CARPATHIAN J. MATH.
Volume **40** (2024), No. 2,
Pages 363 - 380

Online version at https://www.carpathian.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2024.02.09

Modified inertial extragradient algorithm with non-monotonic step sizes for pseudomonotone equilibrium problems and quasi-nonexpansive mapping

THANYALUCK NGAMKHUM, KOMKIND PUNPENG and MANATCHANOK KHONCHALIEW

ABSTRACT. In this paper, we introduce a modified inertial extragradient algorithm with non-monotonic step sizes for approximating a common solution of the pseudomonotone equilibrium problem and the fixed point problem for the quasi-nonexpansive mapping in the framework of a real Hilbert space. Under some constraint qualifications of the scalar sequences, the strong convergence theorem of the introduced algorithm is presented by using the self-adaptive non-monotonic step size without prior information about the Lipschitz constants of bifunction. Some numerical experiments are provided to demonstrate the computational efficiency and advantages of the proposed algorithm.

1. INTRODUCTION

The equilibrium and fixed point problems have a wide range of applications in many mathematical models in the sense that they bring together different mathematical problems, such as optimization problems, variational inequality problems, minimax problems, Nash equilibrium problems, and saddle point problems, see [5, 17, 18, 24, 27, 30], and the references therein. For a real Hilbert space H and a mapping $T : H \to H$, the fixed point problem is a problem of finding a point $x \in H$ such that $Tx = x$. The set of fixed points of the mapping T is represented by $F(T)$.

The majority of methods for finding fixed points of a nonexpansive mapping $T : C \to C$ are derived from the basic Mann iteration as follows:

$$\begin{cases} x_0 \in C, \\ x_{k+1} = (1-\alpha_k)x_k + \alpha_k T x_k, \end{cases} \tag{1.1}$$

where C is a nonempty closed convex subset of H and the sequence $\{\alpha_k\}$ must meet certain conditions. In [26], the authors showed that if T is a quasi-nonexpansive mapping with $I - T$ demiclosed at zero, then the sequence generated by Algorithm 1.1 converges weakly to a fixed point of T.

Furthermore, Ishikawa [19] proposed the following algorithm for finding fixed points of a Lipschitz pseudocontractive mapping T:

$$\begin{cases} x_0 \in C, \\ y_k = (1-\alpha_k)x_k + \alpha_k T x_k, \\ x_{k+1} = (1-\beta_k)x_k + \beta_k T y_k, \end{cases} \tag{1.2}$$

Received: 21.09.2023. In revised form: 10.03.2024. Accepted: 17.03.2024
2010 *Mathematics Subject Classification.* 47H10, 68W10, 90C33.
Key words and phrases. *equilibrium problems, fixed point problems, pseudomonotone bifunction, quasi-nonexpansive mapping, inertial method, extragradient method.*
Corresponding author: Manatchanok Khonchaliew; manatchanok@g.lpru.ac.th

where $0 \leq \beta_k \leq \alpha_k \leq 1$ such that $\sum_{k=0}^{\infty} \alpha_k\beta_k = \infty$ and $\lim_{k\to\infty} \alpha_k = 0$. In [19], the author proved that if C is a convex compact subset of H, then the sequence $\{x_k\}$ generated by Algorithm 1.2 converges strongly to fixed points of T. It was emphasized that Mann iteration may not, in general, be applied for finding fixed points of a Lipschitz pseudocontractive mapping in a Hilbert space, for instance, see [9].

On the other hand, the equilibrium problem is a problem of finding a point $x^* \in C$ such that

$$f(x^*, y) \geq 0, \forall y \in C, \tag{1.3}$$

where C is a nonempty closed convex subset of a real Hilbert space H, and $f\colon H \times H \to \mathbb{R}$ is a bifunction. The solution set of the equilibrium problem (1.3) is denoted by $EP(f, C)$. One of the most popular methods for solving the equilibrium problem (1.3), when f is a monotone bifunction, is the proximal point method, see [10]. However, the proximal point method may not be guaranteed for a weaker assumption, such as a pseudomonotone, see [15]. To overcome this drawback, Tran et al. [36] proposed the following extragradient method for solving the equilibrium problem when the bifunction f is pseudomonotone and satisfies Lipschitz-type continuous with positive constants c_1 and c_2:

$$\begin{cases} x_0 \in C, \\ y_k = \arg\min \left\{\lambda f(x_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\right\}, \\ x_{k+1} = \arg\min \left\{\lambda f(y_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\right\}, \end{cases} \tag{1.4}$$

where $0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$. They proved that the sequence $\{x_k\}$ generated by Algorithm 1.4 converges weakly to a solution of the equilibrium problem (1.3).

Meanwhile, as one of the acceleration approaches, the inertial-type methods based on discrete versions of a second-order dissipative dynamic system [1, 2] have gained a lot of attention from researchers for solving equilibrium problems, for instance, see [14, 37] and the references therein. Indeed, this method is characterized that the next iteration is determined by the combination of the previous two (or more) iterations and can be regarded as a method of speeding up the convergence properties under some suitable conditions, see, e.g., [33, 34] and related references. By using the techniques of inertial and extragradient methods, Rehman et al. [29] proposed the following algorithm for solving the equilibrium problem when the bifunction f is pseudomonotone and satisfies Lipschitz-type continuous with positive constants c_1 and c_2:

$$\begin{cases} x_0, x_1 \in H, \\ w_k = x_k + \theta_k(x_k - x_{k-1}), \\ y_k = \arg\min \left\{\lambda_k f(w_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in C\right\}, \\ x_{k+1} = \arg\min \left\{\sigma\lambda_k f(y_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in C\right\}, \\ \lambda_{k+1} = \min\left\{\zeta, \dfrac{\sigma f(y_k, x_{k+1})}{f(w_k, x_{k+1}) - f(w_k, y_k) - c_1\|w_k - y_k\|^2 - c_2\|x_{k+1} - y_k\|^2 + 1}\right\}, \end{cases} \tag{1.5}$$

where $\lambda_1 > 0$, $0 < \zeta < \min\left\{\frac{1-3\theta}{(1-\theta)^2}, \frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, $\sigma \in (0, \zeta)$, and $0 \leq \theta_k \leq \theta < \frac{1}{3}$. They proved that the sequence $\{x_k\}$ generated by Algorithm 1.5 converges weakly to a solution of the equilibrium problem (1.3). It is important to note that the aforementioned algorithms used step sizes that are dependent on the Lipschitz constants of the bifunction f. This can lead to some limitations in practical applications because the Lipschitz constants of the bifunction are usually unknown or complicated to estimate. Therefore, it is valuable to develop adaptive step sizes that do not require the Lipschitz constants of the bifunction f.

In 2023, Husain and Asad [16] proposed the following algorithm by using the extragradient method for solving the equilibrium and fixed point problems when the bifunction f is pseudomonotone and satisfies Lipschitz-type continuous and the mapping $T\colon C \to H$ is ψ-strongly quasi-nonexpansive with $I - T$ demiclosed at zero:

$$\begin{cases} x_0 \in C, \\ y_k = \arg\min\left\{ f(x_k, y) + \frac{1}{2\lambda_k}\|y - x_k\|^2 : y \in C \right\}, \\ z_k = \arg\min\left\{ f(y_k, y) + \frac{1}{2\lambda_k}\|y - x_k\|^2 : y \in C \right\}, \\ x_{k+1} = \alpha_k h(x_k) + (1-\alpha_k)Tz_k, \\ \lambda_{k+1} = \min\left\{ \lambda_k, \dfrac{\mu(\|x_k - y_k\|^2 + \|z_k - y_k\|^2)}{2\max\{0, f(x_k, z_k) - f(x_k, y_k) - f(y_k, z_k)\}} \right\}, \end{cases} \tag{1.6}$$

where $\lambda_0 > 0$, $\mu \in (0,1)$, $\{\alpha_k\} \subset (0,1)$ such that $\sum\limits_{k=0}^{\infty} \alpha_k = \infty$, $\lim\limits_{k\to\infty} \alpha_k = 0$, and $h\colon H \to H$ is a contraction mapping. They proved that the sequence $\{x_k\}$ generated by Algorithm 1.6 converges strongly to $p = P_{EP(f,C)\cap F(T)}h(p)$. It is worth noting that Algorithm 1.6 used the adaptive step size to deal with the unknown prior information of the Lipschitz constants of the bifunction f. However, this considered step size is a non-increasing sequence, which will further affect the computational efficiency of Algorithm 1.6.

In this paper, we focus on the algorithm for solving the equilibrium and fixed point problems. That is, we introduce a new iterative algorithm for finding the common solution of the pseudomonotone equilibrium problem and the fixed point of quasi-nonexpansive mapping by using the self-adaptive non-monotonic step size. Some numerical examples demonstrate the computational behavior of the proposed algorithm and compare it to some related algorithms in the literature.

This paper is organized as follows: In Section 2, some necessary definitions and properties are reviewed for further use. Section 3 presents the modified inertial extragradient algorithm with non-monotonic step size and proves the strong convergence theorem. In Section 4, the performance of the introduced algorithm is discussed by comparing it with other existing algorithms.

2. PRELIMINARIES

In this section, we collect some basic definitions and important properties that are used in this paper. The notation $\mathbb{R}$ and $\mathbb{N}$ will stand for the set of the real numbers and the natural numbers, respectively. Let H be a real Hilbert space with inner product $\langle \cdot\,, \cdot \rangle$, and its corresponding norm $\|\cdot\|$. The symbols $\to$ and $\rightharpoonup$ are denoted for the strong convergence and the weak convergence in H, respectively.

First, we recall some definitions and results which are related to nonlinear mappings.

Definition 2.1. [7] A mapping $T : H \to H$ is said to be:

i) pseudocontractive if
$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I-T)x - (I-T)y\|^2, \forall x, y \in H,$$
where I denotes the identity operator on H.

ii) Lipschitzian if there exists $L \geq 0$ such that
$$\|Tx - Ty\| \leq L\|x - y\|, \forall x, y \in H.$$
In particular, if $L = 1$, then T is said to be nonexpansive.

iii) ψ-strongly quasi-nonexpansive with $\psi \geq 0$ if $F(T)$ is a nonempty set and

$$\|Tx - p\|^2 \leq \|x - p\|^2 - \psi\|x - Tx\|^2, \forall x \in H, p \in F(T).$$

iv) quasi-nonexpansive if $F(T)$ is a nonempty set and

$$\|Tx - p\| \leq \|x - p\|, \forall x \in H, p \in F(T).$$

Remark 2.1. It is well-known that $F(T)$ is closed and convex when T is a quasi-nonexpansive mapping, see [20].

Definition 2.2. [6] Let C be a nonempty closed convex subset of H. A mapping $T : C \to H$ is said to be demiclosed at $y \in H$ if for any sequence $\{x_k\} \subset C$ with $x_k \rightharpoonup x^* \in C$ and $Tx_k \to y$ imply $Tx^* = y$.

Now, we recall the concerned definitions of the equilibrium problems.

Definition 2.3. [5, 23, 24] Let C be a nonempty closed convex subset of H. A bifunction $f : H \times H \to \mathbb{R}$ is said to be:

i) monotone on C if

$$f(x, y) + f(y, x) \leq 0, \forall x, y \in C;$$

ii) pseudomonotone on C if

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0, \forall x, y \in C;$$

iii) Lipshitz-type continuous on H if there exists two positive constants c_1 and c_2 such that

$$f(x, y) + f(y, z) \geq f(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2, \forall x, y, z \in H. \tag{2.7}$$

Remark 2.2. A monotone bifunction is a pseudomonotone bifunction, but the converse is not true in general, for instance, see [21].

Next, we recall some basic facts in the functional analysis which are referred to in the sequel. For each $x \in H$, we denote the metric projection of x onto a nonempty closed convex subset C of H by $P_C(x)$, that is

$$\|x - P_C(x)\| \leq \|y - x\|, \forall y \in C.$$

Lemma 2.1. [8, 13] *Let C be a nonempty closed convex subset of H. Then,*

i) $P_C(x)$ is singleton and well-defined, for each $x \in H$;
ii) $z = P_C(x)$ if and only if $\langle x - z, y - z\rangle \leq 0, \forall y \in C$.

For a function $g : H \to \mathbb{R}$, the subdifferential of g at $z \in H$ is defined by

$$\partial g(z) = \{w \in H : g(y) - g(z) \geq \langle w, y - z\rangle, \forall y \in H\}.$$

The function g is said to be subdifferentiable at z if $\partial g(z) \neq \emptyset$.

Lemma 2.2. [8] *For any $z \in H$, the subdifferential $\partial g(z)$ of a continuous convex function g is a weakly closed and bounded convex set.*

Lemma 2.3. [12] *Let C be a convex subset of H and $f : C \to \mathbb{R}$ be subdifferentiable on C. Then, x^* is a solution to the following convex problem:*

$$\min\{f(x) : x \in C\}$$

if and only if $0 \in \partial f(x^*) + N_C(x^*)$, *where* $N_C(x^*) := \{y \in H : \langle y, z - x^*\rangle \leq 0, \forall z \in C\}$ *is the normal cone of C at x^*.*

We end this section by recalling some auxiliary lemmas for proving the convergence theorems.

Lemma 2.4. [25] *Let* $\{a_k\}$, $\{b_k\}$ *and* $\{c_k\}$ *be sequences of non-negative real numbers such that* $a_{k+1} \leq a_k b_k + c_k, \forall k \in \mathbb{N}$. *If* $\{b_k\} \subset [1, \infty)$, $\sum\limits_{k=0}^{\infty} (b_k - 1) < \infty$, *and* $\sum\limits_{k=1}^{\infty} c_k < \infty$, *then* $\lim\limits_{k\to\infty} a_k$ *exists.*

Lemma 2.5. [38] *Let* $\{a_k\}$ *and* $\{c_k\}$ *be sequences of non-negative real numbers such that*

$$a_{k+1} \leq (1 - \gamma_k)a_k + \gamma_k b_k + c_k, \forall k \in \mathbb{N} \cup \{0\},$$

where $\{\gamma_k\}$ *is a sequence in* $(0, 1)$ *and* $\{b_k\}$ *is a sequence in* $\mathbb{R}$. *Assume that* $\sum\limits_{k=0}^{\infty} c_k < \infty$. *If* $\sum\limits_{k=0}^{\infty} \gamma_k = \infty$ *and* $\limsup\limits_{k\to\infty} b_k \leq 0$, *then* $\lim\limits_{k\to\infty} a_k = 0$.

Lemma 2.6. [22] *Let* $\{a_k\}$ *be a sequence of real numbers such that there exists a subsequence* $\{a_{k_i}\}$ *of* $\{a_k\}$ *such that* $a_{k_i} < a_{k_i+1}$, *for all* $i \in \mathbb{N}$. *Then, there exists a non-decreasing sequence* $\{m_n\}$ *of positive integers such that* $\lim\limits_{n\to\infty} m_n = \infty$ *and the following properties hold:*

$$a_{m_n} \leq a_{m_n+1} \text{ and } a_n \leq a_{m_n+1},$$

for all (sufficiently large) numbers $n \in \mathbb{N}$. *Indeed,* m_n *is the largest number* k *in the set* $\{1, 2, \ldots, n\}$ *such that*

$$a_k < a_{k+1}.$$

3. Main results

Let C be a nonempty closed convex subset of a real Hilbert space H and $T\colon H \to H$ be a quasi-nonexpansive mapping with $I-T$ demiclosed at zero. The following assumptions on the bifunction $f\colon H \times H \to \mathbb{R}$ will be considered in this paper:

- $(A1)$ $f(\cdot\,, y)$ is sequentially weakly upper semicontinuous on C, for each fixed $y \in C$, that is if $\{x_k\} \subset C$ is a sequence converging weakly to $x \in C$, then $\limsup\limits_{k\to\infty} f(x_k, y) \leq f(x, y)$;
- $(A2)$ $f(x, \cdot\,)$ is convex, subdifferentiable and lower semicontinuous on H, for each fixed $x \in H$;
- $(A3)$ f is psuedomonotone on C;
- $(A4)$ f is Lipshitz-type continuous on H.

Remark 3.3. i) If the bifunction f satisfies the assumptions $(A3)$ and $(A4)$, then $f(x, x) = 0$, for each $x \in C$. Indeed, by using (2.7) and taking $x = y = z \in C$, we have $f(x, x) \geq 0$. It follows from the pseudomonotonic of f that $f(x, x) = 0$, for each $x \in C$.

ii) If the bifunction f satisfies the assumptions $(A1) - (A3)$, then the solution set $EP(f, C)$ is closed and convex, see [4, 28, 36] for more detail.

Now, we introduce the following modified inertial extragradient algorithm with non-monotonic step sizes for solving the equilibrium and fixed point problems.

Algorithm 1: Modified inertial extragradient algorithm with non-monotonic step sizes

Initialization. Choose parameters $\lambda_1 > 0$, $\tau \in [0,1)$, $\mu \in (0,1)$, $\sigma \in \left(0, \frac{1}{2\mu}\right)$, $\eta \in \left[\sigma, \frac{1}{\mu}\right)$, $\{\gamma_k\} \subset [0,1]$ such that $\lim_{k\to\infty} \gamma_k = 1$, $\{\alpha_k\} \subset (0,1)$ with $0 < \inf \alpha_k \leq \sup \alpha_k < 1$, $\{\xi_k\} \subset [1,\infty)$ with $\sum_{k=0}^{\infty} (\xi_k - 1) < \infty$, $\{\rho_k\} \subset [0,\infty)$ with $\sum_{k=0}^{\infty} \rho_k < \infty$, and $\{\epsilon_k\} \subset [0,\infty)$, $\{\beta_k\} \subset (0,1)$ such that $\sum_{k=0}^{\infty} \beta_k = \infty$, $\lim_{k\to\infty} \beta_k = 0$, and $\lim_{k\to\infty} \dfrac{\epsilon_k}{\beta_k} = 0$. Pick $x_0, x_1 \in H$ and set $k = 1$.

Step 1. Choose θ_k such that $0 \leq \theta_k \leq \overline{\theta}_k$, where

$$\overline{\theta}_k = \begin{cases} \min\left\{\tau, \dfrac{\epsilon_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \tau, & \text{otherwise,} \end{cases}$$

and compute

$$w_k = (1-\beta_k)\left(x_k + \theta_k(x_k - x_{k-1})\right).$$

Step 2. Solve the strongly convex program

$$y_k = \arg\min\left\{\eta\lambda_k f(w_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in C\right\}.$$

Step 3. Solve the strongly convex program

$$z_k = \arg\min\left\{\sigma\lambda_k f(y_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in C\right\}.$$

Step 4. Compute

$$\lambda_{k+1} = \begin{cases} \min\left\{\xi_k\lambda_k + \rho_k, \dfrac{\mu(\|w_k - y_k\|^2 + \|z_k - y_k\|^2)}{2\left[f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k)\right]}\right\}, \\ \qquad\qquad\qquad \text{if } f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k) > 0, \\ \xi_k\lambda_k + \rho_k, \qquad\qquad \text{otherwise.} \end{cases}$$

Step 5. Compute

$$v_k = \gamma_k w_k + (1-\gamma_k)Tw_k.$$

Step 6. The next approximation x_{k+1} is defined as

$$x_{k+1} = \alpha_k v_k + (1-\alpha_k)Tz_k.$$

Step 7. Put $k := k+1$ and go to **Step 1**.

Remark 3.4. i) The inertial-type method in Algorithm 1 is reformed, which is different from the inertial method in Algorithm 1.5. This means an inertial factor θ_k is combined in Algorithm 1.5 and the single inertial in Algorithm 1.5 is represented by the term $\theta_k(x_k - x_{k-1})$. Meanwhile, two inertial factors, β_k and θ_k, are included in Algorithm 1 and the double inertial in Algorithm 1 is proposed by the terms $\theta_k(x_k - x_{k-1})$ and $\beta_k(x_k + \theta_k(x_k - x_{k-1}))$, which are intended to accelerate the convergence speed of Algorithm 1. Numerical results assert that inertial terms improve the performance of Algorithm 1 in terms of the number of iterations and the CPU time.

ii) The new parameters η and σ in Algorithm 1 are introduced to modify the extragradient method. Observe that if the parameters $\eta = \sigma = 1$, then the modified extragradient method which is included in Algorithm 1 reduces to the general situation such as presented in [16, 36]. We point out that the choices of parameters η and σ may lead to the superior numerical behavior of Algorithm 1.

iii) The step size λ_k in Algorithm 1 is self-adaptive, which is constructed to implement Algorithm 1 without prior knowledge of the Lipschitz constants of the bifunction and automatically updates the iteration step size with a simple computation by using some previously known information. Furthermore, this step size applies a non-monotonic step size criterion, as highlighted in [35] and related references, which is improved from the non-increasing step size rule such as in Algorithm 1.6, and it is reflected the computational efficiency of Algorithm 1 in the numerical experiments.

The following lemma states the important relations in analyzing the convergence of Algorithm 1.

Lemma 3.7. *Let $f\colon H \times H \to \mathbb{R}$ be a bifunction which satisfies $(A1) - (A4)$. Suppose that the solution set $EP(f, C)$ is nonempty. Let $w_k \in H$. If y_k, z_k, and λ_{k+1} are constructed as in the process of Algorithm 1, then the following result holds:*

$$\|z_k - p\|^2 \leq \|w_k - p\|^2 - \left(\frac{\sigma}{\eta} - \frac{\mu\sigma\lambda_k}{\lambda_{k+1}}\right)\|w_k - y_k\|^2 - \left(\frac{\sigma}{\eta} - \frac{\mu\sigma\lambda_k}{\lambda_{k+1}}\right)\|y_k - z_k\|^2, \forall p \in EP(f, C).$$

Proof. Let $p \in EP(f, C)$. By the definition of y_k and Lemma 2.3, we have

$$0 \in \partial_2 \left\{\eta\lambda_k f(w_k, y_k) + \frac{1}{2}\|y_k - w_k\|^2\right\} + N_C(y_k).$$

Then, there exists $v \in \partial_2 f(w_k, y_k)$ and $q \in N_C(y_k)$ such that

$$\eta\lambda_k v + y_k - w_k + q = 0. \tag{3.8}$$

So, by utilizing the subdifferentiability of f, we obtain that

$$f(w_k, y) - f(w_k, y_k) \geq \langle v, y - y_k\rangle, \forall y \in H. \tag{3.9}$$

Besides, from $q \in N_C(y_k)$, we have

$$\langle q, y_k - y\rangle \geq 0, \forall y \in C.$$

Using this one together with the equality (3.8), we get

$$\langle w_k - y_k, y_k - y\rangle \geq \eta\lambda_k\langle v, y_k - y\rangle, \forall y \in C. \tag{3.10}$$

Thus, the relations (3.9) and (3.10) imply that

$$\langle w_k - y_k, y_k - y\rangle \geq \eta\lambda_k[f(w_k, y_k) - f(w_k, y)], \forall y \in C. \tag{3.11}$$

Note that, from $z_k \in C$, we have

$$\eta\lambda_k[f(w_k, z_k) - f(w_k, y_k)] \geq \langle y_k - w_k, y_k - z_k\rangle. \tag{3.12}$$

Similarly, by the definition of z_k and Lemma 2.3, we can show that

$$\langle w_k - z_k, z_k - y\rangle \geq \sigma\lambda_k[f(y_k, z_k) - f(y_k, y)], \forall y \in C. \tag{3.13}$$

Indeed, since $p \in C$, we have

$$\langle w_k - z_k, z_k - p\rangle \geq \sigma\lambda_k[f(y_k, z_k) - f(y_k, p)].$$

It follows from the pseudomonotonic of f that

$$\langle w_k - z_k, z_k - p\rangle \geq \sigma\lambda_k f(y_k, z_k). \tag{3.14}$$

Combining with the relation (3.12) implies that

$$\begin{aligned} \eta\sigma\lambda_k[f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k)] \quad &\geq \quad \eta\langle z_k - w_k, z_k - p\rangle \\ &\quad + \sigma\langle y_k - w_k, y_k - z_k\rangle. \end{aligned} \tag{3.15}$$

On the other hand, by the definition of λ_{k+1}, we note that

$$f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k) \leq \frac{\mu(\|w_k - y_k\|^2 + \|y_k - z_k\|^2)}{2\lambda_{k+1}}. \tag{3.16}$$

This together with the relation (3.15) yields that

$$\eta\langle w_k - z_k, z_k - p\rangle \geq \sigma\langle y_k - w_k, y_k - z_k\rangle - \frac{\mu\eta\sigma\lambda_k(\|w_k - y_k\|^2 + \|y_k - z_k\|^2)}{2\lambda_{k+1}}.$$

Due to the above relation, we provide the following

$$\begin{aligned} \eta\left(\|w_k - p\|^2 - \|w_k - z_k\|^2 - \|z_k - p\|^2\right) \quad &= \quad 2\eta\langle w_k - z_k, z_k - p\rangle \\ &\geq \quad 2\sigma\langle y_k - w_k, y_k - z_k\rangle \\ &\qquad - \frac{\mu\eta\sigma\lambda_k(\|w_k - y_k\|^2 + \|y_k - z_k\|^2)}{\lambda_{k+1}}. \end{aligned}$$

This implies that

$$\begin{aligned} \|z_k - p\|^2 \quad &\leq \quad \|w_k - p\|^2 - \|w_k - z_k\|^2 - \frac{2\sigma}{\eta}\langle y_k - w_k, y_k - z_k\rangle \\ &\qquad + \frac{\mu\sigma\lambda_k(\|w_k - y_k\|^2 + \|y_k - z_k\|^2)}{\lambda_{k+1}} \\ &= \quad \|w_k - p\|^2 - \|w_k - z_k\|^2 + \frac{\sigma}{\eta}\|w_k - z_k\|^2 - \frac{\sigma}{\eta}\|w_k - y_k\|^2 - \frac{\sigma}{\eta}\|y_k - z_k\|^2 \\ &\qquad + \frac{\mu\sigma\lambda_k(\|w_k - y_k\|^2 + \|y_k - z_k\|^2)}{\lambda_{k+1}} \\ &= \quad \|w_k - p\|^2 - \left(\frac{\sigma}{\eta} - \frac{\mu\sigma\lambda_k}{\lambda_{k+1}}\right)\|w_k - y_k\|^2 - \left(\frac{\sigma}{\eta} - \frac{\mu\sigma\lambda_k}{\lambda_{k+1}}\right)\|y_k - z_k\|^2 \\ &\qquad -(1 - \frac{\sigma}{\eta})\|w_k - z_k\|^2. \end{aligned} \tag{3.17}$$

Then, by using the choices of the parameters σ and η (noting that $\frac{\sigma}{\eta} \in (0, 1]$), we have

$$\|z_k - p\|^2 \quad \leq \quad \|w_k - p\|^2 - \left(\frac{\sigma}{\eta} - \frac{\mu\sigma\lambda_k}{\lambda_{k+1}}\right)\|w_k - y_k\|^2 - \left(\frac{\sigma}{\eta} - \frac{\mu\sigma\lambda_k}{\lambda_{k+1}}\right)\|y_k - z_k\|^2.$$

This completes the proof. □

Now, we are in a position to analyze the strong convergence theorem of Algorithm 1.

Theorem 3.1. *Let $f\colon H \times H \to \mathbb{R}$ be a bifunction which satisfies $(A1) - (A4)$ and $T\colon H \to H$ be a quasi-nonexpansive mapping with $I - T$ demiclosed at zero. Suppose that the solution set $EP(f, C) \cap F(T)$ is nonempty. Then, the sequence $\{x_k\}$ generated by Algorithm 1 converges strongly to the minimum-norm element of $EP(f, C) \cap F(T)$.*

Proof. Let $p \in EP(f,C) \cap F(T)$ be picked. Firstly, by the Lipschitz-type continuity of f on H, there exists two positive constants c_1 and c_2 such that

$$\begin{aligned} f(w_k,z_k) - f(w_k,y_k) - f(y_k,z_k) &\leq c_1\|w_k - y_k\|^2 + c_2\|y_k - z_k\|^2 \\ &\leq \max\{c_1,c_2\}\left(\|w_k - y_k\|^2 + \|y_k - z_k\|^2\right). \end{aligned}$$

Using this one together with the definition of λ_{k+1} and the facts of the sequences $\{\xi_k\}$ and $\{\rho_k\}$, we have

$$\lambda_{k+1} \geq \min\left\{\xi_k\lambda_k + \rho_k, \frac{\mu}{2\max\{c_1,c_2\}}\right\} \geq \min\left\{\lambda_k, \frac{\mu}{2\max\{c_1,c_2\}}\right\}.$$

By induction, we obtain that the sequence $\{\lambda_k\}$ has a lower bound as $\min\left\{\lambda_1, \frac{\mu}{2\max\{c_1,c_2\}}\right\}$.

On the other hand, by the definition of λ_{k+1}, we observe that $\lambda_{k+1} \leq \xi_k\lambda_k + \rho_k$, for each $k \in \mathbb{N}$. It follows from the the properties of the sequences $\{\xi_k\}$, $\{\rho_k\}$ and Lemma 2.4 that $\lim_{k\to\infty} \lambda_k$ exists. This together with the assumptions on the parameters $\sigma \in \left(0, \frac{1}{2\mu}\right)$, $\eta \in \left[\sigma, \frac{1}{\mu}\right)$, and $\mu \in (0,1)$ yields that

$$\lim_{k\to\infty}\left(\frac{\sigma}{\eta} - \frac{\mu\sigma\lambda_k}{\lambda_{k+1}}\right) = \sigma\left(\frac{1}{\eta} - \mu\right) > 0.$$

Thus, there exists $k_0 \in \mathbb{N}$ such that

$$\frac{\sigma}{\eta} - \frac{\mu\sigma\lambda_k}{\lambda_{k+1}} > 0, \forall k \geq k_0.$$

Now, let us consider for each $k \in \mathbb{N}$ such that $k \geq k_0$. By using the useful result of Lemma 3.7 and the above fact, we have

$$\|z_k - p\| \leq \|w_k - p\|. \tag{3.18}$$

In addition, by the definition of v_k and the quasi-nonexpansivity of T, one sees that

$$\begin{aligned} \|v_k - p\| &\leq \gamma_k\|w_k - p\| + (1-\gamma_k)\|Tw_k - p\| \\ &\leq \gamma_k\|w_k - p\| + (1-\gamma_k)\|w_k - p\| \\ &= \|w_k - p\|. \end{aligned} \tag{3.19}$$

Using the above relations, in view of the definition of x_{k+1} and the quasi-nonexpansivity of T, we obtain

$$\begin{aligned} \|x_{k+1} - p\| &\leq \alpha_k\|v_k - p\| + (1-\alpha_k)\|Tz_k - p\| \\ &\leq \alpha_k\|w_k - p\| + (1-\alpha_k)\|z_k - p\| \\ &\leq \alpha_k\|w_k - p\| + (1-\alpha_k)\|w_k - p\| \\ &= \|w_k - p\|. \end{aligned} \tag{3.20}$$

Moreover, due to the definition of w_k, we observe that

$$\begin{aligned} \|w_k - p\| &= \|(1-\beta_k)(x_k - p) + (1-\beta_k)\theta_k(x_k - x_{k-1}) - \beta_k p\| \\ &\leq (1-\beta_k)\|x_k - p\| + (1-\beta_k)\theta_k\|x_k - x_{k-1}\| + \beta_k\|p\| \\ &= (1-\beta_k)\|x_k - p\| + \beta_k\left[(1-\beta_k)\frac{\theta_k}{\beta_k}\|x_k - x_{k-1}\| + \|p\|\right]. \end{aligned} \tag{3.21}$$

It follows from the choices of the sequences $\{\theta_k\}$ that

$$(1-\beta_k)\frac{\theta_k}{\beta_k}\|x_k - x_{k-1}\| \leq (1-\beta_k)\frac{\epsilon_k}{\beta_k}.$$

Applying the fact that $\lim_{k\to\infty} \frac{\epsilon_k}{\beta_k} = 0$, we obtain

$$\lim_{k\to\infty}(1-\beta_k)\frac{\theta_k}{\beta_k}\|x_k - x_{k-1}\| = 0. \tag{3.22}$$

Thus, there exists a constant $M_1 > 0$ such that

$$(1-\beta_k)\frac{\theta_k}{\beta_k}\|x_k - x_{k-1}\| \le M_1. \tag{3.23}$$

Using this one together with the relations (3.20) and (3.21), we have

$$\begin{aligned} \|x_{k+1}-p\| &\le (1-\beta_k)\|x_k - p\| + \beta_k\left(M_1 + \|p\|\right) \\ &\le \max\{\|x_k - p\|, M_1 + \|p\|\} \\ &\le \cdots \\ &\le \max\{\|x_{k_0} - p\|, M_1 + \|p\|\}. \end{aligned}$$

This implies that the sequence $\{\|x_k - p\|\}$ is bounded. Consequently, $\{x_k\}$ is a bounded sequence.

Besides, the relations (3.21) and (3.23) imply that

$$\begin{aligned} \|w_k - p\|^2 &\le \left[(1-\beta_k)\|x_k - p\| + \beta_k(M_1 + \|p\|)\right]^2 \\ &= (1-\beta_k)^2\|x_k - p\|^2 + \beta_k\left[2(1-\beta_k)(M_1+\|p\|)\|x_k - p\| + \beta_k(M_1+\|p\|)^2\right] \\ &\le \|x_k - p\|^2 + \beta_k M_2, \end{aligned} \tag{3.24}$$

where $M_2 = \sup_{k\ge k_0}\left\{2(1-\beta_k)(M_1+\|p\|)\|x_k-p\| + \beta_k(M_1+\|p\|)^2\right\} > 0$. Thus, applying Lemma 3.7 to the above relation, we have

$$\begin{aligned} \|z_k - p\|^2 &\le \|x_k - p\|^2 + \beta_k M_2 - \left(\frac{\sigma}{\eta} - \frac{\mu\sigma\lambda_k}{\lambda_{k+1}}\right)\|w_k - y_k\|^2 \\ &\quad - \left(\frac{\sigma}{\eta} - \frac{\mu\sigma\lambda_k}{\lambda_{k+1}}\right)\|y_k - z_k\|^2. \end{aligned} \tag{3.25}$$

On the other hand, from the definition of x_{k+1} and by utilizing the quasi-nonexpansivity of T and the relation (3.19), we obtain

$$\begin{aligned} \|x_{k+1}-p\|^2 &= \|\alpha_k(v_k - p) + (1-\alpha_k)(Tz_k - p)\|^2 \\ &= \alpha_k\|v_k - p\|^2 + (1-\alpha_k)\|Tz_k - p\|^2 - \alpha_k(1-\alpha_k)\|Tz_k - v_k\|^2 \\ &\le \alpha_k\|w_k - p\|^2 + (1-\alpha_k)\|z_k - p\|^2 - \alpha_k(1-\alpha_k)\|Tz_k - v_k\|^2. \end{aligned}$$

This together with the relations (3.24) and (3.25) yields that

$$\begin{aligned} \|x_{k+1}-p\|^2 &\le \|x_k - p\|^2 + \beta_k M_2 - (1-\alpha_k)\left(\frac{\sigma}{\eta} - \frac{\mu\sigma\lambda_k}{\lambda_{k+1}}\right)\|w_k - y_k\|^2 \\ &\quad -(1-\alpha_k)\left(\frac{\sigma}{\eta} - \frac{\mu\sigma\lambda_k}{\lambda_{k+1}}\right)\|y_k - z_k\|^2 - \alpha_k(1-\alpha_k)\|Tz_k - v_k\|^2. \end{aligned}$$

This implies that

$$\begin{aligned} &(1-\alpha_k)\left[\left(\frac{\sigma}{\eta} - \frac{\mu\sigma\lambda_k}{\lambda_{k+1}}\right)\|w_k - y_k\|^2 + \left(\frac{\sigma}{\eta} - \frac{\mu\sigma\lambda_k}{\lambda_{k+1}}\right)\|y_k - z_k\|^2 + \alpha_k\|Tz_k - v_k\|^2\right] \\ &\le \|x_k - p\|^2 - \|x_{k+1}-p\|^2 + \beta_k M_2. \end{aligned} \tag{3.26}$$

Next, we will show that $\{x_k\}$ converges strongly to $\tilde{p} := P_{EP(f,C)\cap F(T)}(0)$. We investigate the following two possible cases.

Case 1. Suppose that $\|x_{k+1} - \tilde{p}\| \leq \|x_k - \tilde{p}\|$, for all $k \geq k_0$. This means that $\{\|x_k - \tilde{p}\|\}_{k\geq k_0}$ is a non-increasing sequence. Consequently, by using this one together with the boundness property of $\{\|x_k - \tilde{p}\|\}$, we get that the limit of $\|x_k - \tilde{p}\|$ exists. It follows from the relation (3.26) and the properties of the control sequences $\{\alpha_k\}$ and $\{\beta_k\}$ that

$$\lim_{k\to\infty} \|w_k - y_k\| = 0, \tag{3.27}$$

$$\lim_{k\to\infty} \|y_k - z_k\| = 0, \tag{3.28}$$

and

$$\lim_{k\to\infty} \|Tz_k - v_k\| = 0. \tag{3.29}$$

These imply that

$$\lim_{k\to\infty} \|w_k - z_k\| = 0. \tag{3.30}$$

Furthermore, since $\|w_k - x_k\| \leq \theta_k\|x_k - x_{k-1}\| + \beta_k\theta_k\|x_k - x_{k-1}\| + \beta_k\|x_k\|$, it follows from $\lim\limits_{k\to\infty} \theta_k\|x_k - x_{k-1}\| = 0$ and $\lim\limits_{k\to\infty} \beta_k = 0$ that

$$\lim_{k\to\infty} \|w_k - x_k\| = 0. \tag{3.31}$$

Combining with (3.27) implies that

$$\lim_{k\to\infty} \|x_k - y_k\| = 0. \tag{3.32}$$

This together with (3.28) yields that

$$\lim_{k\to\infty} \|x_k - z_k\| = 0. \tag{3.33}$$

Moreover, by the definition of v_k and $\lim_{k\to\infty} \gamma_k = 1$, we have

$$\begin{aligned} \lim_{k\to\infty} \|v_k - w_k\| &= \lim_{k\to\infty} \|\gamma_k w_k + (1-\gamma_k)Tw_k - w_k\| \\ &= \lim_{k\to\infty} (1-\gamma_k)\|Tw_k - w_k\| \\ &= 0. \end{aligned} \tag{3.34}$$

So, since $\|Tz_k - z_k\| \leq \|Tz_k - v_k\| + \|v_k - w_k\| + \|w_k - z_k\|$ and the facts (3.29), (3.30), (3.34), we obtain

$$\lim_{k\to\infty} \|Tz_k - z_k\| = 0. \tag{3.35}$$

On the other hand, we observe that

$$\begin{aligned} \|w_k - \tilde{p}\|^2 &= \|(1-\beta_k)(x_k - \tilde{p}) + (1-\beta_k)\theta_k(x_k - x_{k-1}) - \beta_k\tilde{p}\|^2 \\ &\leq (1-\beta_k)\|x_k - \tilde{p}\|^2 + 2(1-\beta_k)\theta_k\langle x_k - x_{k-1}, w_k - \tilde{p}\rangle + 2\beta_k\langle -\tilde{p}, w_k - \tilde{p}\rangle \\ &\leq (1-\beta_k)\|x_k - \tilde{p}\|^2 + 2(1-\beta_k)\theta_k\|x_k - x_{k-1}\|\|w_k - \tilde{p}\| \\ &\quad +2\beta_k\langle -\tilde{p}, w_k - x_k\rangle + 2\beta_k\langle -\tilde{p}, x_k - \tilde{p}\rangle \\ &\leq (1-\beta_k)\|x_k - \tilde{p}\|^2 + 2(1-\beta_k)\theta_k\|x_k - x_{k-1}\|\|w_k - \tilde{p}\| \\ &\quad +2\beta_k\|\tilde{p}\|\|w_k - x_k\| + 2\beta_k\langle x_k - \tilde{p}, -\tilde{p}\rangle \\ &= (1-\beta_k)\|x_k - \tilde{p}\|^2 + \beta_k\bigg(2(1-\beta_k)\frac{\theta_k}{\beta_k}\|x_k - x_{k-1}\|\|w_k - \tilde{p}\| + 2\|\tilde{p}\|\|w_k - x_k\| \\ &\quad +2\langle x_k - \tilde{p}, -\tilde{p}\rangle\bigg). \end{aligned}$$

It follows from the relation (3.20) that

$$\begin{aligned}\|x_{k+1}-\tilde{p}\|^2 &\leq (1-\beta_k)\|x_k-\tilde{p}\|^2+\beta_k\Bigg(2(1-\beta_k)\frac{\theta_k}{\beta_k}\|x_k-x_{k-1}\|\|w_k-\tilde{p}\| \\ &\quad +2\|\tilde{p}\|\|w_k-x_k\|+2\langle x_k-\tilde{p},-\tilde{p}\rangle\Bigg). \end{aligned} \tag{3.36}$$

Now, we will complete the proof of this theorem by applying the Lemma 2.5. The remaining part of the proof is to show that $\omega_w(x_k)\subset EP(f,C)\cap F(T)$. Let $x^*\in\omega_w(x_k)$ and $\{x_{k_n}\}$ be a subsequence of $\{x_k\}$ such that $x_{k_n}\rightharpoonup x^*$, as $n\to\infty$. We know that, by using (3.31), (3.32), and (3.33), we also have $w_{k_n}\rightharpoonup x^*$, $y_{k_n}\rightharpoonup x^*$, and $z_{k_n}\rightharpoonup x^*$, as $n\to\infty$. Since C is closed and convex set, so C is weakly closed, therefore we can confirm that $x^*\in C$.

Next, in view of the relations (3.12), (3.13), and (3.16), we obtain

$$\begin{aligned}\sigma\lambda_{k_n}f(y_{k_n},y) &\geq \sigma\lambda_{k_n}f(y_{k_n},z_{k_n})+\langle w_{k_n}-z_{k_n},y-z_{k_n}\rangle \\ &\geq \sigma\lambda_{k_n}f(w_{k_n},z_{k_n})-\sigma\lambda_{k_n}f(w_{k_n},y_{k_n})-\frac{\mu\sigma\lambda_{k_n}}{2\lambda_{k_n+1}}\|w_{k_n}-y_{k_n}\|^2 \\ &\quad -\frac{\mu\sigma\lambda_{k_n}}{2\lambda_{k_n+1}}\|y_{k_n}-z_{k_n}\|^2+\langle w_{k_n}-z_{k_n},y-z_{k_n}\rangle \\ &\geq \frac{\sigma}{\eta}\langle y_{k_n}-w_{k_n},y_{k_n}-z_{k_n}\rangle-\frac{\mu\sigma\lambda_{k_n}}{2\lambda_{k_n+1}}\|w_{k_n}-y_{k_n}\|^2 \\ &\quad -\frac{\mu\sigma\lambda_{k_n}}{2\lambda_{k_n+1}}\|y_{k_n}-z_{k_n}\|^2+\langle w_{k_n}-z_{k_n},y-z_{k_n}\rangle, \end{aligned} \tag{3.37}$$

for each $y\in C$. Thus, by using the facts (3.27), (3.28), (3.30), and the boundedness of $\{z_k\}$, we have the right-hand side of the above inequality tends to zero. It follows from the sequentially weakly upper semicontinuity of f and the parameters $\sigma,\lambda_{k_n}>0$ that

$$0\leq\limsup_{n\to\infty}f(y_{k_n},y)\leq f(x^*,y),\forall y\in C.$$

This means that $x^*\in EP(f,C)$. On the other hand, since $z_{k_n}\rightharpoonup x^*$, as $n\to\infty$, and (3.35), then by the demiclosedness at zero of $I-T$, we have $x^*\in F(T)$. Then, we had shown that $x^*\in EP(f,C)\cap F(T)$, and so $\omega_w(x_k)\subset EP(f,C)\cap F(T)$.

Finally, from the properties of $\tilde{p}:=P_{EP(f,C)\cap F(T)}(0)$ and $x^*\in\omega_w(x_k)\subset EP(f,C)\cap F(T)$, we obtain that

$$\limsup_{k\to\infty}\langle x_k-\tilde{p},-\tilde{p}\rangle=\lim_{n\to\infty}\langle x_{k_n}-\tilde{p},-\tilde{p}\rangle=\langle x^*-\tilde{p},-\tilde{p}\rangle\leq 0. \tag{3.38}$$

Hence, by (3.22), (3.31), (3.36), (3.38), and Lemma 2.5, we have

$$\lim_{k\to\infty}\|x_k-\tilde{p}\|=0. \tag{3.39}$$

This completes the proof for the first case.

Case 2. Suppose that there exists a subsequence $\{\|x_{k_i}-\tilde{p}\|\}$ of $\{\|x_k-\tilde{p}\|\}$ such that

$$\|x_{k_i}-\tilde{p}\|<\|x_{k_i+1}-\tilde{p}\|,\ \forall i\in\mathbb{N}.$$

According to Lemma 2.6, there exists a non-decreasing sequence $\{m_n\}\subset\mathbb{N}$ such that $\lim_{n\to\infty}m_n=\infty$, and

$$\|x_{m_n}-\tilde{p}\|\leq\|x_{m_n+1}-\tilde{p}\| \text{ and } \|x_n-\tilde{p}\|\leq\|x_{m_n+1}-\tilde{p}\|,\ \forall n\in\mathbb{N}. \tag{3.40}$$

This together with the relation (3.26) yields that

$$
\begin{aligned}
&(1-\alpha_{m_n})\Bigg[\left(\frac{\sigma}{\eta}-\frac{\mu\sigma\lambda_{m_n}}{\lambda_{m_n+1}}\right)\|w_{m_n}-y_{m_n}\|^2+\left(\frac{\sigma}{\eta}-\frac{\mu\sigma\lambda_{m_n}}{\lambda_{m_n+1}}\right)\|y_{m_n}-z_{m_n}\|^2\\
&+\alpha_{m_n}\|Tz_{m_n}-v_{m_n}\|^2\Bigg]\\
&\le \|x_{m_n}-\tilde{p}\|^2-\|x_{m_n+1}-\tilde{p}\|^2+\beta_{m_n}M_2\\
&\le \|x_{m_n+1}-\tilde{p}\|^2-\|x_{m_n+1}-\tilde{p}\|^2+\beta_{m_n}M_2\\
&=\beta_{m_n}M_2.
\end{aligned}
$$

Following the line proof of Case 1, we can show that

$$\lim_{n\to\infty}\|w_{m_n}-y_{m_n}\|=0,\ \lim_{n\to\infty}\|y_{m_n}-z_{m_n}\|=0, \tag{3.41}$$

$$\lim_{n\to\infty}\|w_{m_n}-z_{m_n}\|=0,\ \lim_{n\to\infty}\|x_{m_n}-y_{m_n}\|=0, \tag{3.42}$$

$$\lim_{n\to\infty}\|w_{m_n}-x_{m_n}\|=0,\ \lim_{n\to\infty}\|Tz_{m_n}-z_{m_n}\|=0, \tag{3.43}$$

$$\limsup_{n\to\infty}\langle x_{m_n}-\tilde{p},-\tilde{p}\rangle\le 0, \tag{3.44}$$

and

$$
\begin{aligned}
\|x_{m_n+1}-\tilde{p}\|^2 \le\ & (1-\beta_{m_n})\|x_{m_n}-\tilde{p}\|^2+\beta_{m_n}\Bigg(2(1-\beta_{m_n})\frac{\theta_{m_n}}{\beta_{m_n}}\|x_{m_n}-x_{m_n-1}\|\|w_{m_n}-\tilde{p}\|\\
&+2\|\tilde{p}\|\|w_{m_n}-x_{m_n}\|+2\langle x_{m_n}-\tilde{p},-\tilde{p}\rangle\Bigg).
\end{aligned}
$$

Combining with the relation (3.40) implies that

$$
\begin{aligned}
\|x_{m_n+1}-\tilde{p}\|^2 \le\ & (1-\beta_{m_n})\|x_{m_n+1}-\tilde{p}\|^2+\beta_{m_n}\Bigg(2(1-\beta_{m_n})\frac{\theta_{m_n}}{\beta_{m_n}}\|x_{m_n}-x_{m_n-1}\|\|w_{m_n}-\tilde{p}\|\\
&+2\|\tilde{p}\|\|w_{m_n}-x_{m_n}\|+2\langle x_{m_n}-\tilde{p},-\tilde{p}\rangle\Bigg).
\end{aligned}
$$

Using this one together with the relation (3.40) again, we obtain

$$
\begin{aligned}
\|x_n-\tilde{p}\|^2 \le\ & 2(1-\beta_{m_n})\frac{\theta_{m_n}}{\beta_{m_n}}\|x_{m_n}-x_{m_n-1}\|\|w_{m_n}-\tilde{p}\|+2\|\tilde{p}\|\|w_{m_n}-x_{m_n}\|\\
&+2\langle x_{m_n}-\tilde{p},-\tilde{p}\rangle.
\end{aligned}
$$

Then, by using (3.22), (3.43), and (3.44), we have

$$\limsup_{n\to\infty}\|x_n-\tilde{p}\|^2\le 0.$$

Hence, we can conclude that the sequence $\{x_n\}$ converges strongly to $\tilde{p}=P_{EP(f,C)\cap F(T)}(0)$. This completes the proof. □

4. Numerical experiments

This section will consider some examples and numerical results to illustrate the convergence of the proposed algorithm. We will compare the introduced algorithm with Algorithm 1.5 in Example 4.1 and Algorithm 1.6 in Example 4.2. All the numerical experiments are written in Matlab R2021b and performed on a MacBook Air with Apple M1 and RAM 8.00 GB. In both two examples Example 4.1 and Example 4.2, for each considered matrix, the $\|\cdot\|$ means the spectral norm.

Example 4.1. Let $H = \mathbb{R}^n$ be a real Hilbert space with the Euclidean norm. We consider the equilibrium and fixed point problems when $T = I_{\mathbb{R}^n}$ is the identity mapping on $\mathbb{R}^n$. It follows that the equilibrium and fixed point problems become the equilibrium problem. In this case, we compare Algorithm 1 with Algorithm 1.5. The bifunction $\tilde{f}$ which is given by the form of Nash-Cournot oligopolistic equilibrium models of electricity markets, see [11, 28], is defined by

$$\tilde{f}(x,y) = \langle Px + Qy, y - x\rangle, \quad \forall x, y \in \mathbb{R}^n, \tag{4.45}$$

where $P, Q \in \mathbb{R}^{n\times n}$ are matrices such that Q is symmetric positive semidefinite and $Q-P$ is negative semidefinite. Notice that $\tilde{f}(x,y)+\tilde{f}(y,x) = (x-y)^T(Q-P)(x-y), \forall x, y \in \mathbb{R}^n$. Thus, by the property of $Q-P$, we have that $\tilde{f}$ is a monotone operator. Now, we consider the bifunction f which is generated by

$$f(x,y) = \begin{cases} \tilde{f}(x,y), & \text{if } (x,y) \in C\times C, \\ 0, & \text{otherwise,} \end{cases} \tag{4.46}$$

where $C = \prod_{i=1}^{n}[-5,5]$ is the constrained box, see [32]. We observe that the bifunction f satisfies Lipschitz-type continuous, see [36].

Here, the numerical experiment is considered under the following setting: the matrices P and Q are randomly chosen from the interval $[-5,5]$ such that they satisfy the above required properties. The control parameters of Algorithm 1 are taken as follows: $\lambda_1 = 0.6$, $\tau = 0.6$, $\mu = 0.4$, $\gamma_k = 1 - \frac{1}{k+2}$, $\alpha_k = 0.01 + \frac{1}{k+1}$, $\epsilon_k = \frac{1}{(k+1)^2}$, and $\theta_k = \overline{\theta}_k$. Besides, the starting points $x_0 = x_1 \in \mathbb{R}^n$ are randomly chosen from the interval $[-5,5]$. Algorithm 1 was tested along with Algorithm 1.5 by using the stopping criteria $\frac{\|x_{k+1}-x_k\|}{\|x_k\|+1} < 10^{-6}$.

In the first experiment, we fix the parameters $\xi_k = 1 + \frac{1}{(k+1)^{1.1}}$, $\rho_k = \frac{1}{(k+1)^{1.1}}$, $\beta_k = \frac{1}{k+1}$, and present results for any collections of parameters $\eta = 0.6, 0.8, 1, 1.2, 1.4$ and $\sigma = 0.6$, 0.8, 1, 1.2 when $n = 10$. We omit the combinations that do not satisfy the assumption in Theorem 3.1 and label it by $-$.

Table 1. Influence of parameters η and σ in Algorithm 1 where $\xi_k = 1 + \frac{1}{(k+1)^{1.1}}$, $\rho_k = \frac{1}{(k+1)^{1.1}}$, and $\beta_k = \frac{1}{k+1}$ for the equilibrium problem in Example 4.1

Algorithm 1	$\eta = 0.6$		$\eta = 0.8$		$\eta = 1$		$\eta = 1.2$		$\eta = 1.4$		Algorithm 1.5	
σ	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time
0.6	91	0.11	122	0.12	143	0.13	165	0.13	184	0.13		
0.8	-	-	96	0.09	121	0.09	141	0.12	154	0.12	1626	1.07
1	-	-	-	-	94	0.09	109	0.09	122	0.10		
1.2	-	-	-	-	-	-	88	0.07	105	0.09		

From Table 1, we presented the number of iterations (Iter) and the CPU time (Time) in seconds. The best choice of the involved parameters for both cases is $\eta = 1.2$ and $\sigma = 1.2$. This means the number of iterations and the CPU time of Algorithm 1 from these cases are better than other all considered cases. Notice that, for each fixed parameter σ, as η increases, both the number of iterations and CPU time of Algorithm 1 increase. Conversely, for each fixed parameter η, as σ increases, both the number of iterations and CPU time of Algorithm 1 reduce, demonstrating that the performance of Algorithm 1 improves. These lead to the conclusion that the superior numerical performance of Algorithm 1 is influenced by the choice of parameters η and σ. However, in all considered cases, the number of iterations as well as the CPU time of Algorithm 1 are better than those of Algorithm 1.5.

In the second experiment, we regard the influence of parameters ξ_k and ρ_k where the parameters $\eta = 1.2$, $\sigma = 1.2$, and $\beta_k = \frac{1}{k+1}$ are fixed. The results are presented for any collections of parameters $\xi_k = 1, 1 + \frac{1}{(k+1)^{1.1}}$ and $\rho_k = 0, \frac{1}{(k+1)^{1.1}}$ when $n = 10$.

TABLE 2. Influence of parameter ξ_k and ρ_k in Algorithm 1 where $\eta = 1.2$, $\sigma = 1.2$, and $\beta_k = \frac{1}{k+1}$ for the equilibrium problem in Example 4.1

Algorithm 1	$\xi_k = 1$		$\xi_k = 1 + \frac{1}{(k+1)^{1.1}}$		Algorithm 1.5	
ρ_k	Iter	Time	Iter	Time	Iter	Time
0	257	0.32	80	0.07	1229	0.83
$\frac{1}{(k+1)^{1.1}}$	69	0.06	68	0.06		

From Table 2, we see that the related parameters $\xi_k = 1 + \frac{1}{(k+1)^{1.1}}$ and $\rho_k = \frac{1}{(k+1)^{1.1}}$ provide better the number of iterations and the CPU time than other cases. Moreover, in the case $\xi_k = 1$ and $\rho_k = 0$, the step size $\{\lambda_k\}$ in Algorithm 1 is reduced to a non-increasing sequence, which affects the efficiency of Algorithm 1, in the terms of both the number of iterations and CPU time. Finally, the number of iterations and the CPU time of Algorithm 1 are better than those of Algorithm 1.5 as in the above experiment.

In the third experiment, the parameters β_k is considered by fixing the best parameters $\eta = 1.2$, $\sigma = 1.2$, and $\xi_k = 1 + \frac{1}{(k+1)^{1.1}}$, $\rho_k = \frac{1}{(k+1)^{1.1}}$. We show results for any collections of parameters $\beta_k = \frac{1}{k+1}, \frac{1}{5(k+1)}, \frac{1}{10(k+1)}$ when $n = 10$.

TABLE 3. Influence of parameter β_k in Algorithm 1 where $\eta = 1.2$, $\sigma = 1.2$, and $\xi_k = 1 + \frac{1}{(k+1)^{1.1}}$, $\rho_k = \frac{1}{(k+1)^{1.1}}$ for the equilibrium problem in Example 4.1

	Algorithm 1		Algorithm 1.5	
β_k	Iter	Time	Iter	Time
$\frac{1}{k+1}$	59	0.06		
$\frac{1}{5(k+1)}$	126	0.09	1364	0.88
$\frac{1}{10(k+1)}$	134	0.11		

From Table 3, we may suggest that the largest size of parameter β_k, as $\beta_k = \frac{1}{k+1}$, yields better the number of iterations and the CPU time than other cases. This implies that the inertial factor β_k included in Algorithm 1 improves the speed of convergence of Algorithm 1 when the appropriate value of another inertial factor θ_k is chosen. Besides, the number of iterations and the CPU time of Algorithm 1 are better than those of Algorithm 1.5, indicating that Algorithm 1 has a better performance.

Example 4.2. Let $H = \mathbb{R}^2$ be a real Hilbert space with the Euclidean norm and $C = \prod_{i=1}^{2}[-10, 10]$ be the constrained box. We consider the operator F which is given by

$$Fx = \begin{pmatrix} \left(x_1^2 + (x_2 - 1)^2\right)(1 + x_2) \\ -x_1^3 - x_1(x_2 - 1)^2 \end{pmatrix},$$

where $x = (x_1, x_2)^T$. Set $\tilde{f}(x,y) = \langle Fx, y-x\rangle, \forall x, y \in \mathbb{R}^2$. We observe that the operator F is pseudomonotone rather than monotone, see [31], and thus the bifunction $\tilde{f}$ is pseudomonotone on C. Notice that $EP(\tilde{f}, C)$ has a unique element as $x^* = (0, -1)^T$.

On the other hand, for a convex function $g : \mathbb{R}^2 \to \mathbb{R}$ such that there is $x \in \mathbb{R}^2$ satisfied $g(x) \le 0$, we consider a mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$, which is defined by

$$Tx = \begin{cases} x - \frac{g(x)}{\|z_x\|^2} z_x, & \text{if } g(x) > 0, \\ x, & \text{otherwise,} \end{cases}$$

where $z_x \in \partial g(x)$. Then, we know that T is a quasi-nonexpansive mapping with $I - T$ demiclosed at zero and $F(T) = \{x \in \mathbb{R}^2 : g(x) \le 0\}$, see [3, 17].

The numerical experiment is considered under the following setting: the bifunction f, and the control parameters are given as in Example 4.1 by fixing the best choice of parameters $\eta = 1.2$, $\sigma = 1.2$, $\xi_k = 1 + \frac{1}{(k+1)^{1.1}}$, $\rho_k = \frac{1}{(k+1)^{1.1}}$, and $\beta_k = \frac{1}{k+1}$. In addition, we consider $g(x) = \max\{0, \langle c, x\rangle + d\}$ where the vector $c \in \mathbb{R}^2$ is randomly chosen from the interval $(0, 2)$ and the real number d is randomly chosen from the interval $(-2, -3)$. Here, the starting points $x_0 = x_1 \in \mathbb{R}^2$ are randomly chosen from the interval $[-10, 10]$ and the results are presented for any collections of parameters $\gamma_k = 1 - \frac{1}{k+2}$, 1 and $\alpha_k = 0.01 + \frac{1}{k+1}$, $0.5 + \frac{1}{k+3}$, $0.9 - \frac{1}{k+2}$. Algorithm 1 was tested along with Algorithm 1.6 by using the stopping criteria $\frac{\|x_{k+1} - x_k\|}{\|x_k\| + 1} < 10^{-4}$.

TABLE 4. Influence of parameters γ_k and α_k in Algorithm 1 for the equilibrium and fixed point problems in Example 4.2

Algorithm 1	$\gamma_k = 1 - \frac{1}{k+2}$		$\gamma_k = 1$		Algorithm 1.6	
α_k	Iter	Time	Iter	Time	Iter	Time
$0.01 + \frac{1}{k+1}$	103	0.11	118	0.18		
$0.5 + \frac{1}{k+3}$	120	0.10	120	0.10	22293	14.79
$0.9 - \frac{1}{k+2}$	861	0.56	861	0.60		

Table 4 shows that the concerned parameters $\gamma_k = 1 - \frac{1}{k+2}$ and $\alpha_k = 0.01 + \frac{1}{k+1}$ provide better the number of iterations and the CPU time than other cases. Furthermore, we observe that the number of iterations and the CPU time of the parameter $\gamma_k = 1$, in which the Ishikawa iteration reduces to the Mann iteration, are greater than or equal to those in other cases. Finally, Algorithm 1 outperforms Algorithm 1.6 in terms of both the number of iterations and CPU time.

5. CONCLUSIONS

We present an algorithm for solving the equilibrium and fixed point problems when the bifunction is pseudomonotone and satisfies Lipschitz-type continuous and the mapping is quasi-nonexpansive in a real Hilbert space. The modified inertial and extragradient methods together with the Ishikawa iteration technique and self-adaptive non-monotonic step size concept are introduced for establishing a sequence which is strongly convergent to a common solution of the pseudomonotone equilibrium problem and the fixed point problem for the quasi-nonexpansive mapping. Some numerical experiments are provided to illustrate the convergence behavior of the proposed algorithm in comparison with some appeared algorithms. One of the goals of the future research directions is to analyze the rate of convergence for the proposed algorithm.

Acknowledgments. This work is supported by Lampang Rajabhat University.

REFERENCES

[1] Alvarez, F. Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert spaces. *SIAM J. Optim.* **9** (2004), 773–782.

[2] Alvarez, F.; Attouch, H. An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator damping. *Set-valued. Anal.* **9** (2001), 3–11.

[3] Bauschke, H. H.; Chen, J.; Wang, X. A projection method for approximating fixed points of quasi nonexpansive mappings without the usual demiclosedness condition. *J. Nonlinear Convex Anal.* **15** (2014), no. 1, 129–135.

[4] Bianchi, M.; Schaible, S. Generalized monotone bifunctions and equilibrium problems. *J. Optim. Theory Appl.* **90** (1996), 31–43.

[5] Blum, E.; Oettli, W. From optimization and variational inequalities to equilibrium problems. *Math. Stud.* **63** (1994), 127–149.

[6] Browder, F. E. Semicontractive and semiaccretive nonlinear mappings in Banach spaces. *Bull. Am. Math. Soc.* **74** (1968), 660–665.

[7] Browder, F. E.; Petryshyn, W.V. Construction of fixed points of nonlinear mappings in Hilbert spaces. *J. Math. Anal. Appl.* **20** (1967), 197–228.

[8] Cegielski, A. Iterative methods for fixed point problems in Hilbert spaces, Lecture notes in mathematics 2057; Springer-Verlag: Berlin, Heidelberg, Germany, 2012.

[9] Chidume, C. E. ; Mutangadura, S.A. An example of the Mann iteration method for Lipschitz pseudocontractions. *Proc. Amer. Math. Soc.* **129** (2001), 2359–2363.

[10] Combettes, P. L.; Hirstoaga, A. Equilibrium programming in Hilbert spaces. *J. Nonlinear Convex Anal.* **6** (2005), 117–136.

[11] Contreras, J.; Klusch, M.; Krawczyk, J. B. Numerical solution to Nash-Cournot equilibria in coupled constraint electricity markets. *EEE Trans. Power Syst.* **19** (2004), 195–206.

[12] Daniele, P.; Giannessi, F.; Maugeri, A. Equilibrium problems and variational models, Kluwer. 2003

[13] Goebel, K.; Reich, S. Uniform convexity, hyperbolic geometry, and nonexpansive mappings; Marcel Dekker: New York, 1984.

[14] Hieu, D. V. An inertial-like proximal algorithm for equilibrium problems. *Math. Meth. Oper. Res.* **88** (2018), 399–415.

[15] Hieu, D. V. Cyclic subgradient extragradient methods for equilibrium problems. *Arab. J. Math.* **5** (2016), 159–175.

[16] Husain, S.; Asad, M. Viscosity-type extragradient algorithm for finding common solution of pseudomonotone equilibrium problem and fixed point problem in Hilbert space. *The Journal of Analysis.* **31** (2023), 1355–1373.

[17] Iiduka, H. Convergence analysis of iterative methods for nonsmooth convex optimization over fixed point sets of quasi-nonexpansive mappings. *Math. Program.* **159** (2016), 509–538.

[18] Iiduka, H.; Yamada, I. A subgradient-type method for the equilibrium problem over the fixed point set and its applications. *Optim.* **58** (2009), 251–261.

[19] Ishikawa, S. Fixed points by a new iteration method. *Proc. Am. Math. Soc.* **40** (1974), 147–150.

[20] Itoh, S.; Takahashi, W. The common fixed point theory of single-valued mappings and multi-valued mappings. *Pac. J. Math.* **79** (1978), 493–508.

[21] Karamardian, S.; Schaible, S.; Crouzeix, J. P. Characterizations of generalized monotone maps. *J. Optim. Theory Appl.* **76** (1993), 399–413.

[22] Mainge, P. E. Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. *Set-Valued Anal.* **16** (2008), 899–912.

[23] Mastroeni, G. On auxiliary principle for equilibrium problems, In *Equilibrium Problems and Variational Models*; Daniele, P.; Giannessi, F.; Maugeri, A., Eds.; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2003; pp. 289–298.

[24] Muu, L. D.; Oettli, W. Convergence of an adaptive penalty scheme for finding constrained equilibria. *Nonlinear Anal.* **18** (1992), 1159–1166.

[25] Osilike, M. O.; Aniagbosor, S. C. Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings. *Math. Comput. Model.* **32** (2000), no. 10, 1181–1191.

[26] Park, J. Y.; Jeong, J. U. Weak convergence to a fixed point of the sequence of Mann type iterates. *J. Math. Anal. Appl.* **184** (1994), 75–81.

[27] Petrot, N.; Rabbani, M.; Khonchaliew, M.; Dadashi, V. A new extragradient algorithm for split equilibrium problems and fixed point problems. *J. Ineq. Appl.* (2019), doi:10.1186/s13660-019-2086-7

[28] Quoc, T. D.; Anh P. N.; Muu, L. D. Dual extragradient algorithms extended to equilibrium problems. *J. Glob. Optim.* **52** (2012), 139–159.

[29] Rehman, H. U.; Kumam, P.; Dong, Q. L.; Cho, Y. J. A modified self-adaptive extragradient methodfor pseudomonotone equilibrium problem in a real Hilbert space with applications. *Math Methods Appl Sci.* **44** (2021), 3527–3547.

[30] Rehman, H. U.; Kumam, W.; Sombut, K. Inertial modification using self-adaptive subgradient extragradient techniques for equilibrium programming applied to variational inequalities and fixed-point problems. *Mathematics* **10** (2022), 1751.

[31] Shehu, Y; Dong, Q. L.; Jiang, D. Single projection method for pseudo-monotone variational inequality in Hilbert spaces. *Optimization* **68** (2019), 385–409.

[32] Suantai, S.; Petrot, N.; Khonchaliew, M. Inertial extragradient methods for solving split equilibrium problems. *Mathematics* **9** (2021), 1884.

[33] Suantai, S.; Pholasa, N.: Cholamjiak, P. The modified inertial relaxed CQ algorithm for solving the split feasibility problems. *J. Ind. Manag. Optim.* **14** (2018), no. 4, 1595–1615.

[34] Suantai, S.; Pholasa, N.: Cholamjiak, P. Relaxed CQ algorithms involving theinertial technique for multiple-sets split feasibility problems. *Revista de la Real Academia de CienciasExactas, Físicas y Naturales. Serie A. Matemáticas* **113** (2019), 1081–1099.

[35] Taddele, G. H.; Kumam, P.; Berinde, V. An extended inertial Halpern-type ball-relaxed CQ algorithm for multiple-setssplit feasibility problem. *Ann. Funct. Anal.* **13** (2022), no. 3, 48.

[36] Tran, D. Q.; Dung, L. M.; Nguyen, V. H. Extragradient algorithms extended to equilibrium problems. *Optimization* **57** (2008), 749–776.

[37] Vinh, N. T.; Muu, L. D. Inertial Extragradient Algorithms for Solving Equilibrium Problems. *Acta Math. Vietnam.* **44** (2019), 639–663.

[38] Xu, H.-K. Iterative algorithms for nonlinear operators. *J. Lond. Math. Soc.* **66** (2002), 240–256.

Department of Mathematics
Faculty of Science
Lampang Rajabhat University
Lampang, 52100, Thailand
Email address: thanyaluck@g.lpru.ac.th
Email address: komkind@g.lpru.ac.th
Email address: manatchanok@g.lpru.ac.th