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# Necessary and sufficient optimality conditions for reverse quasiconvex programs

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ABSTRACT. The goal of this paper is to examine both necessary and sufficient conditions for a specific feasible point to be a global minimizer for reverse quasiconvex programming problems. These results are obtained in terms of adequate approximate subdifferentials and can be viewed as the problem of a convex maximization problem constrained by a convex set. Sufficient conditions for optimality are also established in terms of the Greenberg-Pierskalla subdifferential. Illustrative examples are also given to illustrate the significance of the obtained results.

## 1. INTRODUCTION

Generalized convexity has grown remarkably of considerable interest in nonconvex optimization problems. The most significant one is the category of quasiconvex functions. Recently, there have been substantial improvements in the study of optimality criteria for reverse quasiconvex minimization problems [12], that is, minimizing a quasiconvex function $f : \mathbb{R}^n \to \mathbb{R}$ subject to a reverse quasiconvex constraint $\{x \in \mathbb{R}^n : g(x) \geq 0\}$ defined by a quasiconvex function $g : \mathbb{R}^n \to \mathbb{R}$. However, much work remains to be done.

Reverse quasiconvex optimization can be regarded as an extension of reverse convex optimization. Moreover, reverse quasiconvex constraints are important research aspects in set containment characterization and duality theory [11, 12, 13, 14]. In particular, reverse convex optimization problems constitute an all-encompassing framework for a vast class of nonconvex optimization problems including DC (difference of convex functions) programming problems and convex maximization problems, and have been widely studied in different contexts during the previous four decades (see, e.g., [1, 2, 4, 8, 9, 10, 15, 16, 17, 18] and their references). Interesting conditions that are essential and sufficient for global optimality for reverse convex optimization problems have been proposed [1, 15]. For instance, $\bar{x}$ is a global minimizer of reverse convex programming, $\min_{x\in\mathbb{R}^n}\{f(x) : x \in C,\ g(x) \geq 0\}$ where $f$ and $g$ are convex functions and $C$ is a nonempty closed convex subset of $\mathbb{R}^n$, if and only if

$$\begin{cases} g(\bar{x}) = 0; \\ \partial_\epsilon g(\bar{x}) \subseteq N_\epsilon(\mathrm{lev}_{\leq}(f, f(\bar{x})) \cap C, \bar{x}),\ \forall \epsilon \geq 0, \end{cases} \tag{1.1}$$

where $\mathrm{lev}_{\leq}(f, f(\bar{x})) := \{x \in \mathbb{R}^n : f(x) \leq f(\bar{x})\}$. Additionally, it is remarkable that the global minimizer $\bar{x}$ of the reverse convex programming can be obtained by solving the corresponding equation

$$\max_{\substack{x\in C \\ f(x)\leq f(\bar{x})}} g(x) = 0. \tag{1.2}$$

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However, this property does not generally hold for reverse quasiconvex programming, see Example 3.2. Naturally, we asked ourselves whether global optimality conditions for reverse quasiconvex programming problems can be established similarly as in (1.1). What $\epsilon$-subdifferential would be most relevant for our investigation? Can further conditions be added to (1.2) so that it also gains validity? The present paper provides an affirmative answer to these questions.

To this aim, we use the normal cone type approximate subdifferentials that contribute significantly to establishing necessary and sufficient optimality conditions for maximizing a quasiconvex function on a convex set [19]. We achieve this by using some results in quasiconvex analysis together with a proper separation theorem. We then provide global optimality conditions for reverse quasiconvex programming problems by solving the equation (2.3) under appropriate conditions. Moreover, we also investigate global optimality conditions for reverse quasiconvex programming problems in terms of adequate subdifferentials together with the traditional normal cone. Our proof method essentially is motivated by the approach as the ones in [15, 19]. It is worth mentioning here that Suzuki [12] only examined necessary optimality criteria for reverse quasiconvex programming in terms of Greenberg-Pierskalla subdifferentials. Our investigation yields another way of establishing global optimality conditions for reverse quasiconvex programming problems by employing the continuity properties along with the quasiconvexities of involved functions.

The rest of the paper is organized as follows: in the next section, we recall some definitions and results that will be required in the sequel. In Section 3, we establish conditions that are essential and sufficient for global optimality for reverse quasiconvex minimization problems in terms of suitable $\epsilon$-subdifferentials. We also present some new global optimality conditions in terms of Greenberg-Pierskalla subdifferentials along with normal cones.

## 2. Preliminaries

This section starts off by providing the fundamental concepts and notations that will be utilized throughout the article. Let $\langle u, v\rangle$ denote the inner product of two vectors $u$ and $v$ in $\mathbb{R}^n$. For a real-valued function $f : \mathbb{R}^n \to \mathbb{R}$, we denote by $\mathrm{lev}_*(f, \gamma) := \{x \in \mathbb{R}^n : f(x) * \gamma\}$ for any $\gamma \in \mathbb{R}$ its level sets of $f$ with respect to a binary relation $*$ on $\mathbb{R}$. Recall that the function $f$ is *quasiconvex* if $\mathrm{lev}_{\leq}(f, \gamma)$ is a convex set for all $\gamma \in \mathbb{R}$. Remark that any convex function is a quasiconvex function, while the reverse is typically not true. If a function $f$ is quasiconvex and every local minimizer $x \in \mathbb{R}^n$ of $f$ in $\mathbb{R}^n$ is also a global minimizer, then $f$ is said to be *essentially quasiconvex*. Evidently, any convex function is an essentially quasiconvex. Note that a real-valued continuous quasiconvex function can only be essentially quasiconvex if the following condition is met:

$$f(x_1) < f(x_2) \Longrightarrow f((1-\alpha)x_1 + \alpha x_2) < f(x_2),$$

whenever $x_1, x_2 \in \mathbb{R}^n$, and $\alpha \in ]0, 1[$ (see, e.g., [3, Theorem 3.37]).

For each $\epsilon \geq 0$, one associates the *$\epsilon$-normal set* and the *normal cone* of the nonempty set $E \subseteq \mathbb{R}^n$ at $a \in E$ by

$$N_\epsilon(E, a) := \{u \in \mathbb{R}^n : \langle u, x - a\rangle \leq \epsilon, \ \forall x \in E\}, \ N(E, a) := N_0(E, a).$$

The normal cone type (approximate) subdifferentials of $f$ at $\bar{x}$ are defined by

$$\partial^v_\epsilon f(\bar{x}) := N_\epsilon(\mathrm{lev}_{\leq}(f, f(\bar{x})), \bar{x}), \ \partial^v f(\bar{x}) := \partial^v_0 f(\bar{x}).$$

Let us now review the results that will be relevant in the sequel.

**Lemma 2.1.** [13, Theorem 11] *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous quasiconvex function. If there exists $\gamma \in \mathbb{R}$ such that $\operatorname{int}(\operatorname{lev}_{\leq}(f,\gamma)) \neq \emptyset$ and $\operatorname{int}(\operatorname{lev}_{=}(f,\gamma)) = \emptyset$, then one has*

(i) $\operatorname{lev}_{<}(f,\gamma) = \operatorname{int}(\operatorname{lev}_{\leq}(f,\gamma))$,

(ii) $cl(\operatorname{lev}_{<}(f,\gamma)) = \operatorname{lev}_{\leq}(f,\gamma)$.

The following theorem is directly from [19, Proposition 2].

**Theorem 2.1.** *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $\bar{x} \in \mathbb{R}^n$ be such that $\operatorname{lev}_{<}(h, h(\bar{x})) \neq \emptyset$. For each $\epsilon \geq 0$, it holds that*

$$\partial^v_\epsilon h(\bar{x}) = \{0\} \cup \bigcup_{\lambda>0} \lambda \partial_{\lambda^{-1}\epsilon} h(\bar{x}),$$

*where $\partial_\epsilon h(\bar{x}) := \{u \in \mathbb{R}^n : \langle u, x-\bar{x}\rangle \leq h(x)-h(\bar{x})+\epsilon,\ \forall x \in \mathbb{R}^n\}$ stands for an $\epsilon$-subdifferential of $h$ at $\bar{x}$.*

**Lemma 2.2.** [19, Proposition 1](see also [7]) *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function and let $K$ be a nonempty convex subset of $\mathbb{R}^n$. If $\bar{x}$ is a maximizer of the problem $\max_{x\in\mathbb{R}^n}\{h(x) : x \in K\}$, then*

$$\partial^v_\epsilon h(\bar{x}) \subseteq N_\epsilon(K, \bar{x}),\ \forall \epsilon \geq 0. \tag{2.3}$$

*Moreover, if $\operatorname{lev}_{\leq}(h, h(\bar{x}))$ is a closed convex set and* (2.3) *holds, then $\bar{x}$ is a maximizer of the problem $\max_{x\in\mathbb{R}^n}\{h(x) : x \in K\}$.*

## 3. Main results

Consider an optimization problem of the form

$$\min_{x\in\mathbb{R}^n}\{f(x) : x \in C,\ g(x) \geq 0\}, \tag{P}$$

where $f, g : \mathbb{R}^n \to \mathbb{R}$ are real-valued continuous functions and $C$ is a nonempty closed convex subset of $\mathbb{R}^n$. Throughout this paper, we always assume that $A := \{x \in C : g(x) \geq 0\} \neq \emptyset$ and $f$ attains its minimum on $A$.

In this section, we establish necessary and sufficient optimality conditions for reverse quasiconvex minimization problems in terms of suitable $\epsilon$-subdifferentials. Here, we will proceed by assuming the following:

**(A)** there exists $x_0 \in C$ such that $g(x_0) < 0$ and $f(x_0) < \inf_{x\in A} f(x)$.

**Remark 3.1.** *It should be noted here that if $C$ is bounded, it can be concluded that $A$ is a compact set, and so, $f$ attains its minimum on $A$. Note also that* **(A)** *is equivalent to $\inf_{x\in C} f(x) < \inf_{x\in A} f(x)$ which indicates that the constraint $g(x) \geq 0$ is essential. Indeed, if $\inf_{x\in C} f(x) \geq \inf_{x\in A} f(x)$, we get that $\inf_{x\in C} f(x) = \inf_{x\in A} f(x)$. Consequently, the problem* (P) *becomes quasiconvex minimization with convex constraint.*

In order to derive necessary and sufficient optimality conditions for reverse quasiconvex minimization problems, the following lemma is needed.

**Lemma 3.3.** *Let Assumption* **(A)** *hold and $f$ be essentially quasiconvex. Then, for each $x \in C \cap \operatorname{lev}_{>}(g, 0)$, there exists $\tilde{x} \in ]x_0, x[ \subseteq C$ such that*

$$\tilde{x} \in \operatorname{lev}_{=}(g, 0) \cap \operatorname{lev}_{<}(f, f(x)).$$

*Proof.* Let $x \in C \cap \operatorname{lev}_{>}(g, 0)$ be arbitrary. By defining $r(\alpha) := \alpha x + (1-\alpha)x_0$ and $\phi(\alpha) := g(r(\alpha))$ for all $\alpha \in [0, 1]$, we assert that $\phi$ is continuous on $[0, 1]$, $\phi(0) = g(r(0)) = g(x_0) < 0$ and $\phi(1) = g(r(1)) = g(x) > 0$. Invoking an intermediate value theorem, one can find $\tilde{\alpha} \in ]0, 1[$ such that $\phi(\tilde{\alpha}) = 0$. Since $C$ is convex, we conclude that $\tilde{x} := \tilde{\alpha}x + (1-\tilde{\alpha})x_0 \in ]x_0, x[ \subseteq C$ and so, it holds that $g(\tilde{x}) = g(r(\tilde{\alpha})) = \phi(\tilde{\alpha}) = 0$. Furthermore, as $f(x_0) <$

$\inf_{x\in A} f(x) \leq f(x)$, the essential quasiconvexity of $f$ yields $f(\tilde{x}) < f(x)$, and everything has been proved. □

**Remark 3.2.** Under the assumption **(A)** along with essential quasiconvexity of $f$, if $\bar{x} \in A$ is a minimizer, then $g(\bar{x}) = 0$. Indeed, if $g(\bar{x}) > 0$, it results from Lemma 3.3 that there exists $\tilde{x} \in ]x_0, \bar{x}[ \subseteq C$ such that $\tilde{x} \in \text{lev}_{=}(g, 0) \cap \text{lev}_{<}(f, f(\bar{x}))$. This ensures that $\tilde{x} \in A$ and $f(\tilde{x}) < f(\bar{x})$ which contradicts the hypothesis that $\bar{x}$ is a minimizer of (P).

**Theorem 3.2.** *Let $f$ be a continuous essentially quasiconvex function, $g$ be a continuous quasiconvex function, and Assumption* **(A)** *hold. Let $\bar{x}$ be a feasible point of problem* (P).

(i) *If $\bar{x}$ is a minimizer of* (P)*, then for any $z \in \text{lev}_{\leq}(f, f(\bar{x})) \cap C$ satisfying $g(z) = 0$, one has*

$$\partial_\epsilon^v g(z) \subseteq N_\epsilon(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, z), \ \forall \epsilon \geq 0.$$

(ii) *If $\text{int}(\text{lev}_{=}(g, 0)) = \emptyset$, and*

$$\begin{cases} g(\bar{x}) = 0; \\ \partial_\epsilon^v g(\bar{x}) \subseteq N_\epsilon(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, \bar{x}), \ \forall \epsilon \geq 0, \end{cases} \tag{3.4}$$

*then $\bar{x}$ is a minimizer of* (P).

*Proof.* To justify (i), let us suppose that $\bar{x}$ is a minimizer of (P) and assume contrary to our assertion that there exist $z \in \text{lev}_{\leq}(f, f(\bar{x})) \cap C$, $\epsilon \geq 0$ and $u \in \partial_\epsilon^v g(z)$ such that $g(z) = 0$ and $u \notin N_\epsilon(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, z)$. It then follows that $\langle u, x - z\rangle > \epsilon$ for some $x \in \text{lev}_{\leq}(f, f(\bar{x})) \cap C$. On account of $u \in \partial_\epsilon^v g(z)$, one has

$$\langle u, y - z\rangle \leq \epsilon, \ \forall y \in \text{lev}_{\leq}(g, 0), \tag{3.5}$$

where we should remind that $\text{lev}_{\leq}(g, g(z)) = \text{lev}_{\leq}(g, 0)$. If $x \in \text{lev}_{\leq}(g, 0)$, taking $y := x$ in relation (3.5), we conclude that $\langle u, x - z\rangle \leq \epsilon$ which contradicts to $\langle u, x - z\rangle > \epsilon$. Otherwise, we would have $g(x) > 0$. According to Remark 3.3, there exists $\tilde{x} \in ]x_0, x[ \subseteq C$ such that $g(\tilde{x}) = 0$ and $f(\tilde{x}) < f(x)$. Therefore, $\tilde{x} \in A$ and $f(\tilde{x}) < f(x) \leq f(\bar{x})$, which again contradict to the hypothesis that $\bar{x}$ is a minimizer of (P).

Let us now prove (ii) by assuming that $\text{int}(\text{lev}_{=}(g, 0)) = \emptyset$ and (3.4) hold. We will verify that

$$\text{lev}_{<}(f, f(\bar{x})) \cap C \subseteq \text{lev}_{<}(g, 0). \tag{3.6}$$

Note that both $\text{lev}_{<}(f, f(\bar{x})) \cap C$ and $\text{lev}_{<}(g, 0)$ are nonempty sets due to **(A)**. Note also that $x_0 \in \text{int}(\text{lev}_{\leq}(g, 0)) \neq \emptyset$. Indeed, as $g(x_0) \in ] - \infty, 0[$, by continuity of $g$, there exists $\delta > 0$ such that $g(\mathbb{B}(x_0, \delta)) \subseteq ] - \infty, 0[$, where $\mathbb{B}(x_0, \delta) := \{x \in \mathbb{R}^n : \|x - x_0\| < \delta\}$. This implies that $\mathbb{B}(x_0, \delta) \subseteq \text{lev}_{\leq}(g, 0)$. Now, let $x$ be an arbitrary element in $\text{lev}_{<}(f, f(\bar{x})) \cap C$ fulfilling $x \notin \text{lev}_{<}(g, 0)$. By Lemma 2.1(i), $\text{lev}_{<}(g, 0) = \text{int}(\text{lev}_{\leq}(g, 0))$ is a nonempty open convex set and so, the proper convex separation theorem gives us that there exist $u \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that

$$\langle u, x\rangle > \alpha > \langle u, y\rangle, \ \forall y \in \text{lev}_{<}(g, 0). \tag{3.7}$$

Invoking Lemma 2.1(ii), we have $\text{lev}_{\leq}(g, 0) = \text{cl}(\text{lev}_{<}(g, 0))$ and we obtain that for each $y \in \text{lev}_{\leq}(g, 0)$, there exists a sequence $\{y_k\} \subset \text{lev}_{<}(g, 0)$ such that $y_k \to y$ as $k \to +\infty$. By (3.7) and passing to the limit, we actually have $\alpha \geq \langle u, y\rangle$. As we assume directly that $g(\bar{x}) = 0$, we also have $\alpha \geq \langle u, \bar{x}\rangle$. Putting $\bar{\epsilon} := \alpha - \langle u, \bar{x}\rangle \geq 0$. It follows that for any $y \in \text{lev}_{\leq}(g, 0) = \text{lev}_{\leq}(g, g(\bar{x}))$,

$$\langle u, y - \bar{x}\rangle = \langle u, y\rangle - \langle u, \bar{x}\rangle \leq \alpha - \langle u, \bar{x}\rangle = \bar{\epsilon},$$

and consequently, $u \in \partial^v_{\bar{\epsilon}} g(\bar{x})$. This together with (3.4) yields $u \in N_{\bar{\epsilon}}(\mathrm{lev}_{\leq}(f, f(\bar{x})) \cap C, \bar{x})$ which implies that $\langle u, x - \bar{x}\rangle \leq \bar{\epsilon}$ due to $x \in \mathrm{lev}_{<}(f, f(\bar{x})) \cap C$. This contradicts to the fact that

$$\langle u, x - \bar{x}\rangle = \langle u, x\rangle - \langle u, \bar{x}\rangle > \alpha - \langle u, \bar{x}\rangle = \bar{\epsilon},$$

where the inequality holds by virtue of (3.7). Consequently, (3.6) has been justified. It then follows from (3.6) that $\mathrm{lev}_{\geq}(g, 0) \subseteq (\mathbb{R}^n \backslash C) \cup \mathrm{lev}_{\geq}(f, f(\bar{x}))$, which amounts to

$$C \cap \mathrm{lev}_{\geq}(g, 0) \subseteq C \cap \mathrm{lev}_{\geq}(f, f(\bar{x})) \subseteq \mathrm{lev}_{\geq}(f, f(\bar{x})).$$

Consequently, $\bar{x}$ is a minimizer of (P) as desired. □

**Corollary 3.1.** *Let $f$ be a continuous essentially quasiconvex function, $g$ be a continuous quasiconvex function, and Assumption* **(A)** *hold. Let $\bar{x}$ be a feasible point of problem* (P). *Then necessary condition for $\bar{x}$ to be a minimizer of* (P) *is*

$$\begin{cases} g(\bar{x}) = 0; \\ \partial^v_{\epsilon} g(z) \subseteq N_{\epsilon}(\mathrm{lev}_{\leq}(f, f(\bar{x})) \cap C, z), \ \forall \epsilon \geq 0, \end{cases}$$

*for any $z \in \mathrm{lev}_{\leq}(f, f(\bar{x})) \cap C$ satisfying $g(z) = 0$. If $\mathrm{int}(\mathrm{lev}_{=}(g, 0)) = \emptyset$, the condition is sufficient.*

*Proof.* In view of Theorem 3.2, it is sufficient to establish the sufficient assertion. To this aim, suppose that the above inclusion hold for any $z \in \mathrm{lev}_{\leq}(f, f(\bar{x})) \cap C$ satisfying $g(z) = 0$. By assumption, we get that $\partial^v_{\epsilon} g(\bar{x}) \subseteq N_{\epsilon}(\mathrm{lev}_{\leq}(f, f(\bar{x})) \cap C, \bar{x}), \ \forall \epsilon \geq 0$. Invoking Theorem 3.2(ii), we conclude that $\bar{x}$ is a minimizer of (P). □

**Corollary 3.2.** *Let $f$ be a continuous essentially quasiconvex function, $g$ be a continuous quasiconvex function, and Assumption* **(A)** *hold. Let $\bar{x}$ be a feasible point of problem* (P). *Then necessary conditions for $\bar{x}$ to be a minimizer of* (P) *are*

$$\begin{cases} g(\bar{x}) = 0; \\ \partial^v_{\epsilon} g(\bar{x}) \subseteq N_{\epsilon}(\mathrm{lev}_{\leq}(f, f(\bar{x})) \cap C, \bar{x}), \ \forall \epsilon \geq 0. \end{cases}$$

*If $\mathrm{int}(\mathrm{lev}_{=}(g, 0)) = \emptyset$, these conditions are sufficient.*

*Proof.* In view of Theorem 3.2, it is sufficient to establish the necessary assertion. To do this, suppose now that $\bar{x}$ is a minimizer of (P). Invoking Theorem 3.2(i) allows us to assert that for any $z \in \mathrm{lev}_{\leq}(f, f(\bar{x})) \cap C$ satisfying $g(z) = 0$, one has

$$\partial^v_{\epsilon} g(z) \subseteq N_{\epsilon}(\mathrm{lev}_{\leq}(f, f(\bar{x})) \cap C, z), \ \forall \epsilon \geq 0. \tag{3.8}$$

By Remark 3.2, we actually have $g(\bar{x}) = 0$. This combined with (3.8) which indicates that (3.4) holds, thus yielding the desired results. □

In the case where $C = \mathbb{R}^n$, the following result follows from Corollary 3.2.

**Corollary 3.3.** *Consider the problem* (P) *with $C = \mathbb{R}^n$. Let $f$ be a continuous essentially quasiconvex function, $g$ be a continuous quasiconvex function, and Assumption* **(A)** *hold. Let $\bar{x}$ be a feasible point of problem* (P). *Then necessary conditions for $\bar{x}$ to be a minimizer of* (P) *are*

$$\begin{cases} g(\bar{x}) = 0; \\ \partial^v_{\epsilon} g(\bar{x}) \subseteq \partial^v_{\epsilon} f(\bar{x}), \ \forall \epsilon \geq 0. \end{cases}$$

*If $\mathrm{int}(\mathrm{lev}_{=}(g, 0)) = \emptyset$, these conditions are sufficient.*

*Proof.* This is an immediate consequence of Corollary 3.2 together with the definition of normal cone type approximate subdifferentials of $f$. □

As an application of Corollary 3.2, we obtain optimality requirements that are both necessary and sufficient in terms of some maximization problems with convex constraints.

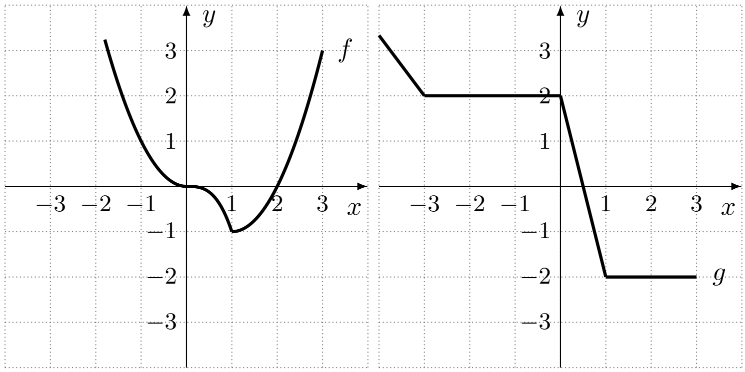


FIGURE 1. Figures for the objective function and constraint function in Example 3.1

**Corollary 3.4.** *Let $f$ be a continuous essentially quasiconvex function, $g$ be a continuous quasiconvex function, and Assumption* **(A)** *hold. Let $\bar{x}$ be a feasible point of problem* (P). *If $\bar{x}$ is a minimizer of* (P), *then*

$$\max_{\substack{x\in C\\ f(x)\le f(\bar{x})}} g(x) = 0. \tag{3.9}$$

*Moreover, if* $\operatorname{int}(\operatorname{lev}_{=}(g,0)) = \emptyset$ *and* (3.9) *hold, then $\bar{x}$ is a minimizer of* (P).

*Proof.* Note that if $\bar{x}$ is a feasible point of problem (P) and (3.9) holds then

$$0 \le g(\bar{x}) \le \max_{\substack{x\in C\\ f(x)\le f(\bar{x})}} g(x) = 0.$$

So, $g(\bar{x}) = 0$ and the result follows directly from Lemma 2.2 and Corollary 3.2. □

The following example illustrates how Corollary 3.4 can be utilized to determine a minimizer of a reverse quasiconvex programming problem.

**Example 3.1.** Consider $C := [-1,1]$ and the continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) := \min\{\max\{-x^3, (x-1)^2 - 1\}, x^2\}$$

and

$$g(x) := \begin{cases} -\frac{4}{3}x - 2 & \text{if } x \in\, ]-\infty, -3[\,; \\ 2 & \text{if } x \in\, [-3, 0[\,; \\ -4x + 2 & \text{if } x \in\, [0, 1[\,; \\ -2 & \text{if } x \in\, [1, +\infty[\,. \end{cases}$$

It can be checked that $f$ is a continuous essentially quasiconvex function and $g$ is a continuous quasiconvex function. We can see that $A = [-1, \frac{1}{2}]$ and $\inf_{x\in A} f(x) = -\frac{1}{8}$. So, Assumption **(A)** holds by taking $x_0 := 1$. We also have $\operatorname{lev}_{=}(g,0) = \{\frac{1}{2}\}$ which implies $\operatorname{int}(\operatorname{lev}_{=}(g,0)) = \emptyset$. In addition, by considering $\bar{x} := \frac{1}{2}$ we get that

$$\max_{\substack{x\in C\\ f(x)\le f(\bar{x})}} g(x) = \max_{x\in[\frac{1}{2},1]} g(x) = 0.$$

By Corollary 3.4, we conclude that $\bar{x}$ is a minimizer of (P).

Noteworthy is the fact that Corollary 3.4 is not valid without the condition $\operatorname{int}(\operatorname{lev}_{=}(g,0)) = \emptyset$. The following example demonstrates this fact.

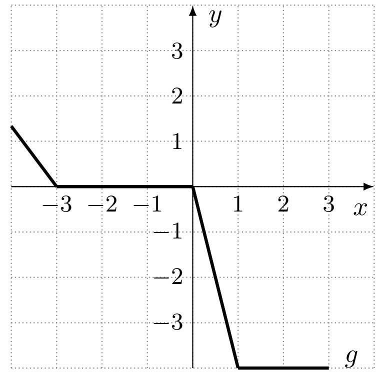


FIGURE 2. The figure of the constraint function in Example 3.2

**Example 3.2.** *Let $f$ be defined as in Example 3.1. Let $C := [-2,2]$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by*

$$g(x) := \begin{cases} -\frac{4}{3}x - 4 & \text{if } x \in ]-\infty, -3[\,; \\ 0 & \text{if } x \in [-3, 0[\,; \\ -4x & \text{if } x \in [0,1[\,; \\ -4 & \text{if } x \in [1, +\infty[\,. \end{cases}$$

*So, $A = [-2,0]$, $\inf_{x\in A} f(x) = 0$ and $\operatorname{lev}_=(g,0) = [-3,0]$. Note that Assumption **(A)** holds by taking $x_0 := 1$. Let us consider $\bar{x} := -2$. We have*

$$\max_{\substack{x\in C \\ f(x)\le f(\bar{x})}} g(x) = \max_{x\in[-2,2]} g(x) = 0.$$

*On the one hand, $\bar{x}$ is not a minimizer, i.e., $f(0) = 0 < 4 = f(\bar{x})$. In light of this, the conclusion of Corollary 3.4 fails to hold. The reason is that $\operatorname{int}(\operatorname{lev}_=(g,0)) = ]-3,0[ \neq \emptyset$.*

**Remark 3.3.** We point out that if there exists $w \in \operatorname{lev}_<(g,0)$ and $g$ is a continuous essentially quasiconvex function, the condition $\operatorname{int}(\operatorname{lev}_=(g,0)) = \emptyset$ is satisfied. Indeed, let us say that there are $x \in \mathbb{R}^n$ and $\delta > 0$ such that $\mathbb{B}(x,\delta) \subseteq \operatorname{lev}_=(g,0)$. As $g(w) < 0 = g(x)$, the essential quasiconvexity of $g$ yields $g((1-\alpha)w + \alpha x) < g(x) = 0$ for all $\alpha \in ]0,1[$. By taking $\alpha_k := 1 - \frac{1}{k}$ and $x_k := (1-\alpha_k)w + \alpha_k x$ for all $k \in \mathbb{N}$, one has $x_k \to x$ as $k \to +\infty$. So, for sufficiently large $k_0 \in \mathbb{N}$, we have $x_{k_0} \in \operatorname{lev}_=(g,0)$, which contradicts to the fact that $g(x_{k_0}) < 0$.

We now deduce necessary and sufficient criteria for optimality of a reverse convex programming problem.

**Corollary 3.5.** *Let $f$ and $g$ be real-valued convex functions on $\mathbb{R}^n$, and Assumption* **(A)** *hold. Let $\bar{x}$ be a feasible point of problem* (P)*. Then, $\bar{x}$ is a minimizer of* (P) *if and only if*

$$\begin{cases} g(\bar{x}) = 0; \\ \partial_\epsilon g(\bar{x}) \subseteq N_\epsilon(\operatorname{lev}_\le(f, f(\bar{x})) \cap C, \bar{x}), \ \forall \epsilon \ge 0. \end{cases}$$

*Proof.* Since a real-valued convex function $g$ is continuous essentially quasiconvex and $\operatorname{lev}_<(g,0) \neq \emptyset$ due to **(A)**, we conclude by Remark 3.3 that $\operatorname{int}(\operatorname{lev}_=(g,0)) = \emptyset$. In addition, in view of Lemma 2.1 and Theorem 2.1, it must yet be proved that

$$\partial_\epsilon g(\bar{x}) \subseteq N_\epsilon(\operatorname{lev}_\le(f, f(\bar{x})) \cap C, \bar{x}), \ \forall \epsilon \ge 0$$

$$\iff \{0\} \cup \bigcup_{\lambda>0} \lambda \partial_{\lambda^{-1}\epsilon} g(\bar{x}) \subseteq N_\epsilon(\operatorname{lev}_\le(f, f(\bar{x})) \cap C, \bar{x}), \ \forall \epsilon \ge 0.$$

It is clear that the implication ($\Leftarrow$) holds. To justify the implication ($\Rightarrow$), let $\epsilon \geq 0$ and $u \in \{0\} \cup \bigcup_{\lambda>0} \lambda\partial_{\lambda^{-1}\epsilon} g(\bar{x})$. If $u = 0$, it then follows by the definition of $\epsilon$-normal set that $u \in N_\epsilon(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, \bar{x})$. Now, let us consider $u \in \bigcup_{\lambda>0} \lambda\partial_{\lambda^{-1}\epsilon} g(\bar{x})$. So, there exists $\lambda > 0$ such that $\lambda^{-1}u \in \partial_{\lambda^{-1}\epsilon} g(\bar{x})$. As $\lambda^{-1}\epsilon \geq 0$, we have by assumption that $\lambda^{-1}u \in N_{\lambda^{-1}\epsilon}(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, \bar{x})$, and hence, $u \in N_\epsilon(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, \bar{x})$. □

**Remark 3.4.** Observe that the necessary assertion in Theorem 3.2 can be stated similarly as in [15, Theorem 2.3], that is,

$$\bar{x} \text{ is a minimizer of (P)} \Longrightarrow \begin{cases} \partial^v g(z) \cap N(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, z) \neq \emptyset, \\ \forall z \in [\text{lev}_{\leq}(f, f(\bar{x})) \cap C] \cap \text{lev}_{=}(g, 0), \end{cases}$$

because of

$$\begin{aligned} &\begin{cases} \partial^v_\epsilon g(z) \subseteq N_\epsilon(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, z), \ \forall \epsilon \geq 0, \\ \forall z \in [\text{lev}_{\leq}(f, f(\bar{x})) \cap C] \cap \text{lev}_{=}(g, 0) \end{cases} \\ \Longrightarrow &\begin{cases} \partial^v g(z) \subseteq N(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, z), \\ \forall z \in [\text{lev}_{\leq}(f, f(\bar{x})) \cap C] \cap \text{lev}_{=}(g, 0) \end{cases} \\ \Longrightarrow &\begin{cases} \partial^v g(z) \cap N(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, z) \neq \emptyset, \\ \forall z \in [\text{lev}_{\leq}(f, f(\bar{x})) \cap C] \cap \text{lev}_{=}(g, 0), \end{cases} \end{aligned}$$

where we remind that $\partial^v g(z)$ and $N(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, z)$ are nonempty sets for all $z \in [\text{lev}_{\leq}(f, f(\bar{x})) \cap C] \cap \text{lev}_{=}(g, 0)$. Note that, unlike in the case of reverse convex programs, the condition

$$\begin{cases} \partial^v g(z) \subseteq N(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, z), \\ \forall z \in [\text{lev}_{\leq}(f, f(\bar{x})) \cap C] \cap \text{lev}_{=}(g, 0) \end{cases}$$

is more suitable than

$$\begin{cases} \partial^v g(z) \cap N(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, z) \neq \emptyset, \\ \forall z \in [\text{lev}_{\leq}(f, f(\bar{x})) \cap C] \cap \text{lev}_{=}(g, 0), \end{cases} \tag{3.10}$$

since (3.10) always holds as $0 \in \partial^v g(z) \cap N(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, z)$ for all $z \in [\text{lev}_{\leq}(f, f(\bar{x})) \cap C] \cap \text{lev}_{=}(g, 0)$.

As we know in Remark 3.4 that (3.10) always holds, in view of the sufficient assertion, now it is interesting to ask whether there are adequate subdifferentials for which the conclusion of Theorem 3.2 holds under the nonempty intersection condition as in [15, Theorem 2.5 ]. In this way, let us recall the *Greenberg-Pierskalla subdifferential* of $f : \mathbb{R}^n \to \mathbb{R}$ at $\bar{x} \in \mathbb{R}^n$ [5] which is the set

$$\partial^* f(\bar{x}) := \{u \in \mathbb{R}^n : \langle u, x - \bar{x} \rangle < 0, \ \forall x \in \text{lev}_{<}(f, f(\bar{x}))\}.$$

We now investigate global optimality conditions for reverse quasiconvex programming problems in terms of normal cone type subdifferentials, which is motivated by [15, Theorem 2.5] and [19, Proposition 4]. In what follows, taking this supposition into account:

**(B)** For any $x \in C \cap \text{lev}_{=}(g, 0)$, it holds that $\partial^v g(x) \nsubseteq N(C, x)$.

**Theorem 3.3.** *Let $f$ be a continuous essentially quasiconvex function, $g$ be a continuous quasiconvex function, and Assumption* **(A)** *hold. Let $\bar{x}$ be a feasible point of problem* (P).

(i) *If $\bar{x}$ is a minimizer of* (P), *then for any $z \in \text{lev}_{\leq}(f, f(\bar{x})) \cap C$ satisfying $g(z) = 0$, one has*

$$\partial^v g(z) \subseteq N(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, z).$$

(ii) *If Assumption* **(B)** *holds and*

$$\partial^* g(z) \cap N(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, z) \neq \emptyset, \tag{3.11}$$

*for all $z \in \text{lev}_{\leq}(f, f(\bar{x})) \cap C$ satisfying $g(z) = 0$, then $\bar{x}$ is a minimizer of* (P).

*Proof.* (i) It can be obtained directly from Theorem 3.2 by taking $\epsilon = 0$.

(ii) Now, assume that the assumptions of the theorem are satisfied, but $\bar{x}$ is not a minimizer of (P). So, there is $x \in C \cap \mathrm{lev}_{\geq}(g, 0)$ such that $f(x) < f(\bar{x})$.

If $g(x) = 0$, it follows from **(B)** that there exist $u \in \partial^v g(x)$ and $y \in C$ such that $\langle u, y - x\rangle > 0$. Thus, for any $\alpha \in \,]0, 1[$,

$$\langle u, (1-\alpha)y + \alpha x - x\rangle = (1-\alpha)\langle u, y - x\rangle > 0,$$

and so, by virtue of $u \in \partial^v g(x)$,

$$g((1-\alpha)y + \alpha x) > g(x) = 0. \tag{3.12}$$

In particular, for each $k \in \mathbb{N}$, by taking $\alpha := 1 - \frac{1}{k}$ in relation (3.12), we have $g(x_k) > 0$ for all $k \in \mathbb{N}$ where $x_k := \frac{1}{k}y + (1 - \frac{1}{k})x$, and $x_k \to x$ as $k \to +\infty$. On the one hand, as $f(x) < f(\bar{x})$ and the continuity of $f$, there exists $\delta > 0$ such that

$$f(z) < f(\bar{x}),\ \forall z \in \mathbb{B}(x, \delta).$$

Thus, for sufficiently large $k_0 \in \mathbb{N}$, one has $g(x_{k_0}) > 0$ and $f(x_{k_0}) < f(\bar{x})$.

While maintaining generality, by replacing $x_{k_0}$ with $x$ if necessary, we may assume that there exists $x \in C$ such that $g(x) > 0$ and $f(x) < f(\bar{x})$. Then by Lemma 3.3 there exists $\alpha \in \,]0, 1[$ such that $\tilde{x} := \alpha x + (1-\alpha)x_0 \in C$, $g(\tilde{x}) = 0$ and $f(\tilde{x}) < f(x)$. On the one hand, taking $z := \tilde{x}$ in the relation (3.11), there exists $u \in \partial^* g(\tilde{x})$ fulfilling $u \in N(\mathrm{lev}_{\leq}(f, f(\bar{x})) \cap C, \tilde{x})$. Hence,

$$\begin{aligned}&\Big[x_0 \in \mathrm{lev}_{<}(g, g(\tilde{x}))\Big] \bigwedge \Big[u \in \partial^* g(\tilde{x})\Big]\\ \Longrightarrow\ &\alpha\langle u, x_0 - x\rangle = \langle u, x_0 - \tilde{x}\rangle < 0\end{aligned}$$

and

$$\begin{aligned}&\Big[x \in \mathrm{lev}_{\leq}(f, f(\bar{x})) \cap C\Big] \bigwedge \Big[u \in N(\mathrm{lev}_{\leq}(f, f(\bar{x})) \cap C, \tilde{x})\Big]\\ \Longrightarrow\ &(1-\alpha)\langle u, x - x_0\rangle = \langle u, x - \tilde{x}\rangle \leq 0.\end{aligned}$$

We get the contradiction that $\langle u, x_0 - x\rangle < 0 \leq \langle u, x_0 - x\rangle$ and therefore $\bar{x}$ is a minimizer of (P), which completes the proof. ☐

**Remark 3.5.** In view of Remark 3.2 and Corollary 3.2, the necessary assertion in Theorem 3.3 can be stated as "If $\bar{x}$ is a minimizer of (P), then $g(\bar{x}) = 0$ and $\partial^v g(\bar{x}) \subseteq N(\mathrm{lev}_{\leq}(f, f(\bar{x})) \cap C, \bar{x})$."

**Remark 3.6.** It is important to notice that the conclusions of Theorem 3.2 and Theorem 3.3 also hold by considering $z \in \mathbb{R}^n$ for which $g(z) = 0$, that is, a point $z$ may not even lie in $\mathrm{lev}_{\leq}(f, f(\bar{x})) \cap C$, and by considering the normal cone $N(E, a)$ where $a \notin E$ as follows, see [15],

$$N(E, a) := \{u \in \mathbb{R}^n : \langle u, y - a\rangle \leq 0,\ \forall y \in E\}.$$

However, this appears to be unnatural since we need to remain concerned regarding $\mathbb{R}^n \backslash C$ when we are actually minimizing over $C$. Also, the mainstream literature on convex analysis of the normal cone $N(E, a)$ is defined as the emptyset if $a \notin E$.

Next let us provide an example illustrating Theorem 3.3.

**Example 3.3.** Let $f, g$ and $C$ be defined as in Example 3.1. It can be verified that $C \cap \mathrm{lev}_{=}(g, 0) = \{\frac{1}{2}\}$, $\partial^v g(\frac{1}{2}) = \,]-\infty, 0]$ and $N(C, \frac{1}{2}) = \{0\}$. So, $\partial^v g(\frac{1}{2}) \not\subseteq N(C, \frac{1}{2})$, showing that assumption **(B)** is satisfied. Consider $\bar{x} := \frac{1}{2}$, we get $\mathrm{lev}_{\leq}(f, f(\bar{x})) \cap C = [\frac{1}{2}, 1]$. We can see that

$$\partial^v g(\bar{x}) = \,]-\infty, 0] \subseteq \,]-\infty, 0] = N(\mathrm{lev}_{\leq}(f, f(\bar{x})) \cap C, \bar{x}).$$

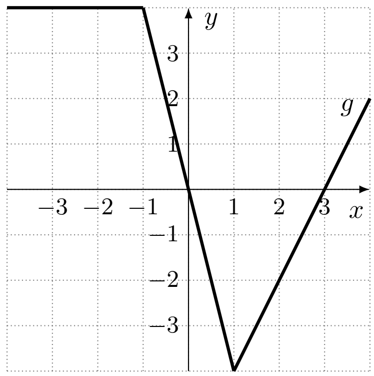


FIGURE 3. The figure of the constraint function in Example 3.4

As $z_0 := \frac{1}{2} \in \text{lev}_{\leq}(f, f(\bar{x})) \cap C$ satisfying $g(z_0) = 0$, we also have $N(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, z_0) = ]-\infty, 0]$. Thus,

$$\partial^* g(z_0) \cap N(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, z_0) = ]-\infty, 0[ \neq \emptyset.$$

By Theorem 3.3, we conclude that $\bar{x}$ is a minimizer of (P).

Noteworthy is the fact that the conclusion of Theorem 3.3 may fail without the condition **(B)** as the following illustrated example shows.

**Example 3.4.** Let $f$ be defined as in Example 3.1. Let $C := [0, +\infty[$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) := \begin{cases} 4 & \text{if } x \in ]-\infty, -1[\,; \\ -4x & \text{if } x \in [-1, 1[\,; \\ 2x - 6 & \text{if } x \in [1, +\infty[\,. \end{cases}$$

We see that $A = \{0\} \cup [3, +\infty[$ and $\inf_{x \in A} f(x) = 0$. By taking $x_0 := 1$, assumption **(A)** holds. By considering $\bar{x} := 3$, we get that $\text{lev}_{\leq}(f, f(\bar{x})) \cap C = [0, 3]$. Now, consider $z \in \text{lev}_{\leq}(f, f(\bar{x})) \cap C$ such that $g(z) = 0$.

If $z := 0$, then $\partial^* g(z) = ]-\infty, 0[$ and $N(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, z) = ]-\infty, 0]$. Thus, $\partial^* g(z) \cap N(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, z) = ]-\infty, 0[ \neq \emptyset$.

Similarly, if $z := 3$, we have $\partial^* g(z) = ]0, +\infty[$ and $N(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, z) = [0, +\infty[$. So, $\partial^* g(z) \cap N(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, z) = ]0, +\infty[ \neq \emptyset$. These show that (3.11) is satisfied. However, $\bar{x}$ is not a minimizer of (P), i.e., $f(\bar{x}) = 3 > 0 = f(0)$. As a result, the conclusion of Theorem 3.3 is invalid. The reason is that assumption **(B)** does not hold, i.e., one has $C \cap \text{lev}_{=}(g, 0) = \{0, 3\}$, and $\partial^v g(0) = ]-\infty, 0] \subseteq ]-\infty, 0] = N(C, 0)$.

To this end, we can easily obtain the following global optimality conditions for reverse convex programs.

**Corollary 3.6.** *Let $f$ and $g$ be convex functions, and Assumption* **(A)** *hold. Let $\bar{x}$ be a feasible point of problem* (P).

(i) *If $\bar{x}$ is a minimizer of* (P), *then for any $z \in \text{lev}_{\leq}(f, f(\bar{x})) \cap C$ satisfying $g(z) = 0$, one has*

$$\partial g(z) \subseteq N(\text{lev}_{\leq}(f, f(\bar{x})) \cap C, z),$$

*where $\partial g(z) := \{u \in \mathbb{R}^n : \langle u, x - z \rangle \leq g(x) - g(z),\ \forall x \in \mathbb{R}^n\}$ stands for a subdifferential of $g : \mathbb{R}^n \to \mathbb{R}$ at $z \in \mathbb{R}^n$,*

(ii) *If for any* $x \in C \cap \mathrm{lev}_{=}(g,0)$*, it holds that* $\partial g(x) \nsubseteq N(C,x)$ *and*

$$\partial g(z) \cap N(\mathrm{lev}_{\leq}(f, f(\bar{x})) \cap C, z) \neq \emptyset,$$

*for all* $z \in \mathrm{lev}_{\leq}(f, f(\bar{x})) \cap C$ *satisfying* $g(z) = 0$*, then* $\bar{x}$ *is a minimizer of* (P).

*Proof.* (i) This is an immediate consequence of Theorem 3.3(i) since $\partial g(z) \subseteq \partial^v g(z)$ for any $z \in \mathrm{lev}_{\leq}(f, f(\bar{x})) \cap C$ satisfying $g(z) = 0$.

(ii) As $\partial g(z) \subseteq \partial^* g(z)$, we can see that

$$\partial g(z) \cap N(\mathrm{lev}_{\leq}(f, f(\bar{x})) \cap C, z) \neq \emptyset \Longrightarrow \partial^* g(z) \cap N(\mathrm{lev}_{\leq}(f, f(\bar{x})) \cap C, z) \neq \emptyset,$$

for all $z \in \mathrm{lev}_{\leq}(f, f(\bar{x})) \cap C$ satisfying $g(z) = 0$. Now for each $x \in C \cap \mathrm{lev}_{=}(g,0)$, if there exists $u \in \partial g(x)$ such that $u \notin N(C,x)$, we also have $\partial^v g(x) \nsubseteq N(C,x)$ due to $\partial g(x) \subseteq \partial^v g(x)$. Thus, Assumption **(B)** is satisfied, and so, the conclusion follows from Theorem 3.3. □

## 4. Conclusions

In this paper, we have employed proper separation theorem together with some obtained results from set containment characterization for quasiconvex programming to provide necessary and sufficient optimality conditions for reverse quasiconvex programs. They also provide an alternative to solve the considered class of reverse quasiconvex programs via maximization problems with linked constraints. Moreover, sufficient conditions for optimality in terms of Greenberg-Pierskalla subdifferential have also been provided.

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