the algorithm (1.3) becomes the popular gradient projection algorithm, which is defined as follows. For an initial guess $x_1 \in H$,

$$x_{n+1} = P_C(I - \gamma\nabla g)(x_n), \quad \forall n \geq 1. \tag{1.4}$$

Recently, Guo and Cui [8] modified the proximal-gradient algorithm with viscosity technique as follows: For arbitrarily given $x_1 \in H$, let the sequences $\{x_n\}$ be generated iteratively by

$$x_{n+1} = \alpha_n h(x_n) + (1 - \alpha_n) \operatorname{Prox}_{\mu_n f}(I - \mu_n \nabla g)(x_n) + e(x_n), \quad \forall n \geq 1, \tag{1.5}$$

where $\{\alpha_n\}$ is a real number sequence in $[0, 1]$, $0 < a = \inf_n \mu_n \leq \mu_n < \frac{2}{L}$, $h : H \to H$ a ρ-contractive operator with $\rho \in (0, 1)$, and $e : H \to H$ represents a perturbation operator and satisfies $\sum_{n=1}^{\infty} \|e(x_n)\| < +\infty$. Under some appropriate conditions, they proved that the algorithm (1.5) converges strongly to a solution of (1.1) in a real Hibert space.

Currently, one of the best methods for solving the fixed point problem of nonlinear mapping is to use the intermixed algorithm, proposed by Yao et al.[32]. This algorithm has the following features: the definition of the sequence $\{x_n\}$ is involved in the sequence $\{y_n\}$ and the definition of the sequence $\{y_n\}$ is also involved in the sequence $\{x_n\}$. In recent years, the intermixed algorithm has attracted a significant amount of interest from authors, who improved it in various ways (see, e.g., [26], [23], [11]). In particular, Yao et al.[32] introduced the intermixed algorithm for two strict pseudo-contractions as follows: For arbitrarily given $x_1 \in C$, $y_1 \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by

$$\begin{cases} x_{n+1} = (1 - \delta_n)x_n + \delta_n P_C[\alpha_n h_1(y_n) + (1 - k - \alpha_n)x_n + k\mathcal{T}x_n], & \forall n \geq 1, \\ y_{n+1} = (1 - \delta_n)y_n + \delta_n P_C[\alpha_n h_2(x_n) + (1 - k - \alpha_n)y_n + k\mathcal{S}y_n], & \forall n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$, $\{\delta_n\}$ are two real number sequences in $(0, 1)$, $\mathcal{S}, \mathcal{T} : C \to C$ are λ-strictly pseudo-contractions with $k \in (0, 1 - \lambda)$, and $h_1, h_2 : C \to H$ are ρ_1, ρ_2-contractions, respectively. Moreover, Yao et al. also proved in [32] that the sequence $\{x_n\}$ generated by (1.6) weakly converges to a fixed point of two strict pseudo-contractions under some appropriate conditions.

The Krasnoselskii-Mann algorithm (see, [15], [16], [21]) is one of the most well-known fixed point algorithms. In recent years, several researchers have increasingly investigated the Krasnoselskii-Mann algorithm in various directions, for example [24], [7], [33], [9] and the references therein. In particular, Kanzow and Shehu [12] proposed the following inexact Krasnoselskii–Mann algorithm for finding a fixed point of a nonexpansive mapping T in a real Hilbert space: For arbitrarily given $x_1 \in H$, let $\{x_n\}$ be a sequence generated iteratively by

$$x_{n+1} = \alpha_n x_n + \beta_n T x_n + r_n, \quad \forall n \geq 1, \tag{1.7}$$

where $T : H \to C$ is a nonexpansive mapping, r_n denotes the residual vector and $\{\alpha_n\}$, $\{\beta_n\}$ are two real number sequences in $[0, 1]$ such that $\alpha_n + \beta_n \leq 1$. They proved that if $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$, $\sum_{n=1}^{\infty} \|r_n\| < \infty$, and $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$, then the sequence $\{x_n\}$ generated by (1.7) converges weakly to a fixed point of T.

In this paper, based on the problems (1.1) and (1.2), we modify a proximal operator and introduce a new mathematical tool relevant to the modified proximal operator in Hilbert spaces. Inspired and motivated by previous works, we introduce an intermixed algorithm with viscosity technique to find the solution of fixed point problem of two proximal operators in a real Hilbert space. Using the mathematical tool above, a strong convergence theorem for the proposed algorithm is established under some mild conditions. Applications to the split feasibility problems are also considered. Finally, we provide some numerical experiments to verify the theoretical results of this paper. In summary,

- Applying the convex minimization problem (1.1) and the fixed point equation (1.2), we propose a new mathematical tool related to two proximal operators;
- We propose an intermixed algorithm to find the solution of fixed point problem of two proximal operators in a real Hilbert space and prove a strong convergence theorem for the proposed algorithm under some mild conditions;
- Our algorithm combine the proximal-gradient algorithm with viscosity technique in Guo and Cui [8], the intermixed algorithm in Yao et al.[32] and the Krasnoselskii–Mann algorithm in Kanzow and Shehu [12].

The organization of our paper is as follows: In section 2, we first recall some basic definitions and lemmas. We also give a new lemma related to two proximal operators (see Lemma 2.4 below). In section 3, we prove the strong convergence theorem of our proposed algorithm under some mild conditions. In section 4, we consider the application of our main result to solve the split feasibility problems. In section 5, we provide numerical examples to support the main result.

2. Preliminary

For the purpose of proving our theorem, we provide several definitions and lemmas in this section. For convenience, the following notations are used throughout the paper:

- H denotes a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and an induced norm $\| \cdot \|$;
- C denotes a nonempty closed convex subset of H;
- $\Gamma_0(H)$ denotes a class of convex, lower semicontinuous, and proper functions from a Hilbert space H to $(-\infty, +\infty]$;
- $x_n \to q$ ($x_n \rightharpoonup q$) denote the strong (weak) convergence of a sequence $\{x_n\}$ to q in H, respectively;
- $Fix(S)$ denotes the set of all fixed points of S.

Recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

Lemma 2.1. [18] *For given $x \in H$ and let $P_C : H \to C$ be a metric projection. Then*

(a) $w = P_C x$ if and only if $\langle x - w, y - w \rangle \leq 0, \quad \forall y \in C$;

(b) $w = P_C x$ if and only if $\|x - w\|^2 \leq \|x - y\|^2 - \|y - w\|^2, \quad \forall y \in C$;

(c) $\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H$.

Definition 2.1. A mapping $S : C \to C$ is called nonexpansive, if

$$\|Sx - Sy\| \leq \|x - y\| \quad \forall x, y \in C.$$

Definition 2.2. A mapping $A : C \to H$ is called

(i) Monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C;$$

(ii) η-Strongly monotone if there exists a positive real number η such that

$$\langle Ax - Ay, x - y \rangle \geq \eta \|x - y\|^2, \quad \forall x, y \in C;$$

(iii) L-Lipschitz continuous if there exists $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C;$$

(iv) α-inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Clearly, If A is η-Strongly monotone and L-Lipschitz continuous, then A is $\frac{\eta}{L^2}$-inverse strongly monotone. If A is an α-inverse strongly monotone mapping, then $\frac{1}{\alpha}$-Lipschitz continuous.

Lemma 2.2. [12] *Let X be a real inner product space.Then*

(a) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X;$

(b) $\|px + qy\|^2 = p(p + q)\|x\|^2 + q(p + q)\|y\|^2 - pq\|x - y\|^2, \quad \forall x, y \in X, \forall p, q \in \mathbb{R}.$

The following Lemma, which comes from [17], [30], will be used to prove our strong convergence result.

Lemma 2.3. [17], [30] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n + \mu_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a real sequence. Assume $\sum_{n=1}^{\infty} |\mu_n| < \infty$. Then, the following results hold:

(i) If $\delta_n \leq \alpha_n M$ for some $M \geq 0$ then $\{a_n\}$ is a bounded sequence;

(ii) If $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.

Let the function $f \in \Gamma_0(H)$. The set

$$\partial f(x) = \{z \in H : \langle z, y - x\rangle + f(x) \leq f(y),\ \forall y \in H\}$$

is called a *subdifferential* of f at $x \in H$. The function f is said to be *subdifferentiable* at x if $\partial f(x) \neq \emptyset$. The elements of $\partial f(x)$ are called the *subgradients* of f at x. If the function f is continuously differentiable, then $\partial f(x) = \{\nabla f(x)\}$; this is the gradient of f. It is well known that the subdifferential ∂f is a maximal monotone operator. It is notable that a point $x^* \in H$ minimizes f if and only if $0 \in \partial f(x^*)$. Let x and p be in H. The proximity operator of f is characterized by the inclusion

$$p = \operatorname{Prox}_{\gamma f} x \Leftrightarrow x - p \in \gamma\partial f(p).$$

In other words,

$$\operatorname{Prox}_{\gamma f} = (I + \gamma\partial f)^{-1}.$$

Moreover, the proximity operator of f is firmly nonexpansive, namely,

$$\langle \operatorname{Prox}_{\gamma f}(x) - \operatorname{Prox}_{\gamma f}(y), x - y\rangle \geq \|\operatorname{Prox}_{\gamma f}(x) - \operatorname{Prox}_{\gamma f}(y)\|^2 \tag{2.8}$$

for all $x, y \in H$, which is equivalent to

$$\|\operatorname{Prox}_{\gamma f}(x) - \operatorname{Prox}_{\gamma f}(y)\|^2 \leq \|x - y\|^2 - \|(I - \operatorname{Prox}_{\gamma f})(x) - (I - \operatorname{Prox}_{\gamma f})(y)\|^2 \tag{2.9}$$

for all $x, y \in H$. For general information on proximal operator, see Combettes and Pesquet [3].

Proposition 2.1. [2] *Let the function $f \in \Gamma_0(H)$ and let $x, p \in H$. Then*

$$p = Prox_f x \Leftrightarrow \langle y - p, x - p\rangle + f(p) \leq f(y),$$

for all $y \in H$.

Lemma 2.4. *Let the function $f \in \Gamma_0(H)$. Let $A, B : C \to H$ be δ^A and δ^B- inverse strongly monotone operators, respectively, with $\delta = \min\{\delta^A, \delta^B\}$ and $Fix(Prox_{\gamma f}(I - \gamma A)) \cap Fix(Prox_{\gamma f}(I - \gamma B)) \neq \emptyset$. Then*

$$Fix(Prox_{\gamma f}(I-\gamma A))\cap Fix(Prox_{\gamma f}(I-\gamma B)) = Fix(Prox_{\gamma f}(I-\gamma(aA+(1-a)B))), \tag{2.10}$$

for all $a \in (0,1)$ and $\gamma \in (0, 2\delta)$.

Proof. From Proposition 2.1, it is easy to see that

$$Fix(Prox_{\gamma f}(I-\gamma A))\cap Fix(Prox_{\gamma f}(I-\gamma B)) \subseteq Fix(Prox_{\gamma f}(I-\gamma(aA+(1-a)B))). \tag{2.11}$$

Let $x_0 \in Fix(Prox_{\gamma f}(I-\gamma(aA+(1-a)B)))$ and $x^* \in Fix(Prox_{\gamma f}(I-\gamma A))\cap Fix(Prox_{\gamma f}(I-\gamma B))$. So, we have

$$x^* \in Fix(Prox_{\gamma f}(I - \gamma(aA + (1-a)B))).$$

By the definitions of A, B, we have

$$\begin{aligned}
\|x_0 - x^*\|^2 =& \|Prox_{\gamma f}(I - \gamma(aA + (1-a)B))x_0 - Prox_{\gamma f}(I - \gamma(aA + (1-a)B))x^*\|^2 \\
\leq& \|x_0 - x^* - \gamma(a(Ax_0 - Ax^*) + (1-a)(Bx_0 - Bx^*))\|^2 \\
=& \|x_0 - x^*\|^2 - 2\gamma\langle a(Ax_0 - Ax^*) + (1-a)(Bx_0 - Bx^*), x_0 - x^*\rangle \\
&+ \gamma^2\|a(Ax_0 - Ax^*) + (1-a)(Bx_0 - Bx^*)\|^2 \\
\leq& \|x_0 - x^*\|^2 - 2\gamma a\langle Ax_0 - Ax^*, x_0 - x^*\rangle - 2\gamma(1-a)\langle Bx_0 - Bx^*, x_0 - x^*\rangle \\
&+ \gamma^2\left(a\|Ax_0 - Ax^*\|^2 + (1-a)\|Bx_0 - Bx^*\|^2\right) \\
\leq& \|x_0 - x^*\|^2 - \gamma a(2\delta - \gamma)\|Ax_0 - Ax^*\|^2 - \gamma(1-a)(2\delta - \gamma)\|Bx_0 - Bx^*\|^2. \qquad (2.12)
\end{aligned}$$

It implies that

$$Ax_0 = Ax^*, Bx_0 = Bx^*.$$

Let $y \in H$. By applying Proposition 2.1 and $x^* \in Fix(Prox_{\gamma f}(I - \gamma A))$, we obtain

$$\langle y - x^*, (I - \gamma A)x^* - x^*\rangle + \gamma f(x^*) \leq \gamma f(y),$$

which implies that

$$f(y) - f(x^*) + \langle y - x^*, Ax^*\rangle \geq 0. \tag{2.13}$$

Since $x^* \in Fix(Prox_{\gamma f}(I - \gamma B))$, we also get

$$f(y) - f(x^*) + \langle y - x^*, Bx^*\rangle \geq 0. \tag{2.14}$$

Since $x_0 \in Fix(Prox_{\gamma f}(I - \gamma(aA + (1-a)B))$ and by Proposition 2.1, we get

$$\langle y - x_0, (I - \gamma(aA + (1-a)B))x_0 - x_0\rangle + \gamma f(x_0) \leq \gamma f(y),$$

which implies that

$$f(y) - f(x_0) + \langle y - x_0, (aA + (1-a)B)x_0\rangle \geq 0. \tag{2.15}$$

From (2.13) and $Ax_0 = Ax^*$, we obtain

$$\begin{aligned}
\langle y - x_0, Ax_0\rangle + f(y) - f(x_0) &= \langle y - x^*, Ax^*\rangle + \langle x^* - x_0, Ax_0\rangle + f(y) - f(x^*) \\
&\quad + f(x^*) - f(x_0) \\
&= \langle y - x^*, Ax^*\rangle + f(y) - f(x^*) + \langle x^* - x_0, Ax_0\rangle \\
&\quad + f(x^*) - f(x_0) \\
&\geq \langle x^* - x_0, Ax_0\rangle + f(x^*) - f(x_0). \qquad (2.16)
\end{aligned}$$

From $Bx_0 = Bx^*$, (2.14), and (2.15), we get

$$\begin{aligned}\langle x^* - x_0, aAx_0\rangle + &af(x^*) - af(x_0)\\ &= \langle x^* - x_0, aAx_0 + (1-a)Bx_0\rangle - \langle x^* - x_0, (1-a)Bx_0\rangle\\ &\quad + af(x^*) - af(x_0)\\ &= \langle x^* - x_0, aAx_0 + (1-a)Bx_0\rangle + f(x^*) - f(x_0)\\ &\quad - f(x^*) + f(x_0) - \langle x^* - x_0, (1-a)Bx_0\rangle\\ &\quad + af(x^*) - af(x_0)\\ &\geq \langle x_0 - x^*, (1-a)Bx^*\rangle + (1-a)f(x_0) - (1-a)f(x^*)\\ &= (1-a)\left(\langle x_0 - x^*, Bx^*\rangle + f(x_0) - f(x^*)\right)\\ &\geq 0.\end{aligned}$$

Since $a \in (0,1)$, we have

$$\langle x^* - x_0, Ax_0\rangle + f(x^*) - f(x_0) \geq 0. \tag{2.17}$$

From (2.16) and (2.17), we obtain

$$\langle y - x_0, Ax_0\rangle + f(y) - f(x_0) \geq 0, \forall y \in H. \tag{2.18}$$

It follows from (2.18) and Proposition 2.1,

$$\langle y - x_0, x_0 - (I - \gamma A)x_0\rangle + \gamma f(y) - \gamma f(x_0) \geq 0, \forall y \in H.$$

It implies that

$$x_0 \in Fix(Prox_{\gamma f}(I - \gamma A)). \tag{2.19}$$

Using the same method as (2.19), we also have

$$x_0 \in Fix(Prox_{\gamma f}(I - \gamma B)).$$

So, we can conclude that

$$Fix(Prox_{\gamma f}(I - \gamma(aA + (1-a)B))) \subseteq Fix(Prox_{\gamma f}(I - \gamma A)) \cap Fix(Prox_{\gamma f}(I - \gamma B)). \tag{2.20}$$

From (2.11) and (2.20), we deduce that

$$Fix(Prox_{\gamma f}(I - \gamma A)) \cap Fix(Prox_{\gamma f}(I - \gamma B)) = Fix(Prox_{\gamma f}(I - \gamma(aA + (1-a)B))). \tag{2.21}$$

□

Lemma 2.5. [31] *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

3. Main results

In this section, we introduce an intermixed algorithm and prove a strong convergence of the proposed algorithm to find the solution of fixed point problem of two proximal operators.

For every $i = 1, 2$, let the functions $f_i \in \Gamma_0(H)$, let $A_i, B_i : C \to H$ be δ_i^A and δ_i^B-inverse strongly monotone operators, respectively, with $\delta_i = \min\{\delta_i^A, \delta_i^B\}$. Assume that $\Omega_i = Fix(Prox^i_{\gamma f}(I - \gamma_i A_i)) \cap Fix(Prox^i_{\gamma f}(I - \gamma_i B_i)) \neq \emptyset$, for all $i = 1, 2$. Let $\xi_1, \xi_2 : H \to H$ be σ_1 and σ_2-contraction mappings with $\sigma_1, \sigma_2 \in (0,1)$ and $\sigma = \max\{\sigma_1, \sigma_2\}$.

Now, we introduce an intermixed algorithm with viscosity technique for solving a common fixed point of proximal operators as follows:

Algorithm 1: An intermixed algorithm with viscosity technique for solving a common fixed point of two proximal operators.

Initialization: Given $x_1, y_1 \in C$ be arbitrary.
Iterative Steps: Given the current iterate $\{x_n\}$, $\{y_n\}$, calculate $\{x_{n+1}\}$, $\{y_{n+1}\}$ as follows:
Step 1: Compute

$$\begin{cases} v_n = Prox^2_{\gamma f}(y_n - \gamma_2(a_2A_2 + (1-a_2)B_2)y_n) \\ u_n = Prox^1_{\gamma f}(x_n - \gamma_1(a_1A_1 + (1-a_1)B_1)x_n) \end{cases}$$

Step 2: Compute

$$\begin{cases} y_{n+1} = \mu_n y_n + \beta_n P_C(\alpha_n \xi_2(x_n) + (1-\alpha_n)v_n) \\ x_{n+1} = \mu_n x_n + \beta_n P_C(\alpha_n \xi_1(y_n) + (1-\alpha_n)u_n), \end{cases}$$

where $\{\mu_n\}, \{\beta_n\}, \{\alpha_n\} \subseteq [0,1]$ with $\mu_n + \beta_n \leq 1$, $\gamma_i \in (0, 2\delta_i)$, $a_i \in (0,1)$ and $Prox^i_{\gamma f} : H \to H$ is the proximity operator, for all $i = 1, 2$.
Set $n := n+1$ and go to Step 1.

Theorem 3.1. *For every $i = 1, 2$, let the functions $f_i \in \Gamma_0(H)$, let $A_i, B_i : C \to H$ be δ_i^A and δ_i^B- inverse strongly monotone operators, respectively, with $\delta_i = \min\{\delta_i^A, \delta_i^B\}$. Assume that $\Omega_i = Fix(Prox^i_{\gamma f}(I - \gamma_i A_i)) \cap Fix(Prox^i_{\gamma f}(I - \gamma_i B_i)) \neq \emptyset$, for all $i = 1, 2$. Let $\xi_1, \xi_2 : H \to H$ be σ_1 and σ_2-contraction mappings with $\sigma_1, \sigma_2 \in (0,1)$ and $\sigma = \max\{\sigma_1, \sigma_2\}$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by Algorithm 1, satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) *there are* $\overline{\varepsilon}, l > 0$ *with* $0 < \overline{\varepsilon} \leq \mu_n, \beta_n \leq l < 1$ *for all* $n \in \mathbb{N}$;
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} (1 - \mu_n - \beta_n) < \infty.$

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to $x^* = P_{\Omega_1}\xi_1(y^*)$ and $y^* = P_{\Omega_2}\xi_2(x^*)$, respectively.

Proof. Putting $K_i = \text{Prox}^i_{\gamma f}(I - \gamma_i(a_iA_i + (1-a_i)B_i)))$ for all $i = 1, 2$. First, we will show that K_i is nonexpansive mapping for all $i = 1, 2$. For every $i = 1, 2$, let $x^*, x_0 \in C$. Using the same method as (2.12), we have

$$\begin{aligned} \|K_i x^* - K_i x_0\| =& \| \text{Prox}^i_{\gamma f}(I - \gamma_i(a_iA_i + (1-a_i)B_i)))x^* \\ & - \text{Prox}^i_{\gamma f}(I - \gamma_i(a_iA_i + (1-a_i)B_i)))x_0\|^2 \\ \leq& \|x^* - x_0\|^2 + a_i\gamma_i(\gamma_i - 2\delta_i)\|A_i x^* - A_i x_0\|^2 \\ & + (1-a_i)\gamma_i(\gamma_i - 2\delta_i)\|B_i x^* - B_i x_0\|^2 \\ \leq& \|x^* - x_0\|^2. \end{aligned} \tag{3.22}$$

Thus, $\text{Prox}^i_{\gamma f}(I - \gamma_i(a_iA_i + (1-a_i)B_i)))$ is nonexpansive mapping for all $i = 1, 2$. Assume that $x^* \in \Omega_1$ and $y^* \in \Omega_2$.

From the definition of x_n, u_n and the nonexpansiveness of K_i, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| =& \|\mu_n x_n + \beta_n P_C(\alpha_n \xi_1(y_n) + (1-\alpha_n)u_n) - x^*\| \\
=& \|\mu_n(x_n - x^*) + \beta_n(P_C(\alpha_n \xi_1(y_n) + (1-\alpha_n)u_n) - x^*) \\
& - (1-\mu_n-\beta_n)x^*\| \\
\leq& \mu_n\|x_n - x^*\| + \beta_n\|\alpha_n\xi_1(y_n) + (1-\alpha_n)u_n - x^*\| \\
& + (1-\mu_n-\beta_n)\|x^*\| \\
\leq& \mu_n\|x_n - x^*\| + \beta_n(\alpha_n\|\xi_1(y_n) - x^*\| + (1-\alpha_n)\|u_n - x^*\|) \\
& + (1-\mu_n-\beta_n)\|x^*\| \\
=& \mu_n\|x_n - x^*\| + \beta_n(\alpha_n\|\xi_1(y_n) - x^*\| + (1-\alpha_n)\|K_1 x_n - x^*\|) \\
& + (1-\mu_n-\beta_n)\|x^*\| \\
\leq& \mu_n\|x_n - x^*\| + \beta_n(\alpha_n\|\xi_1(y_n) - x^*\| + (1-\alpha_n)\|x_n - x^*\|) \\
& + (1-\mu_n-\beta_n)\|x^*\| \\
\leq& (1-\beta_n)\|x_n - x^*\| + \beta_n\alpha_n\|\xi_1(y_n) - x^*\| + \beta_n(1-\alpha_n)\|x_n - x^*\| \\
& + (1-\mu_n-\beta_n)\|x^*\| \\
=& (1-\alpha_n\beta_n)\|x_n - x^*\| + \alpha_n\beta_n\|\xi_1(y_n) - x^*\| \\
& + (1-\mu_n-\beta_n)\|x^*\| \\
\leq& (1-\alpha_n\beta_n)\|x_n - x^*\| + \alpha_n\beta_n(\|\xi_1(y_n) - \xi_1(y^*)\| + \|\xi_1(y^*) - x^*\|) \\
& + (1-\mu_n-\beta_n)\|x^*\| \\
\leq& (1-\alpha_n\beta_n)\|x_n - x^*\| + \alpha_n\beta_n\sigma_1\|y_n - y^*\| + \alpha_n\beta_n\|\xi_1(y^*) - x^*\| \\
& + (1-\mu_n-\beta_n)\|x^*\| \\
\leq& (1-\alpha_n\beta_n)\|x_n - x^*\| + \alpha_n\beta_n\sigma\|y_n - y^*\| + \alpha_n\beta_n\|\xi_1(y^*) - x^*\| \\
& + (1-\mu_n-\beta_n)\|x^*\|. \qquad (3.23)
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\|y_{n+1} - y^*\| \leq& (1-\alpha_n\beta_n)\|y_n - y^*\| + \alpha_n\beta_n\sigma\|x_n - x^*\| + \alpha_n\beta_n\|\xi_2(x^*) - y^*\| \\
& + (1-\mu_n-\beta_n)\|y^*\|. \qquad (3.24)
\end{aligned}$$

Combining (3.23) and (3.24), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| \leq& (1-\alpha_n\beta_n)\|x_n - x^*\| + \alpha_n\beta_n\sigma\|y_n - y^*\| \\
& + \alpha_n\beta_n\|\xi_1(y^*) - x^*\| + (1-\mu_n-\beta_n)\|x^*\| \\
& + (1-\alpha_n\beta_n)\|y_n - y^*\| + \alpha_n\beta_n\sigma\|x_n - x^*\| \\
& + \alpha_n\beta_n\|\xi_2(x^*) - y^*\| + (1-\mu_n-\beta_n)\|y^*\| \\
=& (1-\alpha_n\beta_n)\left(\|x_n - x^*\| + \|y_n - y^*\|\right) \\
& + \alpha_n\beta_n\sigma\left(\|x_n - x^*\| + \|y_n - y^*\|\right) \\
& + \alpha_n\beta_n(\|\xi_1(y^*) - x^*\| + \|\xi_2(x^*) - y^*\|) \\
& + (1-\mu_n-\beta_n)(\|x^*\| + \|y^*\|)
\end{aligned}$$

$$
\begin{aligned}
&=(1-\alpha_n\beta_n(1-\sigma))(\|x_n-x^*\|+\|y_n-y^*\|) \\
&\quad+\alpha_n\beta_n(\|\xi_1(y^*)-x^*\|+\|\xi_2(x^*)-y^*\|) \\
&\quad+(1-\mu_n-\beta_n)(\|x^*\|+\|y^*\|) \\
&=(1-\alpha_n\beta_n(1-\sigma))(\|x_n-x^*\|+\|y_n-y^*\|) \\
&\quad+\alpha_n\beta_n(1-\sigma)\left(\frac{\|\xi_1(y^*)-x^*\|+\|\xi_2(x^*)-y^*\|}{1-\sigma}\right) \\
&\quad+(1-\mu_n-\beta_n)(\|x^*\|+\|y^*\|).
\end{aligned}
$$

By Lemma 2.3, we get that $\{x_n\}$ and $\{y_n\}$ are bounded. Next, we will show that $\{u_n\}$, $\{v_n\}$, $\{\xi_1(y_n)\}$, and $\{\xi_2(x_n)\}$ are bounded. From the definition of u_n and the nonexpansiveness of K_1, we have

$$
\begin{aligned}
\|u_n-x^*\| &=\|K_1x_n-x^*\| \\
&\leq\|x_n-x^*\|.
\end{aligned}
$$

Since $\{x_n\}$ is bounded, then $\{u_n\}$ is bounded. Using the same method, we establish that the sequence v_n is bounded. Observe that

$$
\begin{aligned}
\|\xi_1(y_n)-x^*\| &\leq\|\xi_1(y_n)-\xi_1(x^*)\|+\|\xi_1(x^*)-x^*\| \\
&\leq\sigma_1\|x_n-x^*\|+\|\xi_1(x^*)-x^*\|.
\end{aligned}
$$

Since $\{x_n\}$ is bounded, then $\{\xi_1(y_n)\}$ is bounded. Using the same method, we show that the sequence $\{\xi_2(x_n)\}$ is bounded.

Setting $T_n=P_C(\alpha_n\xi_1(y_n)+(1-\alpha_n)u_n)$ and $T_n^*=P_C(\alpha_n\xi_2(x_n)+(1-\alpha_n)v_n)$. We will show that $\{T_n\}$ and $\{T_n^*\}$ are bounded. Observe that

$$
\begin{aligned}
\|T_n-x^*\| &=\|P_C(\alpha_n\xi_1(y_n)+(1-\alpha_n)u_n)-x^*\| \\
&\leq\|\alpha_n\xi_1(y_n)+(1-\alpha_n)u_n-x^*\| \\
&\leq\alpha_n\|\xi_1(y_n)-x^*\|+(1-\alpha_n)\|u_n-x^*\| \\
&\leq\alpha_n\|\xi_1(y_n)-\xi_1(x^*)\|+\alpha_n\|\xi_1(x^*)-x^*\|+(1-\alpha_n)\|u_n-x^*\| \\
&\leq\alpha_n\sigma_1\|x_n-x^*\|+\alpha_n\|\xi_1(x^*)-x^*\|+(1-\alpha_n)\|x_n-x^*\| \\
&=(1-\alpha_n(1-\sigma_1))\|x_n-x^*\|+\alpha_n\|\xi_1(x^*)-x^*\|.
\end{aligned}
$$

Since $\{x_n\}$ is bounded, then $\{T_n\}$ is bounded. Using the same method, we show that the sequence $\{T_n^*\}$ is bounded.

Next, we show that $\|x_{n+1}-x_n\|\to 0$ and $\|y_{n+1}-y_n\|\to 0$ as $n\to\infty$.
By the nonexpansiveness of K_i, we have

$$\begin{aligned}
\|T_n-T_{n-1}\| =&\|P_C(\alpha_n\xi_1(y_n)+(1-\alpha_n)u_n)-P_C(\alpha_{n-1}\xi_1(y_{n-1})+(1-\alpha_{n-1})u_{n-1})\|\\
\le&\|(\alpha_n\xi_1(y_n)+(1-\alpha_n)u_n)-(\alpha_{n-1}\xi_1(y_{n-1})+(1-\alpha_{n-1})u_{n-1})\|\\
=&\|\alpha_n\xi_1(y_n)-\alpha_n\xi_1(y_{n-1})+\alpha_n\xi_1(y_{n-1})+(1-\alpha_n)u_n-(1-\alpha_n)u_{n-1}\\
&+(1-\alpha_n)u_{n-1}-\alpha_{n-1}\xi_1(y_{n-1})-(1-\alpha_{n-1})u_{n-1}\|\\
=&\|\alpha_n(\xi_1(y_n)-\xi_1(y_{n-1}))+(\alpha_n-\alpha_{n-1})\xi_1(y_{n-1})+(1-\alpha_n)(u_n-u_{n-1})\\
&+(\alpha_{n-1}-\alpha_n)u_{n-1}\|\\
\le&\alpha_n\|\xi_1(y_n)-\xi_1(y_{n-1})\|+|\alpha_n-\alpha_{n-1}|\|\xi_1(y_{n-1})\|\\
&+(1-\alpha_n)\|K_1x_n-K_1x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|u_{n-1}\|\\
\le&\alpha_n\sigma_1\|y_n-y_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|\xi_1(y_{n-1})\|+(1-\alpha_n)\|x_n-x_{n-1}\|\\
&+|\alpha_n-\alpha_{n-1}|\|u_{n-1}\|\\
\le&\alpha_n\sigma\|y_n-y_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|\xi_1(y_{n-1})\|+(1-\alpha_n)\|x_n-x_{n-1}\|\\
&+|\alpha_n-\alpha_{n-1}|\|u_{n-1}\|. \qquad (3.25)
\end{aligned}$$

From the definition of x_n and (3.25), we have

$$\begin{aligned}
\|x_{n+1}-x_n\| =&\|\mu_nx_n+\beta_nT_n-(\mu_{n-1}x_{n-1}+\beta_{n-1}T_{n-1})\|\\
\le&\mu_n\|x_n-x_{n-1}\|+|\mu_n-\mu_{n-1}|\|x_{n-1}\|\\
&+\beta_n\|T_n-T_{n-1}\|+|\beta_n-\beta_{n-1}|\|T_{n-1}\|\\
\le&\mu_n\|x_n-x_{n-1}\|+|\mu_n-\mu_{n-1}|\|x_{n-1}\|\\
&+\beta_n\Big(\alpha_n\sigma\|y_n-y_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|\xi_1(y_{n-1})\|\\
&+(1-\alpha_n)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|u_{n-1}\|\Big)\\
&+|\beta_n-\beta_{n-1}|\|T_{n-1}\|\\
\le&(1-\alpha_n\beta_n)\|x_n-x_{n-1}\|+|\mu_n-\mu_{n-1}|\|x_{n-1}\|\\
&+\alpha_n\beta_n\sigma\|y_n-y_{n-1}\|+|\alpha_n-\alpha_{n-1}|\Big(\|\xi_1(y_{n-1})\|+\|u_{n-1}\|\Big)\\
&+|\beta_n-\beta_{n-1}|\|T_{n-1}\|. \qquad (3.26)
\end{aligned}$$

Using the same method as derived in (3.26), we have

$$\begin{aligned}
\|y_{n+1}-y_n\| \le&(1-\alpha_n\beta_n)\|y_n-y_{n-1}\|+|\mu_n-\mu_{n-1}|\|y_{n-1}\|+\alpha_n\beta_n\sigma\|x_n-x_{n-1}\|\\
&+|\alpha_n-\alpha_{n-1}|\Big(\|\xi_2(x_{n-1})\|+\|v_{n-1}\|\Big)+|\beta_n-\beta_{n-1}|\|T^*_{n-1}\|. \qquad (3.27)
\end{aligned}$$

From (3.26) and (3.27), we have

$$\begin{aligned}
\|x_{n+1}-x_n\|+\|y_{n+1}-y_n\| \le&(1-(1-\sigma)\beta_n\alpha_n)\left(\|x_n-x_{n-1}\|+\|y_n-y_{n-1}\|\right)\\
&+|\mu_n-\mu_{n-1}|\left(\|x_{n-1}\|+\|y_{n-1}\|\right)\\
&+|\beta_n-\beta_{n-1}|\left(\|T_{n-1}\|+\|T^*_{n-1}\|\right)\\
&+|\alpha_n-\alpha_{n-1}|\big(\|\xi_1(y_{n-1})\|+\|u_{n-1}\|\\
&+\|\xi_2(x_{n-1})\|+\|v_{n-1}\|\big).
\end{aligned}$$

Applying Lemma 2.3 and the condition (iii), we can conclude that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0 \text{ and } \lim_{n\to\infty}\|y_{n+1}-y_n\|=0. \qquad (3.28)$$

Next, we show that $\|x_n - U_n\| \to 0$ as $n \to \infty$ where $U_n = \alpha_n \xi_1(y_n) + (1-\alpha_n)u_n$, $\|y_n - V_n\| \to 0$ where $V_n = \alpha_n \xi_2(x_n) + (1-\alpha_n)v_n$ as $n \to \infty$.
From the definition of x_n, we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 =& \|\mu_n x_n + \beta_n P_C U_n - x^*\|^2 \\
=& \|\mu_n(x_n - x^*) + \beta_n(P_C U_n - x^*) - (1-\mu_n-\beta_n)x^*\|^2 \\
\le& \|\mu_n(x_n - x^*) + \beta_n(P_C U_n - x^*)\|^2 \\
& - 2(1-\mu_n-\beta_n)\langle x^*, x_{n+1} - x^*\rangle \\
\le& \|\mu_n(x_n - x^*) + \beta_n(P_C U_n - x^*)\|^2 \\
& + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
=& \mu_n(\mu_n+\beta_n)\|x_n - x^*\|^2 + \beta_n(\mu_n+\beta_n)\|P_C U_n - x^*\|^2 \\
& - \mu_n\beta_n\|x_n - P_C U_n\|^2 + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\le& \mu_n(\mu_n+\beta_n)\|x_n - x^*\|^2 + \beta_n(\mu_n+\beta_n)\|U_n - x^*\|^2 \\
& - \mu_n\beta_n\|x_n - P_C U_n\|^2 + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
=& \mu_n(\mu_n+\beta_n)\|x_n - x^*\|^2 \\
& + \beta_n(\mu_n+\beta_n)\|\alpha_n(\xi_1(y_n) - u_n) + (u_n - x^*)\|^2 \\
& - \mu_n\beta_n\|x_n - P_C U_n\|^2 + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\le& \mu_n(\mu_n+\beta_n)\|x_n - x^*\|^2 + \beta_n(\mu_n+\beta_n)\Big(\|u_n - x^*\|^2 \\
& + 2\alpha_n\langle \xi_1(y_n) - u_n, \alpha_n\xi_1(y_n) + (1-\alpha_n)u_n - x^*\rangle\Big) \\
& - \mu_n\beta_n\|x_n - P_C U_n\|^2 + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\le& \mu_n(\mu_n+\beta_n)\|x_n - x^*\|^2 + \beta_n(\mu_n+\beta_n)\Big(\|u_n - x^*\|^2 \\
& + 2\alpha_n\|\xi_1(y_n) - u_n\|\|\alpha_n\xi_1(y_n) + (1-\alpha_n)u_n - x^*\|\Big) \\
& - \mu_n\beta_n\|x_n - P_C U_n\|^2 + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\le& \|x_n - x^*\|^2 + 2\alpha_n\beta_n\|\xi_1(y_n) - u_n\|\|\alpha_n\xi_1(y_n) + (1-\alpha_n)u_n - x^*\| \\
& - \mu_n\beta_n\|x_n - P_C U_n\|^2 + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\|.
\end{aligned} \tag{3.29}$$

It follows from (3.29) that

$$\begin{aligned}
\mu_n\beta_n\|x_n - P_C U_n\|^2 \le& \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
& + 2\alpha_n\beta_n\|\xi_1(y_n) - u_n\|\|\alpha_n\xi_1(y_n) + (1-\alpha_n)u_n - x^*\| \\
& + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\le& \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
& + 2\alpha_n\beta_n\|\xi_1(y_n) - u_n\|\|\alpha_n\xi_1(y_n) + (1-\alpha_n)u_n - x^*\| \\
& + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\|.
\end{aligned}$$

By (3.28), the condition (i)- (iii), we get

$$\lim_{n\to\infty} \|P_C U_n - x_n\| = 0. \tag{3.30}$$

From definition of y_n and applying the same method as (3.30), we have

$$\lim_{n\to\infty} \|P_C V_n - y_n\| = 0. \tag{3.31}$$

From Lemma 2.1, we obtain

$$\|P_C U_n - x^*\|^2 \leq \|U_n - x^*\|^2 - \|U_n - P_C U_n\|^2. \tag{3.32}$$

From the definition of U_n, we get

$$\begin{aligned}
\|U_n - x^*\|^2 =& \|\alpha_n(\xi_1(y_n) - x^*) + (1-\alpha_n)(u_n - x^*)\|^2 \\
\leq& \alpha_n\|\xi_1(y_n) - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 \\
\leq& \alpha_n\|\xi_1(y_n) - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2.
\end{aligned} \tag{3.33}$$

From (3.32), (3.33), and Lemma 2.2, we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 =& \|\mu_n x_n + \beta_n P_C U_n - x^*\|^2 \\
\leq& \mu_n(\mu_n + \beta_n)\|x_n - x^*\|^2 + \beta_n(\mu_n + \beta_n)\|P_C U_n - x^*\|^2 \\
& - \mu_n\beta_n\|x_n - P_C U_n\|^2 + 2(1 - \mu_n - \beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\leq& \mu_n(\mu_n + \beta_n)\|x_n - x^*\|^2 + \beta_n(\mu_n + \beta_n)(\|U_n - x^*\|^2 - \|U_n - P_C U_n\|^2) \\
& - \mu_n\beta_n\|x_n - P_C U_n\|^2 + 2(1 - \mu_n - \beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\leq& \mu_n(\mu_n + \beta_n)\|x_n - x^*\|^2 + \beta_n(\mu_n + \beta_n)(\alpha_n\|\xi_1(y_n) - x^*\|^2 \\
& + (1-\alpha_n)\|x_n - x^*\|^2 - \|U_n - P_C U_n\|^2) - \mu_n\beta_n\|x_n - P_C U_n\|^2 \\
& + 2(1 - \mu_n - \beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
=& \mu_n\|x_n - x^*\|^2 + \beta_n\alpha_n\|\xi_1(y_n) - x^*\|^2 + \beta_n(1-\alpha_n)\|x_n - x^*\|^2 \\
& - \beta_n(\mu_n + \beta_n)\|U_n - P_C U_n\|^2 - \mu_n\beta_n\|x_n - P_C U_n\|^2 \\
& + 2(1 - \mu_n - \beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
=& (\mu_n + \beta_n(1-\alpha_n))\|x_n - x^*\|^2 + \beta_n\alpha_n\|\xi_1(y_n) - x^*\|^2 \\
& - \beta_n(\mu_n + \beta_n)\|U_n - P_C U_n\|^2 - \mu_n\beta_n\|x_n - P_C U_n\|^2 \\
& + 2(1 - \mu_n - \beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\leq& \|x_n - x^*\|^2 + \beta_n\alpha_n\|\xi_1(y_n) - x^*\|^2 - \beta_n(\mu_n + \beta_n)\|U_n - P_C U_n\|^2 \\
& + 2(1 - \mu_n - \beta_n)\|x^*\|\|x_{n+1} - x^*\|,
\end{aligned}$$

it follows that

$$\begin{aligned}
\beta_n(\mu_n + \beta_n)\|U_n - P_C U_n\|^2 \leq& \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n\beta_n\|\xi_1(y_n) - x^*\|^2 \\
& + 2(1 - \mu_n - \beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\leq& \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
& + \alpha_n\beta_n\|\xi_1(y_n) - x^*\|^2 \\
& + 2(1 - \mu_n - \beta_n)\|x^*\|\|x_{n+1} - x^*\|.
\end{aligned}$$

From $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ and the condition (i), (ii), we have

$$\lim_{n\to\infty} \|U_n - P_C U_n\| = 0. \tag{3.34}$$

Applying the same argument as (3.34) to the definition of V_n also yields

$$\lim_{n\to\infty} \|V_n - P_C V_n\| = 0. \tag{3.35}$$

Consider

$$\begin{aligned}\|x_n - U_n\| =&\|x_n - P_C U_n + P_C U_n - U_n\|\\ \leq&\|x_n - P_C U_n\| + \|P_C U_n - U_n\|.\end{aligned}$$

From (3.30) and (3.34), we have

(3.36) $$\lim_{n\to\infty} \|x_n - U_n\| = 0.$$

Using the same methodology as (3.36) and the definition of y_n, we also have

(3.37) $$\lim_{n\to\infty} \|y_n - V_n\| = 0.$$

Next, we show that $\|x_n - u_n\| \to 0$ as $n \to \infty$ and $\|y_n - v_n\| \to 0$ as $n \to \infty$. Observe that

$$U_n - x_n = \alpha_n(\xi_1(y_n) - x_n) + (1 - \alpha_n)(u_n - x_n),$$

this implies that

(3.38) $$(1 - \alpha_n)\|u_n - x_n\| \leq \|U_n - x_n\| + \alpha_n\|\xi_1(y_n) - x_n\|.$$

From (3.36), (3.38), and the condition (i), we have

(3.39) $$\lim_{n\to\infty} \|u_n - x_n\| = \lim_{n\to\infty} \| \operatorname{Prox}^1_{\gamma f}(I - \gamma_1(a_1A_1 + (1 - a_1)B_1))x_n - x_n\| = 0.$$

Applying the same argument as (3.39), we also obtain

(3.40) $$\lim_{n\to\infty} \|v_n - y_n\| = \lim_{n\to\infty} \| \operatorname{Prox}^2_{\gamma f}(I - \gamma_2(a_2A_2 + (1 - a_2)B_2))y_n - y_n\| = 0.$$

Next, we show that $\limsup_{n\to\infty}\langle\xi_1(y^*) - x^*, U_n - x^*\rangle \leq 0$, where $x^* = P_{\Omega_1}\xi_1(y^*)$ and $\limsup_{n\to\infty}\langle\xi_2(x^*) - y^*, V_n - y^*\rangle \leq 0$, where $y^* = P_{\Omega_2}\xi_2(x^*)$. Indeed, take a subsequence $\{U_{n_k}\}$ of $\{U_n\}$ such that

$$\limsup_{n\to\infty}\langle\xi_1(y^*) - x^*, U_n - x^*\rangle = \lim_{k\to\infty}\langle\xi_1(y^*) - x^*, U_{n_k} - x^*\rangle.$$

Since $\{x_n\}$, $\{y_n\}$ are bounded, without loss of generality, we may assume that $x_{n_k} \rightharpoonup \bar{x}$ and $y_{n_k} \rightharpoonup \bar{y}$ as $k \to \infty$, respectively. Since C is closed and convex, C is weakly closed. So, we obtain $\bar{x}, \bar{y} \in C$.

Since $\operatorname{Prox}^i_{\gamma f}(I - \gamma_i(a_iA_i + (1 - a_i)B_i))$ is nonexpansive, for all $i = 1, 2$, (3.39), and (3.40), it follows from Lemma 2.5 that $\bar{x} \in Fix(\operatorname{Prox}^1_{\gamma f}(I - \gamma_1(a_1A_1 + (1 - a_1)B_1)))$ and $\bar{y} \in Fix(\operatorname{Prox}^2_{\gamma f}(I - \gamma_2(a_2A_2 + (1 - a_2)B_2)))$.

By Lemma 2.4, we have

(3.41) $$\bar{x} \in Fix(Prox^1_{\gamma f}(I - \gamma_1 A_1)) \cap Fix(Prox^1_{\gamma f}(I - \gamma_1 B_1)) = \Omega_1.$$

and

(3.42) $$\bar{y} \in Fix(Prox^2_{\gamma f}(I - \gamma_2 A_2)) \cap Fix(Prox^2_{\gamma f}(I - \gamma_2 B_2)) = \Omega_2.$$

From (3.36), we obtain $U_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$. Since $U_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$, $\bar{x} \in \Omega_1$ and Lemma 2.1, we can derive that

$$\begin{aligned}\limsup_{n\to\infty}\langle\xi_1(y^*) - x^*, U_n - x^*\rangle =& \lim_{k\to\infty}\langle\xi_1(y^*) - x^*, U_{n_k} - x^*\rangle\\ =&\langle\xi_1(y^*) - x^*, \bar{x} - x^*\rangle\\ \leq&0.\end{aligned} \tag{3.43}$$

Similarly, indeed, take a subsequence $\{V_{n_k}\}$ of $\{V_n\}$ such that

$$\limsup_{n\to\infty}\langle\xi_2(x^*) - y^*, V_n - y^*\rangle = \lim_{k\to\infty}\langle\xi_2(x^*) - y^*, V_{n_k} - y^*\rangle.$$

From (3.37), we obtain $V_{n_k} \rightharpoonup \bar{y}$ as $k \to \infty$.
Following the same method as (3.43), we easily obtain that

$$\limsup_{n\to\infty}\langle \xi_2(x^*) - y^*, V_n - y^*\rangle \leq 0. \tag{3.44}$$

Finally, we show that $\{x_n\}$ converses strongly to x^*, where $x^* = P_{\Omega_1}\xi_1(y^*)$ and $\{y_n\}$ converses strongly to y^*, where $y^* = P_{\Omega_2}\xi_2(x^*)$.
Since $U_n = \alpha_n\xi_1(y_n) + (1-\alpha_n)u_n$ and $V_n = \alpha_n\xi_2(x_n) + (1-\alpha_n)v_n$ and the definition of x_n, we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 =& \|\mu_n x_n + \beta_n P_C U_n - x^*\|^2 \\
\leq& \mu_n(\mu_n+\beta_n)\|x_n - x^*\|^2 + \beta_n(\mu_n+\beta_n)\|\alpha_n(\xi_1(y_n) - x^*) \\
&+ (1-\alpha_n)(u_n - x^*)\|^2 + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\leq& \mu_n(\mu_n+\beta_n)\|x_n - x^*\|^2 + \beta_n(\mu_n+\beta_n)\Big((1-\alpha_n)\|u_n - x^*\|^2 \\
&+ 2\alpha_n\langle \xi_1(y_n) - x^*, U_n - x^*\rangle\Big) \\
&+ 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\leq& \mu_n(\mu_n+\beta_n)\|x_n - x^*\|^2 + \beta_n(\mu_n+\beta_n)\Big((1-\alpha_n)\|x_n - x^*\|^2 \\
&+ 2\alpha_n\langle \xi_1(y_n) - x^*, U_n - x^*\rangle\Big) \\
&+ 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\leq& \mu_n\|x_n - x^*\|^2 + \beta_n(1-\alpha_n)\|x_n - x^*\|^2 \\
&+ 2\alpha_n\beta_n(\mu_n+\beta_n)\langle \xi_1(y_n) - x^*, U_n - x^*\rangle \\
&+ 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
=& (\mu_n + \beta_n(1-\alpha_n))\|x_n - x^*\|^2 + 2\alpha_n\beta_n(\mu_n+\beta_n)\langle \xi_1(y_n) - x^*, U_n - x^*\rangle \\
&+ 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
=& ((\mu_n+\beta_n) - \alpha_n\beta_n)\|x_n - x^*\|^2 \\
&+ 2\alpha_n\beta_n(\mu_n+\beta_n)\Big(\langle \xi_1(y_n) - \xi_1(y^*), U_n - x^*\rangle + \langle \xi_1(y^*) - x^*, U_n - x^*\rangle\Big) \\
&+ 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\leq& (1-\alpha_n\beta_n)\|x_n - x^*\|^2 \\
&+ 2\alpha_n\beta_n(\mu_n+\beta_n)(\|\xi_1(y_n) - \xi_1(y^*)\|\|U_n - x^*\| + \langle \xi_1(y^*) - x^*, U_n - x^*\rangle) \\
&+ 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\leq& (1-\alpha_n\beta_n)\|x_n - x^*\|^2 \\
&+ 2\alpha_n\beta_n\|\xi_1(y_n) - \xi_1(y^*)\|(\|U_n - x_{n+1}\| + \|x_{n+1} - x^*\|) \\
&+ 2\alpha_n\beta_n(\mu_n+\beta_n)\langle \xi_1(y^*) - x^*, U_n - x^*\rangle \\
&+ 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\|
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n\beta_n)\|x_n - x^*\|^2 \\
&\quad + 2\alpha_n\beta_n\sigma\|y_n - y^*\|\|U_n - x_{n+1}\| + 2\alpha_n\beta_n\sigma\|y_n - y^*\|\|x_{n+1} - x^*\| \\
&\quad + 2\alpha_n\beta_n(\mu_n + \beta_n)\langle \xi_1(y^*) - x^*, U_n - x^*\rangle \\
&\quad + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
&\leq (1-\alpha_n\beta_n)\|x_n - x^*\|^2 \\
&\quad + 2\alpha_n\beta_n\sigma\|y_n - y^*\|\|U_n - x_{n+1}\| + \alpha_n\beta_n\sigma\left(\|y_n - y^*\|^2 + \|x_{n+1} - x^*\|^2\right) \\
&\quad + 2\alpha_n\beta_n(\mu_n + \beta_n)\langle \xi_1(y^*) - x^*, U_n - x^*\rangle \\
&\quad + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\|,
\end{aligned}
$$

which yields that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 \leq& \frac{1-\alpha_n\beta_n}{1-\alpha_n\beta_n\sigma}\|x_n - x^*\|^2 + \frac{2\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|y_n - y^*\|\|U_n - x_{n+1}\| \\
&+ \frac{\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|y_n - y^*\|^2 + \frac{2\alpha_n\beta_n(\mu_n+\beta_n)}{1-\alpha_n\beta_n\sigma}\langle \xi_1(y^*) - x^*, U_n - x^*\rangle \\
&+ \frac{2(1-\mu_n-\beta_n)}{1-\alpha_n\beta_n\sigma}\|x^*\|\|x_{n+1} - x^*\| \\
=&\Big(1 - \frac{\alpha_n\beta_n - \alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\Big)\|x_n - x^*\|^2 + \frac{2\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|y_n - y^*\|\|U_n - x_{n+1}\| \\
&+ \frac{\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|y_n - y^*\|^2 + \frac{2\alpha_n\beta_n(\mu_n+\beta_n)}{1-\alpha_n\beta_n\sigma}\langle \xi_1(y^*) - x^*, U_n - x^*\rangle \\
&+ \frac{2(1-\mu_n-\beta_n)}{1-\alpha_n\beta_n\sigma}\|x^*\|\|x_{n+1} - x^*\| \\
=&\Big(1 - \frac{\alpha_n\beta_n(1-\sigma)}{1-\alpha_n\beta_n\sigma}\Big)\|x_n - x^*\|^2 + \frac{2\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|y_n - y^*\|\|U_n - x_{n+1}\| \\
&+ \frac{\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|y_n - y^*\|^2 + \frac{2\alpha_n\beta_n(\mu_n+\beta_n)}{1-\alpha_n\beta_n\sigma}\langle \xi_1(y^*) - x^*, U_n - x^*\rangle \\
&+ \frac{2(1-\mu_n-\beta_n)}{1-\alpha_n\beta_n\sigma}\|x^*\|\|x_{n+1} - x^*\|,
\end{aligned}
$$

there exists $\hat{M} > 0$, such that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 \leq& \Big(1 - \frac{\alpha_n\beta_n(1-\sigma)}{1-\alpha_n\beta_n\sigma}\Big)\|x_n - x^*\|^2 + \frac{2\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|y_n - y^*\|\|U_n - x_{n+1}\| \\
&+ \frac{\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|y_n - y^*\|^2 + \frac{2\alpha_n\beta_n(\mu_n+\beta_n)}{1-\alpha_n\beta_n\sigma}\langle \xi_1(y^*) - x^*, U_n - x^*\rangle \\
&+ (1-\mu_n-\beta_n)\hat{M}\|x^*\|\|x_{n+1} - x^*\|.
\end{aligned}
\tag{3.45}
$$

Similarly, as previously stated, there exists $\bar{M} > 0$, such that

$$
\begin{aligned}
\|y_{n+1} - y^*\|^2 \leq& \Big(1 - \frac{\alpha_n\beta_n(1-\sigma)}{1-\alpha_n\beta_n\sigma}\Big)\|y_n - y^*\|^2 + \frac{2\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|x_n - x^*\|\|V_n - y_{n+1}\| \\
&+ \frac{\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|x_n - x^*\|^2 + \frac{2\alpha_n\beta_n(\mu_n+\beta_n)}{1-\alpha_n\beta_n\sigma}\langle \xi_2(x^*) - y^*, V_n - y^*\rangle \\
&+ (1-\mu_n-\beta_n)\bar{M}\|y^*\|\|y_{n+1} - y^*\|.
\end{aligned}
\tag{3.46}
$$

From (3.45), (3.46), and choose $M = \max\{\hat{M}, \bar{M}\}$, we get

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 \\
\le&\Big(1 - \frac{\alpha_n\beta_n(1-\sigma)}{1-\alpha_n\beta_n\sigma}\Big)(\|x_n - x^*\|^2 + \|y_n - y^*\|^2) \\
&+ \frac{2\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}(\|y_n - y^*\|\|U_n - x_{n+1}\| + \|x_n - x^*\|\|V_n - y_{n+1}\|) \\
&+ \frac{\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}(\|x_n - x^*\|^2 + \|y_n - y^*\|^2) \\
&+ \frac{2\alpha_n\beta_n(\mu_n+\beta_n)}{1-\alpha_n\beta_n\sigma}(\langle \xi_1(y^*) - x^*, U_n - x^*\rangle + \langle \xi_2(x^*) - y^*, V_n - y^*\rangle) \\
&+ (1-\mu_n-\beta_n)M\,(\|x^*\|\|x_{n+1} - x^*\| + \|y^*\|\|y_{n+1} - y^*\|) \\
=&\Big(1 - \frac{\alpha_n\beta_n(1-2\sigma)}{1-\alpha_n\beta_n\sigma}\Big)(\|x_n - x^*\|^2 + \|y_n - y^*\|^2) \\
&+ \frac{2\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}(\|y_n - y^*\|\|U_n - x_{n+1}\| + \|x_n - x^*\|\|V_n - y_{n+1}\|) \\
&+ \frac{2\alpha_n\beta_n(\mu_n+\beta_n)}{1-\alpha_n\beta_n\sigma}(\langle \xi_1(y^*) - x^*, U_n - x^*\rangle + \langle \xi_2(x^*) - y^*, V_n - y^*\rangle) \\
&+ (1-\mu_n-\beta_n)M\,(\|x^*\|\|x_{n+1} - x^*\| + \|y^*\|\|y_{n+1} - y^*\|)\,.
\end{aligned} \tag{3.47}$$

By (3.28), (3.36), (3.37), (3.43), (3.44), the condition (i), (iii) and Lemma 2.3, we have $\lim_{n\to\infty}(\|x_n - x^*\| + \|y_n - y^*\|) = 0$. It implies that the sequences $\{x_n\}$, $\{y_n\}$ converge to $x^* = P_{\Omega_1}\xi_1(y^*)$, $y^* = P_{\Omega_2}\xi_2(x^*)$, respectively. This completes the proof. □

Remark 3.1. We have the following observations for the offered Algorithms 1.

(1) It should be noted that we use a new mathematical tool (Lemma 2.4) related to two proximal operators that exploits the information of v_n and u_n, which actually draws inspiration from Xu [29] and Guo and Cui [8].

(2) By combining the proximal-gradient algorithm with viscosity technique in Guo and Cui [8], the intermixed algorithm in Yao et al.[32] and the Krasnoselskii–Mann algorithm in Kanzow and Shehu [12], the algorithm presented in this paper provide a strong convergence theorem in real Hilbert spaces.

4. Applications

In this section, we reduce our main problem to the following the split feasibility problems.

4.1. **The Split Feasibility Problem.** Let C and Q be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. The *split feasibility problem* is to find a point

$$x \in C \text{ such that } Ax \in Q, \tag{4.48}$$

where $A : H_1 \to H_2$ is a bounded linear operator. The set of all solution (SFP) is denoted by $\Gamma = \{x \in C; Ax \in Q\}$. The split feasibility problem is the first example of the split inverse problem, which was first introduced by Censor and Elfving [5] in Euclidean spaces.

Proposition 4.2. *([6]) Given $x^* \in \mathcal{H}_1$, the following statements are equivalent.*

(i) x^* solves the Γ;

(ii) $P_C(I - \lambda A^*(I - P_Q)A)x^* = x^*$, where A^* is the adjoint of A;

(iii) x^* *solves the variational inequality problem of finding* $x^* \in C$ *such that*

$$\langle \nabla \mathcal{Q}(x^*), x - x^* \rangle \geq 0, \qquad \forall x \in C, \tag{4.49}$$

where $\nabla \mathcal{Q} = A^*(I - P_Q)A$.

If C is a closed convex subset of H and the function f is the indicator function of C then it is well known that $\text{Prox}_{\gamma f} = P_C$, the projection operator of H, onto the closed convex set C and putting $A_i = B_i$ for all $i = 1, 2$ in Theorem 3.1. Consequently, the following result can be obtain from Theorem 3.1.

Algorithm 2: An intermixed algorithm with viscosity technique for solving the split feasibility problems.

Initialization: Given $x_1, y_1 \in C$ be arbitrary.

Iterative Steps: Given the current iterate $\{x_n\}$, $\{y_n\}$, calculate $\{x_{n+1}\}$, $\{y_{n+1}\}$ as follows:

Step 1: Compute

$$\begin{cases} v_n = P_C(I - \gamma_2 \nabla \mathcal{Q}_2) y_n) \\ u_n = P_C(I - \gamma_1 \nabla \mathcal{Q}_1) x_n) \end{cases}$$

Step 2: Compute

$$\begin{cases} y_{n+1} = \mu_n y_n + \beta_n P_C(\alpha_n \xi_2(x_n) + (1 - \alpha_n) v_n) \\ x_{n+1} = \mu_n x_n + \beta_n P_C(\alpha_n \xi_1(y_n) + (1 - \alpha_n) u_n), \end{cases}$$

where $\nabla \mathcal{Q}_i = A_i^*(I - P_Q)A_i$, $\gamma_i \in (0, \frac{2}{\|A_i\|^2})$, $\{\mu_n\}$, $\{\beta_n\}, \{\alpha_n\} \subseteq [0, 1]$, with $\mu_n + \beta_n \leq 1$.

Set $n := n + 1$ and go to Step 1.

From Proposition 4.2, it is clear that the solution of the problem (4.49) is the same as the problem (4.48). By applying the aforementioned technique, it is possible to find the solution to the two-split feasibility problem, as demonstrated in the following theorem.

Theorem 4.2. *Let* H_1 *and* H_2 *be real Hilbert spaces and let* C*,* Q *be two nonempty closed convex subsets of real Hilbert spaces* H_1 *and* H_2*, respectively. Let* $A_1, A_2 : H_1 \to H_2$ *be bounded linear operators with* A_1^*, A_2^* *are adjoint of* A_1 *and* A_2*, respectively. Assume that* $\Gamma_1 = \{x^* \in C; A_1 x^* \in Q\} \neq \emptyset$ *and* $\Gamma_2 = \{y^* \in C; A_1 y^* \in Q\} \neq \emptyset$*. Let* $\xi_1, \xi_2 : H \to H$ *be* σ_1 *and* σ_2*-contraction mappings with* $\sigma_1, \sigma_2 \in (0, 1)$ *and* $\sigma = \max\{\sigma_1, \sigma_2\}$*. Let* $\{x_n\}$ *and* $\{y_n\}$ *be the sequences generated by Algorithm 2, satisfying the following conditions:*

(i) $\lim\limits_{n \to \infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) there are $\overline{\varepsilon}, l > 0$ *with* $0 < \overline{\varepsilon} \leq \mu_n, \beta_n \leq l < 1$ *for all* $n \in \mathbb{N}_+$*;*

(iii) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum\limits_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty, \sum\limits_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum\limits_{n=1}^{\infty} (1 - \mu_n - \beta_n) < \infty.$

Then $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $x^* = P_{\Gamma_1} \xi_1(y^*)$ *and* $y^* = P_{\Gamma_2} \xi_2(x^*)$*, respectively.*

Proof. Let $x^*, x_0 \in C$ and $\nabla \mathcal{Q}_i = A_i^*(I - P_Q)A_i$, for all $i = 1, 2$. First, we show that $\nabla \mathcal{Q}_i$ is $\dfrac{1}{\|A_i\|^2}$-inverse strongly monotone for all $i = 1, 2$. Since P_Q is firmly nonexpansive, then

P_Q is $\frac{1}{2}$-averaged mapping, thus $I - P_Q$ is 1-inverse strongly monotone. Observe that

$$\begin{aligned}\langle \nabla \mathcal{Q}_i(x^*) - \nabla \mathcal{Q}_i(x_0), x^* - x_0\rangle =& \langle A_i^*(I-P_Q)A_i x^* - A_i^*(I-P_Q)A_i x_0, x^* - x_0\rangle \\ =& \langle (I-P_Q)A_i x^* - (I-P_Q)A_i x_0, A_i x^* - A_i x_0\rangle \\ \geq& \|(I-P_Q)A_i x^* - (I-P_Q)A_i x_0\|^2 \\ \geq& \frac{1}{\|A_i\|^2}\cdot \|A_i^*(I-P_Q)A_i x^* - A_i^*(I-P_Q)A_i x_0\|^2 \\ =& \frac{1}{\|A_i\|^2}\cdot \|\nabla \mathcal{Q}_i(x^*) - \nabla \mathcal{Q}_i(x_0)\|^2.\end{aligned}$$

Then $\nabla \mathcal{Q}_i$ is $\frac{1}{\|A_i\|^2}$-inverse strongly monotone, for all $i = 1, 2$. So, we can conclude of Theorem 4.2 from Proposition 4.2 and Theorem 3.1. □

5. NUMERICAL EXAMPLES

In this section, we give some numerical examples to support our main theorem. All the numerical results are completed on Apple MacBook Pro with 2 GHz Quad-Core Intel Core i5. The program is implemented in Python 3.10.4.

Example 5.1. We consider our problem in the infinite-dimensional Hilbert space $H = L_2([0, 1])$ with the inner product defined by

$$\langle x, y\rangle := \int_0^1 x(t)y(t)dt,\ \forall x, y \in H$$

and the induced norm by

$$\|x\|_2 := \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}},\ \forall x \in H.$$

Let $C := \{x \in L_2([0, 1]) : \|x\| \leq 1\}$ be the unit ball. Then, we have

$$P_C(x(t)) = \begin{cases} x(t), & \text{if } \|x(t)\|_2 \leq 1, \\ \dfrac{x(t)}{\|x(t)\|_2}, & \text{if } \|x(t)\|_2 > 1. \end{cases} \tag{5.50}$$

Now take $f = \|\cdot\|_2$, the norm in $L_2([0, 1])$. Then, the proximal operator is given by

$$\mathrm{Prox}_{\gamma f}(x(t)) = \begin{cases} \left(1 - \dfrac{\gamma}{\|x(t)\|_2}\right) x(t), & \text{if } \|x(t)\|_2 \geq \gamma, \\ 0, & \text{if } \|x(t)\|_2 < \gamma. \end{cases} \tag{5.51}$$

This proximal operator is also known as the block soft thresholding operator.
For every $i = 1, 2$, let $A_i, B_i : C \to H$ defined by

$$A_1(x(t)) = x(t), A_2(x(t)) = \frac{x(t)}{3}, B_1(x(t)) = 5x(t) \text{ and } B_2(x(t)) = \frac{22x(t)}{7},$$

for all $t \in [0, 1]$, $x \in C$. For every $i = 1, 2$, we take the operator $\xi_i : H \to H$ to be defined as $\xi_1(x(t)) = \frac{x(t)}{25}$ and $\xi_2(x(t)) = \frac{x(t)}{36}$, for all $t \in [0, 1]$, $x \in H$. In Algorithm 1, choose $\alpha_n = \frac{1}{5n}$, $\mu_n = \frac{1}{(n+1)^2} - \frac{1}{(n+1)^4}$, $\beta_n = 1 - \frac{1}{(n+1)^2}$, $a_1 = 0.70$ and $a_2 = 0.20$, for all

$n \in \mathbb{N}$. So our Algorithm 1 has the following form:

(5.52)

$$\begin{cases} v_n = \mathrm{Prox}^2_{\gamma f}(y_n - \gamma_2(0.2A_2 + 0.8B_2)y_n) \\ u_n = \mathrm{Prox}^1_{\gamma f}(x_n - \gamma_1(0.7A_1 + 0.3B_1)x_n) \\ y_{n+1} = \left(\frac{1}{(n+1)^2} - \frac{1}{(n+1)^4}\right) y_n + \left(1 - \frac{1}{(n+1)^2}\right) P_C\left(\frac{1}{5n}\xi_2(x_n) + \left(1 - \frac{1}{5n}\right) v_n\right) \\ x_{n+1} = \left(\frac{1}{(n+1)^2} - \frac{1}{(n+1)^4}\right) x_n + \left(1 - \frac{1}{(n+1)^2}\right) P_C\left(\frac{1}{5n}\xi_1(y_n) + \left(1 - \frac{1}{5n}\right) u_n\right). \end{cases}$$

We test the Algorithm 1 for three different starting points and use $\|x_{n+1} - x_n\|_2 < 10^{-10}$ and $\|y_{n+1} - y_n\|_2 < 10^{-10}$ as stopping criterion.

Case 1: $x_1 = \frac{t}{5}$ and $y_1 = \frac{t}{7}$;

Case 2: $x_1 = e^{-5t}$ and $y_1 = \frac{t^2}{2}$;

Case 3: $x_1 = \sin(2t)$ and $y_1 = \cos(2t)$.

According to the definition of A_i, B_i, f_i, for all $i = 1, 2$, then the solution of the problem is $x^*(t) = \{\mathbf{0}\}$. The computational experiments, using our Algorithm 1, for each case are reported in Tables 1, 2, 3, and Figures 1, 2, 3. The convergence behavior of the error $\|x_n - x_{n-1}\|_2$ and $\|y_n - y_{n-1}\|_2$ for each case is shown in Figure 4.

TABLE 1. Computational result of **Case 1** for Example 5.1.

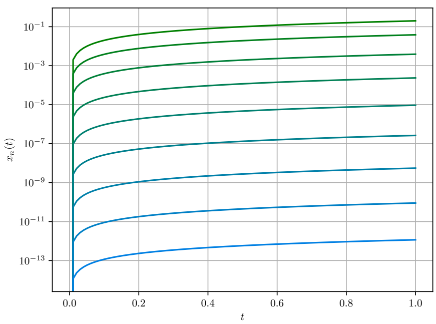
n	$x_n(t)$	$y_n(t)$	$\|x_{n+1} - x_n\|_2$	$\|y_n - y_{n-1}\|_2$
1	$0.2t$	$0.14286t$	-	-
2	$0.038357t$	$0.027619t$	0.093325	0.066533
3	$0.0038866t$	$0.0028225t$	0.019902	0.014316
4	$0.00023478t$	$0.00017213t$	0.0021084	0.0015302
5	$9.3462 \cdot 10^{-6}t$	$6.9228 \cdot 10^{-6}t$	0.00013016	$9.5382 \cdot 10^{-5}$
6	$2.6317 \cdot 10^{-7}t$	$1.9705 \cdot 10^{-7}t$	$5.2441 \cdot 10^{-6}$	$3.8831 \cdot 10^{-6}$
7	$5.5187 \cdot 10^{-9}t$	$4.1782 \cdot 10^{-9}t$	$1.4876 \cdot 10^{-7}$	$1.1136 \cdot 10^{-7}$
8	$8.9582 \cdot 10^{-11}t$	$6.8575 \cdot 10^{-11}t$	$3.1345 \cdot 10^{-9}$	$2.3727 \cdot 10^{-9}$
9	$1.16 \cdot 10^{-12}t$	$8.976 \cdot 10^{-13}t$	$5.1051 \cdot 10^{-11}$	$3.9074 \cdot 10^{-11}$

TABLE 2. Computational result of **Case 2** for Example 5.1.

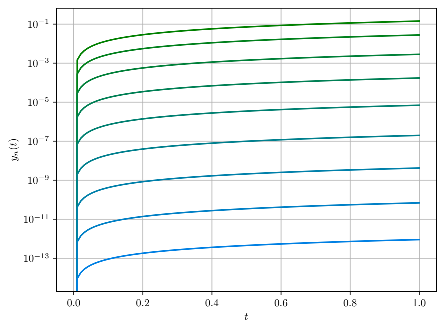
n	$x_n(t)$	$y_n(t)$	$\|x_n - x_{n-1}\|_2$	$\|y_n - y_{n-1}\|_2$
1	e^{-5t}	$0.2t^2$	-	-
2	$0.0012t^2 + 0.1875e^{-5t}$	$0.0375t^2 + 0.0041667e^{-5t}$	0.25688	0.072554
3	$0.00025185t^2 + 0.018533e^{-5t}$	$0.0037067t^2 + 0.00087449e^{-5t}$	0.053474	0.015251
4	$2.4024 \cdot 10^{-5}t^2 + 0.0010881e^{-5t}$	$0.00021762t^2 + 8.3415 \cdot 10^{-5}e^{-5t}$	0.0055276	0.0016045
5	$1.3403 \cdot 10^{-6}t^2 + 4.1944 \cdot 10^{-5}e^{-5t}$	$8.3888 \cdot 10^{-6}t^2 + 4.654 \cdot 10^{-6}e^{-5t}$	0.00033198	$9.9186 \cdot 10^{-5}$
6	$4.9247 \cdot 10^{-8}t^2 + 1.14 \cdot 10^{-6}e^{-5t}$	$2.28 \cdot 10^{-7}t^2 + 1.71 \cdot 10^{-7}e^{-5t}$	$1.2973 \cdot 10^{-5}$	$4.044 \cdot 10^{-6}$
7	$1.2823 \cdot 10^{-9}t^2 + 2.3014 \cdot 10^{-8}e^{-5t}$	$4.6027 \cdot 10^{-9}t^2 + 4.4525 \cdot 10^{-9}e^{-5t}$	$3.5598 \cdot 10^{-7}$	$1.1746 \cdot 10^{-7}$
8	$2.4901 \cdot 10^{-11}t^2 + 3.5898 \cdot 10^{-10}e^{-5t}$	$7.1796 \cdot 10^{-11}t^2 + 8.6463 \cdot 10^{-11}e^{-5t}$	$7.2412 \cdot 10^{-9}$	$2.5624 \cdot 10^{-9}$
9	$3.7454 \cdot 10^{-13}t^2 + 4.4625 \cdot 10^{-12}e^{-5t}$	$8.925 \cdot 10^{-13}t^2 + 1.3005 \cdot 10^{-12}e^{-5t}$	$1.1372 \cdot 10^{-10}$	$4.3587 \cdot 10^{-11}$
10	$4.4933 \cdot 10^{-15}t^2 + 4.5323 \cdot 10^{-14}e^{-5t}$	$9.0647 \cdot 10^{-15}t^2 + 1.5602 \cdot 10^{-14}e^{-5t}$	$1.4228 \cdot 10^{-12}$	$5.9411 \cdot 10^{-13}$

TABLE 3. Computational result of **Case 3** for Example 5.1.

n	$x_n(t)$	$y_n(t)$	$\|x_n - x_{n-1}\|_2$	$\|y_n - y_{n-1}\|_2$
1	$\sin(2t)$	$\cos(2t)$	-	-
2	$0.1875 \sin(2t) + 0.006 \cos(2t)$	$0.0041667 \sin(2t) + 0.4676 \cos(2t)$	0.62492	0.33764
3	$0.018533 \sin(2t) + 0.0022552 \cos(2t)$	$0.0012487 \sin(2t) + 0.088194 \cos(2t)$	0.13131	0.24253
4	$0.0010891 \sin(2t) + 0.00035263 \cos(2t)$	$0.00010534 \sin(2t) + 0.0051715 \cos(2t)$	0.014005	0.053239
5	$4.2022 \cdot 10^{-5} \sin(2t) + 2.347 \cdot 10^{-5} \cos(2t)$	$5.4972 \cdot 10^{-6} \sin(2t) + 0.00019906 \cos(2t)$	0.00091556	0.0031992
6	$1.1434 \cdot 10^{-6} \sin(2t) + 9.4349 \cdot 10^{-7} \cos(2t)$	$1.9385 \cdot 10^{-7} \sin(2t) + 5.4012 \cdot 10^{-6} \cos(2t)$	$3.975 \cdot 10^{-5}$	0.00012508
7	$2.3112 \cdot 10^{-8} \sin(2t) + 2.5917 \cdot 10^{-8} \cos(2t)$	$4.9126 \cdot 10^{-9} \sin(2t) + 1.0883 \cdot 10^{-7} \cos(2t)$	$1.2299 \cdot 10^{-6}$	$3.4336 \cdot 10^{-6}$
8	$3.6101 \cdot 10^{-10} \sin(2t) + 5.2106 \cdot 10^{-10} \cos(2t)$	$9.3615 \cdot 10^{-11} \sin(2t) + 1.6942 \cdot 10^{-9} \cos(2t)$	$2.8427 \cdot 10^{-8}$	$6.9863 \cdot 10^{-8}$
9	$4.4943 \cdot 10^{-12} \sin(2t) + 8.0267 \cdot 10^{-12} \cos(2t)$	$1.3891 \cdot 10^{-12} \sin(2t) + 2.1015 \cdot 10^{-11} \cos(2t)$	$5.0783 \cdot 10^{-10}$	$1.0972 \cdot 10^{-9}$
10	$4.5716 \cdot 10^{-14} \sin(2t) + 9.7958 \cdot 10^{-14} \cos(2t)$	$1.6498 \cdot 10^{-14} \sin(2t) + 2.1296 \cdot 10^{-13} \cos(2t)$	$7.1996 \cdot 10^{-12}$	$1.3724 \cdot 10^{-11}$

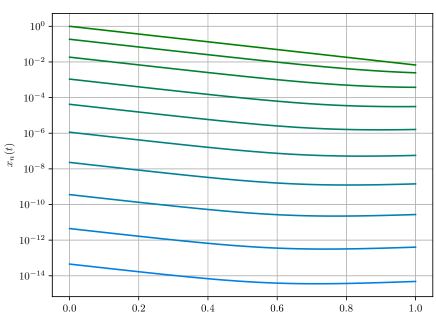


(A) **Case 1** : $x_1 = \frac{t}{5}$ for $n = 1, 2, 3, ..., 8.$ and

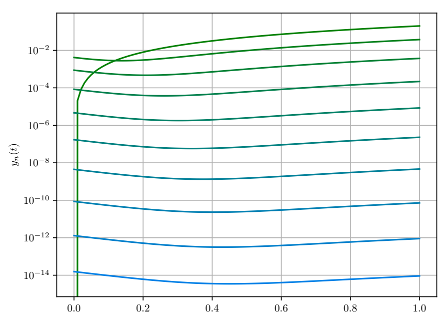


(B) **Case 1** : $y_1 = \frac{t}{7}$ for $n = 1, 2, 3, ..., 8.$

FIGURE 1. The convergence behavior of $\{x_n(t)\}$ and $\{y_n(t)\}$ with **Case 1** in Example 5.1 and y-axis is illustrated in Log scale.

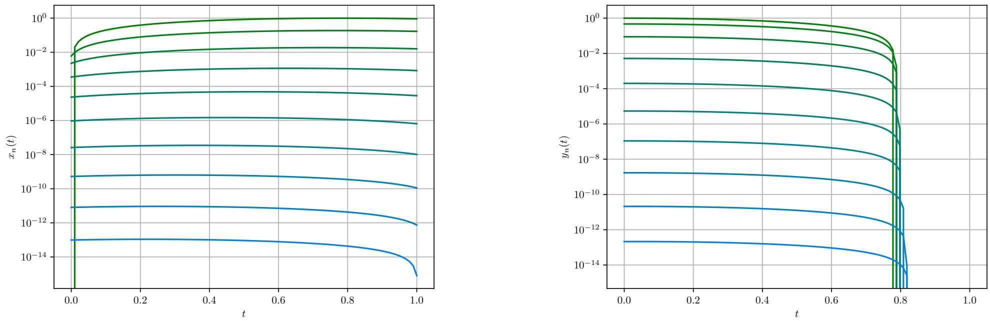


(A) **Case 2** : $x_1 = e^{-5t}$ for $n = 1, 2, 3, ..., 9.$



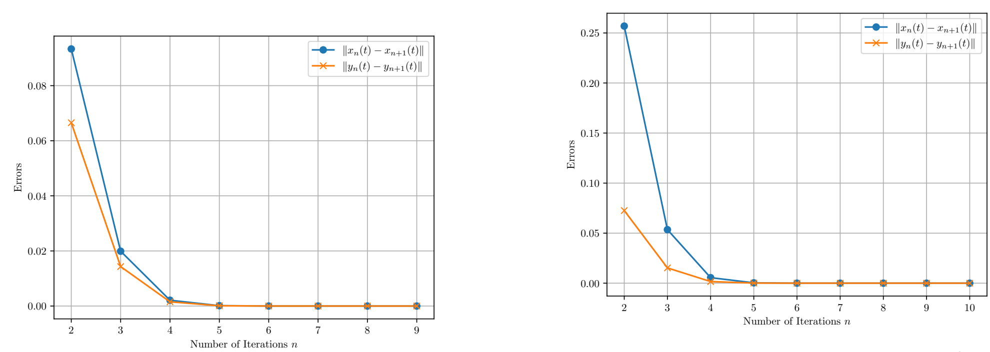
(B) **Case 2** : $y_1 = \frac{t^2}{2}$ for $n = 1, 2, 3, ..., 9.$

FIGURE 2. The convergence behavior of $\{x_n(t)\}$ and $\{y_n(t)\}$ with **Case 2** in Example 5.1 and y-axis is illustrated in logscale.

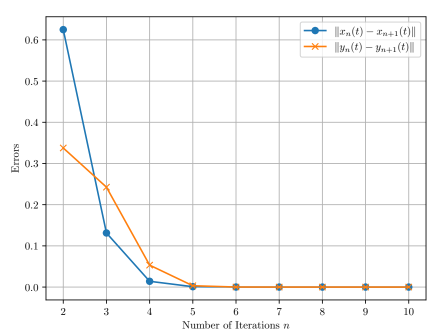


(A) **Case 3** : $x_1 = \sin(2t)$ for $n = 1, 2, 3, ..., 9$. (B) **Case 3** : $y_1 = \cos(2t)$ for $n = 1, 2, 3, ..., 9$.

FIGURE 3. The convergence behavior of $\{x_n(t)\}$ and $\{y_n(t)\}$ with **Case 3** in Example 5.1 and y-axis is illustrated in logscale.



(A) Case 1 : $x_1 = 0.2t$ and $y_1 = 0.8t$. (B) Case 2 : $x_1 = e^{-2t}$ and $y_1 = \dfrac{t^2}{2}$.



(C) Case 3 : $x_1 = \sin(2t)$ and $y_1 = \cos(2t)$

FIGURE 4. Error plotting of $\|x_n - x_{n-1}\|_2$ and $\|y_n - y_{n-1}\|_2$ in Example 5.1.

Moreover, we also provide the comparison (in terms of convergence and the CPU time) of the sequences μ_n and β_n on Algorithm 1 by choosing different μ_n and β_n with $\mu_n + \beta_n \leq$

1 satisfying the conditions (ii), (iii) in the following choices.

Choice 1: $\mu_n = \dfrac{1}{(n+1)^2} - \dfrac{1}{(n+1)^4}$ and $\beta_n = 1 - \dfrac{1}{(n+1)^2}$;

Choice 2: $\mu_n = \dfrac{n}{(8n+9)} - \dfrac{1}{(n^2+1)}$ and $\beta_n = 1 - \dfrac{1}{(n^2+1)}$;

Choice 3: $\mu_n = \dfrac{1}{n+1} - \dfrac{1}{(n+1)^{20}}$ and $\beta_n = 1 - \dfrac{1}{n+1}$;

Choice 4: $\mu_n = \dfrac{1}{n^2+1}$ and $\beta_n = 1 - \dfrac{1}{n^2+1}$.

It is emphasized that all these sequences of μ_n and β_n are to satisfy conditions (ii) and (iii). The results are reported in Table 4.

TABLE 4. Comparison of Algorithm 1 for Example 5.1 with different cases of μ_n and β_n.

Starting point		Choice 1	Choice 2	Choice 3	Choice 4
$x_1 = \frac{t}{5}$	No. of Iter.	9	10	14	16
$y_1 = \frac{t}{7}$	CPU Time (s)	0.56213975	0.69141006	0.92488122	1.04418993

Remark 5.2. By testing the convergence behavior of Algorithm 1, we see in Example 5.1 that

(1) Tables 1, 2, 3 and Figures 1, 2, 3, 4 show that $\{x_n\}$ and $\{y_n\}$ converge to $x(t) = \mathbf{0}$, where $\mathbf{0} \in Fix(\text{Prox}_{\gamma f}(I - \gamma A)) \cap Fix(\text{Prox}_{\gamma f}(I - \gamma B))$, for all $i = 1, 2$. The convergence of $\{x_n\}$ and $\{y_n\}$ of Example 5.1 can be guaranteed by Theorem 3.1.

(2) From the discussion of Tables 1, 2, and 3, we see that the sequences $\{x_n\}$ and $\{y_n\}$ in **Case 1** on algorithm 1 converge the fastest.

(3) The sequences $\mu_n = \frac{1}{(n+1)^2} - \frac{1}{(n+1)^4}$ and $\beta_n = 1 - \frac{1}{(n+1)^2}$ with $\mu_n + \beta_n \leq 1$ satisfy the conditions (ii), (iii) in Theorem 3.1.

(4) From the discussion of Table 4, we see that the sequences $\{x_n\}$ and $\{y_n\}$ in **Choice 1** on algorithm 1 converge the fastest and the least time.

Next, we use the Algorithm 2 in Theorem 4.2 to solve a system of linear equations. Systems of linear equations are used in a wide range of fields, including traffic analysis, economics, and electrical engineering.

Example 5.2. We assume that $H_1 = H_2 = \mathbb{R}^4$. Solving a system of linear equations $A_i\mathbf{x} = b_i$ for all $i = 1, 2$. In the following, we take:

$$A_1 = \begin{pmatrix} 2 & -1 & 3 & -1 \\ 1 & -2 & 1 & -3 \\ 2 & -1 & -1 & 1 \\ 2 & 0 & -2 & 3 \end{pmatrix}, \; b_1 = \begin{pmatrix} 1 \\ -15 \\ 7 \\ 10 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} -1 & -3 & 0 & -3 \\ 1 & -1 & 2 & -2 \\ 1 & 2 & 1 & 0 \\ 2 & -2 & 8 & 6 \end{pmatrix}, \; b_2 = \begin{pmatrix} -22 \\ -6 \\ 9 \\ 36 \end{pmatrix}.$$

Then the split feasibility problem can be formulated as the problem of finding a point $\mathbf{x}^*$ with the property $\mathbf{x}^* \in C$ and $A_i\mathbf{x}^* \in Q$, where $C = \mathbb{R}^4$, $Q = \{b_i\}$, for all $i = 1, 2$. That is,

$\mathbf{x}^*$ is the solution of the system of linear equations $A_i\mathbf{x} = b_i$, and

$$\mathbf{x}^* = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}.$$

For every $i = 1, 2$, we take the operator $\xi_i : \mathbb{R}^4 \to \mathbb{R}^4$ to be defined as $\xi_1(\mathbf{x}) = \frac{\mathbf{x}}{49}$ and $\xi_2(\mathbf{x}) = \frac{\mathbf{x}}{64}$. In Algorithm 2, choose $\alpha_n = \frac{1}{5n}$, $\mu_n = \frac{1}{(n+1)^2} - \frac{1}{(n+1)^4}$ and $\beta_n = 1 - \frac{1}{(n+1)^2}$ for all $n \in \mathbb{N}$. So our Algorithm 2 becomes

$$\begin{cases} \mathbf{v}_n = \mathbf{y}_n - \frac{1}{100}A_2^*(I - P_Q)A_2\mathbf{y}_n \\ \mathbf{u}_n = \mathbf{x}_n - \frac{2}{100}A_1^*(I - P_Q)A_1\mathbf{x}_n \\ \mathbf{y}_{n+1} = \left(\frac{1}{(n+1)^2} - \frac{1}{(n+1)^4}\right)\mathbf{y}_n + \left(1 - \frac{1}{(n+1)^2}\right)\left(\frac{1}{5n}\xi_2(\mathbf{x}_n) + \left(1 - \frac{1}{5n}\right)\mathbf{v}_n\right) \\ \mathbf{x}_{n+1} = \left(\frac{1}{(n+1)^2} - \frac{1}{(n+1)^4}\right)\mathbf{x}_n + \left(1 - \frac{1}{(n+1)^2}\right)\left(\frac{1}{5n}\xi_1(\mathbf{y}_n) + \left(1 - \frac{1}{5n}\right)\mathbf{u}_n\right). \end{cases} \tag{5.53}$$

According to the definition of A_i, for all $i = 1, 2$, then the solution of the problem is $\mathbf{x}^* = (1, 3, 2, 4)^T$. From Theorem 4.2, we can conclude that the sequences x_n and y_n converge strongly to $\mathbf{x}^*$. The numerical results, using our Algorithm 2, for the sequences $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ are reported in Tables 5 and 6.

TABLE 5. The numerical results for the sequence $\{\mathbf{x}_n\}$ of Example 5.2.

n	$\mathbf{x}_n = (x_n^1, x_n^2, x_n^3, x_n^4)^T$	$\|\mathbf{x}_n - \mathbf{x}_{n-1}\|_2$
1	(10, 10, 10, 10)	-
100	(1.01101939, 2.96006399, 1.9079274, 3.93186252)	2.0827e-03
500	(0.99900056, 2.98427758, 1.98454128, 3.9924486)	4.7872e-05
1000	(0.99952084, 2.99227651, 1.99233583, 3.99621442)	1.1675e-05
5000	(0.99990674, 2.99847538, 1.99847759, 3.9992422)	4.5827e-07
10000	(0.99995351, 2.99923891, 1.99923946, 3.99962108)	1.1430e-07

TABLE 6. The numerical results for the sequence $\{\mathbf{y}_n\}$ of Example 5.2.

n	$\mathbf{y}_n = (y_n^1, y_n^2, y_n^3, y_n^4)^T$	$\|\mathbf{y}_n - \mathbf{y}_{n-1}\|_2$
1	(-10, -10, -10, -10)	-
100	(0.74657296, 3.05961817, 2.10596020, 3.95701678)	1.9015e-03
500	(0.97220178, 3.00338976, 2.01114426, 3.99416767)	2.3602e-04
1000	(1.02507278, 2.98613767, 1.98783547, 4.00236647)	5.3877e-05
5000	(1.01697453, 2.99245005, 1.99212248, 4.00209563)	4.7330e-06
10000	(1.00763210, 2.99656705, 1.99645073, 4.00093167)	1.0058e-06

Remark 5.3. By testing the convergence behavior of Algorithm 2, we see in Example 5.2 that

(1) Tables 5 and 6 show that $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ converge to $\mathbf{x}^* = (1, 3, 2, 4)^T$. The convergence of $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ of Example 5.2 can be guaranteed by Theorem 4.2.
(2) Tables 5 and 6, we see that $\mathbf{x}_{10000} = (0.99995351, 2.99923891, 1.99923946, 3.99962108)^T$ is an approximation of the system of linear equations with an error $1.1430e-07$ and $\mathbf{y}_{10000} = (1.00763210, 2.99656705, 1.99645073, 4.00093167)^T$ is an approximation of the system of linear equations with an error $1.0058e-06$, respectively.

6. Conclusion

In this paper, we introduce an intermixed algorithm with viscosity technique for solving a common fixed point of two proximal operators in a real Hilbert space. The strong convergence theorem of our proposed algorithm, Theorem 3.1, has been established and proven under some mild conditions. However, we should like remark the following:

(1) We modify the results of Yao et al.[32] from strict pseudo-contraction mappings to proximal operators of in Hilbert spaces. Further, we also give the new mathematical tool related to proximal operators by using the concept of the convex minimization problem and the fixed point equation (1.2) (see Lemma 2.4).
(2) Our result is proved with a new assumption on the control conditions $\{\mu_n\}$ and $\{\beta_n\}$ such that $\mu_n + \beta_n \leq 1$.
(3) We apply our theorem to solve the split feasibility problem by using an intermixed algorithm with viscosity technique.
(4) We give a numerical example that shows the efficiency and implementation of our main result in the space L_2 as shown in Example 5.1. Moreover, we present a numerical example of the algorithm 2 for solving the system of linear equations in Example 5.2.

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