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On Coupled Systems of Hilfer-Hadamard Sequential Fractional Differential Equations with Three-Point Boundary Conditions

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ABSTRACT. This paper investigates the existence and uniqueness of solutions for a system of Hilfer-Hadamard sequential fractional differential equations using standard fixed-point theorems. We apply the Leray-Schauder alternative and Banach's contraction mapping principle to obtain the existence and uniqueness results for the given problem. Additionally, we discuss illustrative examples.

1. INTRODUCTION

The significance of fractional calculus has been discovered in the past few decades due to its accurate mathematical modeling compared to classical calculus. Fractional differential equations are widely used in applied science, engineering, technical science, and more. This inspiration has led mathematicians in the past century to introduce many new fractional derivatives, including Riemann-Liouville fractional derivative, Caputo derivative, Hadamard derivative, Hilfer derivative, Hilfer-Hadamard derivative, and many more. Coupled systems of such fractional differential equations provide precise mathematical models for physical phenomena like anomalous diffusion, disease models, secure communication and control processing, Chua circuit, ecological effects, and others. For applications of fractional derivatives, please refer to [9], [10], [13], [15], [18], [20],[21], [22], [24], [26], [27],[28], [31].

The following list includes some of the research articles related to coupled systems of fractional differential equations. Alsaedi *et al.* [7] studied the existence of solutions for a Riemann-Liouville coupled system of nonlinear fractional integro-differential equations

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), v(t), (\phi_1 u)(t), (\psi_1 v)(t)), \ \ t \in [0, T], \\ D^{\beta}v(t) = g(t, u(t), v(t), (\phi_2 u)(t), (\psi_2 v)(t)), \ \ 1 < \alpha, \beta \le 2, \end{cases}$$

with coupled Riemann-Liouville integro-differential boundary conditions

$$\begin{cases} D^{\alpha-2}u(0^+) = 0, \ \ D^{\alpha-1}u(0^+) = \nu I^{\alpha-1}v(\eta), \ \ 0 < \eta < T, \\ D^{\beta-2}v(0^+) = 0, \ \ D^{\beta-1}v(0^+) = \mu I^{\beta-1}u(\sigma), \ \ 0 < \sigma < T, \end{cases}$$

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where $D^{(.)}, I^{(.)}$ denote the Riemann-Liouville derivatives and integral of fractional order (.), respectively, $f, g : [0, T] \times \mathbb{R}^4 \to \mathbb{R}$ are given continuous functions, ν, μ are real constants and

$$(\phi_1 u)(t) = \int_0^t \gamma_1(t,s)u(s)ds, \ \ (\phi_2 u)(t) = \int_0^t \gamma_2(t,s)u(s)ds$$
$$(\psi_1 v)(t) = \int_0^t \delta_1(t,s)v(s)ds, \ \ (\psi_2 v)(t) = \int_0^t \delta_2(t,s)v(s)ds,$$

with γ_i and δ_i $(i = 1, 2)$ are continuous function on $[0, T] \times [0, T]$.

Alsulami *et al.* [8] studied a system of coupled Caputo type fractional differential equations

$$\begin{cases} {}^cD^\alpha u(t) = f(t, u(t), v(t)), \ \ t \in [0, T], \ \ 1 < \alpha \le 2, \\ {}^cD^\beta v(t) = g(t, u(t), v(t)), \ \ t \in [0, T], \ \ 1 < \beta \le 2, \end{cases}$$

with non-separated coupled boundary conditions

$$\begin{cases} u(0) = \lambda_1 v(T), \ \ u'(0) = \lambda_2 v'(T), \\ v(0) = \mu_1 u(T), \ \ v'(0) = \mu_2 u'(T), \end{cases}$$

where ${}^cD^\alpha$, ${}^cD^\beta$ denote the Caputo fractional derivatives of order α and β, respectively, $f, g : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are appropriate functions and λ_i, μ_i, $i = 1, 2$ are real constants with $\lambda_i \mu_i \neq 1$, $i = 1, 2$.

Ahmad *et al.* [2] studied a coupled system of nonlinear fractional differential equations with three-point boundary conditions. Applying the Schauder fixed point theorem, an existence result is proved for the following system

$$\begin{cases} D^\alpha u(t) = f(t, u(t), D^p v(t)), \ t \in (0, 1), \\ D^\beta v(t) = g(t, u(t), D^q v(t)), \ t \in (0, 1), \\ u(0 = 0, \ u(1) = \gamma u(\eta), \ v(0) = 0, \ v(1) = \gamma v(\eta), \end{cases}$$

where $1 < \alpha, \beta < 2, \ \ p, q, \gamma > 0, \ \ 0 < \eta < 1, \ \ \alpha - q \ge 1, \ \ \beta - p \ge 1, \ \ \gamma\eta^{\alpha-1} < 1, \ \ \gamma\eta^{\beta-1} < 1, D^\chi$ $(\chi = \alpha, \beta, p, q)$ is the standard Riemann–Liouville fractional derivative and $f, g : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \times\mathbb{R}$ are given continuous functions. It is important to note that the nonlinear terms in the coupled system involve the fractional derivatives of the unknown functions, and later [3], they studied the existence and uniqueness of solutions for nonlinear Caputo sequential fractional differential equations

$$\begin{cases} ({}^cD^\alpha + k_1 {}^cD^{\alpha-1})u(t) = f(t, u(t), v(t)), \ \ 1 < \alpha \le 2, \ \ t \in (0, T), \\ ({}^cD^\beta + k_2 {}^cD^{\beta-1})v(t) = g(t, u(t), v(t)), \ \ 1 < \beta \le 2, \ \ t \in (0, T), \end{cases}$$

supplemented with coupled boundary conditions

$$\begin{cases} u(0) = a_1 v(T), \ \ u'(0) = a_2 v'(T), \\ v(0) = b_1 u(T), \ \ v'(0) = b_2 u'(T), \end{cases}$$

where ${}^cD^\alpha, {}^c D^\beta$ denotes the Caputo fractional derivative of order α and β, respectively, $k_1, k_2 \in \mathbb{R}_+$, $T > 0$, $f, g : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and a_1, a_2, b_1 and b_2 are real constants with $a_1 b_1 \neq 1$, and $a_2 b_2 e^{-(k_1 T + k_2 T)} \neq 1$.

Aljoudi *et al.* [5] studied a coupled system of Hadamard type sequential fractional differential equations with coupled strip conditions given by

$$\begin{cases}(D^q+kD^{q-1})u(t)=f(t,u(t),v(t),D^{\alpha}v(t)), \quad k>0,\ 1<q\le 2,\ 0<\alpha<1,\\ (D^p+kD^{p-1})v(t)=g(t,u(t),v(t),D^{\delta}u(t)),\ 1<p\le 2,\ 0<\delta<1,\\ u(1)=0,\ u(e)=I^{\gamma}v(\eta)=\dfrac{1}{\Gamma(\gamma)}\displaystyle\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\gamma-1}\frac{v(s)}{s}ds,\ \gamma>0,\ 1<\eta<e,\\ v(1)=0,\ v(e)=I^{\beta}v(\zeta)=\dfrac{1}{\Gamma(\beta)}\displaystyle\int_1^{\zeta}\left(\log\frac{\zeta}{s}\right)^{\beta-1}\frac{u(s)}{s}ds,\ \beta>0,\ 1<\zeta<e,\end{cases}$$

where $D^{(\cdot)}$ and $I^{(\cdot)}$ denote the Hadamard fractional derivative and Hadamard fractional integral, respectively and $f,g:[1,e]\times\mathbb{R}^3\to\mathbb{R}$ are given continuous functions. For other recent results we refer to [16, 33, 6, 1] and references cited therein.

Recently, in [30] the authors studied existence and uniqueness of solutions for a class of system of Hilfer-Hadamard sequential fractional differential equations

$$\begin{cases}({}_HD_{1^+}^{\alpha_1,\beta_1}+k_1\,{}_HD_{1^+}^{\alpha_1-1,\beta_1})u(t)=f(t,u(t),v(t)),\ 1<\alpha_1\le 2,\ t\in[1,e],\\ ({}_HD_{1^+}^{\alpha_2,\beta_2}+k_2\,{}_HD_{1^+}^{\alpha_2-1,\beta_2})v(t)=g(t,u(t),v(t)),\ 1<\alpha_2\le 2,\ t\in[1,e],\end{cases}$$

with two point boundary conditions

$$\begin{cases}u(1)=0,\ u(e)=A_1,\\ v(1)=0,\ v(e)=A_2,\end{cases}$$

where ${}_HD^{\alpha_i,\beta_i}$ is the Hilfer-Hadamard fractional derivative of order $\alpha_i\in(1,2]$ and type $\beta_i\in[0,1]$ for $i\in\{1,2\}$, $k_1,k_2,A_1,A_2\in\mathbb{R}_+$ and $f,g:[1,e]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given continuous functions.

Very recently, in [34], the authors studied the existence and uniqueness of solutions for boundary value problems for sequential Hilfer-Hadamard fractional differential equations with three-point boundary conditions,

$$({}_HD_{1+}^{\alpha,\beta}+k\,{}_HD_{1+}^{\alpha-1,\beta})u(t)=f(t,u(t)),\ 1<\alpha\le 2,\ t\in[1,e],$$
$$u(1)=0,\ u(e)=\lambda u(\theta),\ \theta\in(1,e),$$

where ${}_HD_{1+}^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order $\alpha\in(1,2]$ and type $\beta\in[0,1]$, $\gamma=\alpha+n\beta-\alpha\beta$, $n-1<\gamma\le n$, $n=[\alpha]+1$, $\mathrm{k}\in\mathbb{R}^+:=[0,\infty)$, $\lambda\in\mathbb{R}\setminus\{\frac{1}{(\log\ \theta)^{\gamma-1}}\}$ and $f:[1,e]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function. However, it has been observed that the literature on Hilfer-Hadamard sequential fractional differential equations of order in $(1,2]$ is scarce and needs to be developed further.

Motivated by the research going on in this direction, in the present paper we extend the results of [34] to sequential fractional coupled system

$$\begin{cases}({}_HD_{1^+}^{\alpha_1,\beta_1}+k_1\ {}_HD_{1^+}^{\alpha_1-1,\beta_1})u(t)=f(t,u(t),v(t)),\ 1<\alpha_1\le 2,\ t\in[1,e],\\ ({}_HD_{1^+}^{\alpha_2,\beta_2}+k_2\ {}_HD_{1^+}^{\alpha_2-1,\beta_2})v(t)=g(t,u(t),v(t)),\ 1<\alpha_2\le 2,\ t\in[1,e],\end{cases}\tag{1.1}$$

with three-point coupled boundary conditions

$$\begin{aligned}&u(1)=0,\ u(e)=\lambda v(\theta),\ 1<\theta<e,\\ &v(1)=0,\ v(e)=\mu u(\eta),\ 1<\eta<e,\end{aligned}\tag{1.2}$$

where ${}_HD_{1^+}^{\alpha_i,\beta_i}$ is the Hilfer-Hadamard fractional derivatives of order $\alpha_i\in(1,2]$ and type $\beta_i\in[0,1]$ for $i\in\{1,2\}$, $k_1,k_2\in\mathbb{R}^+$, $f,g:[1,e]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given continuous functions and λ and μ are real constants.

Concerning the significance of problem (1.1)-(1.2), we recall that the Hilfer fractional derivative generalize both Riemann–Liouville and Caputo fractional derivatives and in fact interpolates between them. Analogously, the Hilfer-Hadamard type fractional derivative covers the cases of the Riemann-Liouville-Hadamard and Caputo-Hadamard fractional derivatives. Therefore, the present study will be useful for improving the works related to glass forming materials [19], fractional glassy relaxation [23], turbulent flow model [35], etc. An example of a physical system modeled by means of the Hilfer fractional derivative is described in [19], while the Hilfer fractional advection–diffusion equation with the power-law initial condition is studied in [4]. In [11, 12], the Hilfer–Prabhakar and Hilfer fractional derivatives are used to model filtration processes. In a recent work [36], the authors discussed the attractivity for Hilfer fractional stochastic evolution equations. One can find the application of Hilfer fractional derivative operator in the cobweb economics model in [29]. The concept of the Hilfer-Hadamard fractional derivative operator is quite a recent one, and it is expected that the models based on the Hilfer fractional derivative operators will be considered with the Hilfer-Hadamard fractional derivatives to find more insight into these models.

It is well known that the nonlocal conditions are more appropriate than the local conditions to describe several problems in applied mathematics and physics more appropriately, see the survey paper [25]. We emphasize that in the present paper we study coupled systems of Hilfer-Hadamard sequential fractional differential equations of order in $(1,2]$. Our results are new and enrich the new research area on Hilfer-Hadamard coupled systems of the order in $(1,2]$. The used method is standard, but its configuration in the problem (1.1)-(1.2) is new.

The rest of the paper is organized into three sections. In Section 2, we recall some definitions and notations that will be used throughout the paper. The main results regarding the existence and uniqueness of solutions for the coupled system (1.1) with the boundary conditions (1.2) are presented in Section 3. The final section, Section 4, contains examples that illustrate our main findings.

2. Preliminaries

In this section, some basic definitions and theorems are mentioned.

Definition 2.1 (Hadamard fractional integral [22])**.** The Hadamard fractional integral of order $\alpha \in \mathbb{R}_+$ for a function $f : [a,\infty) \to \mathbb{R}$ is defined as

$$ {}_HI^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{f(\tau)}{\tau} d\tau, \ \ (t > a) \tag{2.3}$$

provided the integral exists, where $\log(.) = \log_e(.)$.

Definition 2.2 (Hadamard fractional derivative [22])**.** The Hadamard fractional derivative of order $\alpha > 0$, applied to the function $f : [a,\infty) \to \mathbb{R}$ is defined as

$$ {}_HD^{\alpha}_{a^+} f(t) = \delta^m({}_HI^{m-\alpha}_{a^+} f(t)), \ \ m-1 < \alpha < m, \ \ m = [\alpha]+1, \tag{2.4}$$

where $\delta^m = (t\frac{d}{dt})^m$ and $[\alpha]$ denotes the integer part of the real number α.

Definition 2.3 (Hilfer-Hadamard fractional derivative [32])**.** Let $m-1 < \alpha < m$ and $0 \le \beta \le 1$, $f \in L^1(a,b)$. The Hilfer-Hadamard fractional derivative of order α and type β of f is defined as

$$\begin{aligned} ({}_HD^{\alpha,\beta}_{a^+} f)(t) &= ({}_HI^{\beta(m-\alpha)}_{a^+} \delta^m \ {}_HI^{(m-\alpha)(1-\beta)}_{a^+} f)(t) \\ &= ({}_HI^{\beta(m-\alpha)}_{a^+} \delta^m \ {}_HI^{m-\gamma}_{a^+} f)(t); \ \ \gamma = \alpha + m\beta - \alpha\beta \\ &= ({}_HI^{\beta(m-\alpha)}_{a^+} \ {}_HD^{\gamma}_{a^+} f)(t), \end{aligned}$$

where ${}_HI^{(.)}_{a+}$ and ${}_HD^{(.)}_{a+}$ are the Hadamard fractional integral and derivative defined by (2.3) and (2.4), respectively.

The Hilfer-Hadamard fractional derivative interpolates between the Hadamard fractional derivative and Caputo fractional derivative depending on the value of β. When $\beta = 0$, it reduces to Hadamard fractional derivative and when $\beta = 1$, it reduces to Caputo fractional derivative.

We utilize certain theorem related to the Hilfer-Hadamard fractional integral and derivative.

Theorem 2.1 ([32]). *Let $\alpha > 0$, $0 \le \beta \le 1$, $\gamma = \alpha + m\beta - \alpha\beta$, $m - 1 < \gamma \le m$, $m = [\alpha] + 1$ and $0 < a < b < \infty$. If $f \in L^1(a, b)$ and $({}_HI^{m-\gamma}_{a+} f)(t) \in AC^m_\delta[a, b]$, then*

$$\begin{aligned} {}_HI^{\alpha}_{a+}\,({}_HD^{\alpha,\beta}_{a+} f)(t) &= {}_HI^{\gamma}_{a+}\,({}_HD^{\gamma}_{a+} f)(t) \\ &= f(t) - \sum_{j=0}^{m-1} \frac{(\delta^{(m-j-1)}({}_HI^{m-\gamma}_{a+} f))(a)}{\Gamma(\gamma - j)} \left(\log\frac{t}{a}\right)^{\gamma-j-1}. \end{aligned}$$

3. Existence and uniqueness results

In this section, we prove existence and uniqueness of solutions for system of Hilfer-Hadamard sequential fractional differential equations (1.1) with the boundary conditions (1.2).

3.1. **An auxiliary lemma.** In this subsection we first prove an auxiliary result, concerning a linear variant of the problem (1.1)-(1.2), that plays a key role in transforming the given problem into a fixed point problem.

Lemma 3.1. *Let $h_1, h_2 \in AC([1, e], \mathbb{R})$ and $\Delta \neq 0$. Then $u, v \in AC^2([1, e], \mathbb{R})$ are solutions of the system of fractional differential equations*

$$(3.5)\qquad \begin{cases} ({}_HD^{\alpha_1,\beta_1}_{1^+} + k_1\,{}_HD^{\alpha_1-1,\beta_1}_{1^+})u(t) = h_1(t), & 1 < \alpha_1 \le 2,\ t \in [1, e], \\ ({}_HD^{\alpha_2,\beta_2}_{1^+} + k_2\,{}_HD^{\alpha_2-1,\beta_2}_{1^+})v(t) = h_2(t), & 1 < \alpha_2 \le 2,\ t \in [1, e] \end{cases}$$

supplemented with three-point coupled boundary conditions (1.2) if and only if

$$\begin{aligned} u(t) &= \frac{1}{\Delta}\Bigg\{\Bigg[k_1\int_1^e \frac{u(s)}{s}ds - \frac{1}{\Gamma(\alpha_1)}\int_1^e \frac{h_1(s)}{s}\left(\log\frac{e}{s}\right)^{\alpha_1-1} ds \\ &\quad +\lambda\Bigg(-k_2\int_1^\theta \frac{v(s)}{s}ds + \frac{1}{\Gamma(\alpha_2)}\int_1^\theta \frac{h_2(s)}{s}\left(\log\frac{\theta}{s}\right)^{\alpha_2-1} ds\Bigg)\Bigg] \\ &\quad -A\Bigg[k_2\int_1^e \frac{v(s)}{s}ds - \frac{1}{\Gamma(\alpha_2)}\int_1^e \frac{h_2(s)}{s}\left(\log\frac{e}{s}\right)^{\alpha_2-1} ds \\ &\quad +\mu\Bigg(-k_1\int_1^\eta \frac{u(s)}{s}ds + \frac{1}{\Gamma(\alpha_1)}\int_1^\eta \frac{h_1(s)}{s}\left(\log\frac{\eta}{s}\right)^{\alpha_1-1} ds\Bigg)\Bigg]\Bigg\}(\log t)^{\gamma_1-1} \\ &\quad -k_1\int_1^t \frac{u(s)}{s}ds + \frac{1}{\Gamma(\alpha_1)}\int_1^t \frac{h_1(s)}{s}\left(\log\frac{t}{s}\right)^{\alpha_1-1} ds \qquad (3.6) \end{aligned}$$

and

$$\begin{aligned} v(t) &= \frac{1}{\Delta}\Bigg\{\Bigg[k_2\int_1^e \frac{v(s)}{s}ds - \frac{1}{\Gamma(\alpha_2)}\int_1^e \frac{h_2(s)}{s}\left(\log\frac{e}{s}\right)^{\alpha_2-1} ds \\ &\quad +\mu\Bigg(-k_1\int_1^\eta \frac{u(s)}{s}ds + \frac{1}{\Gamma(\alpha_1)}\int_1^\eta \frac{h_1(s)}{s}\left(\log\frac{\eta}{s}\right)^{\alpha_1-1} ds\Bigg)\Bigg] \end{aligned}$$

$$-B\bigg[k_1\int_1^e \frac{u(s)}{s}ds - \frac{1}{\Gamma(\alpha_1)}\int_1^e \frac{h_1(s)}{s}\bigg(\log\frac{e}{s}\bigg)^{\alpha_1-1}ds$$
$$+\lambda\bigg(-k_2\int_1^\theta \frac{v(s)}{s}ds + \frac{1}{\Gamma(\alpha_2)}\int_1^\theta \frac{h_2(s)}{s}\bigg(\log\frac{\theta}{s}\bigg)^{\alpha_2-1}ds\bigg)\bigg]\bigg\}(\log t)^{\gamma_2-1}$$
$$-k_2\int_1^t \frac{v(s)}{s}ds + \frac{1}{\Gamma(\alpha_2)}\int_1^t \frac{h_2(s)}{s}\bigg(\log\frac{t}{s}\bigg)^{\alpha_2-1}ds, \tag{3.7}$$

where

$$A = -\lambda(\log\theta)^{\gamma_2-1},\ B = -\mu(\log\eta)^{\gamma_1-1},\ \Delta = 1 - AB.$$

Proof. Applying the Hadamard fractional integral operator of orders α_1, α_2 from 1 to t on both sides of Hilfer-Hadamard fractional differential equations in (3.5) and using Theorem 2.1, we find that

$$u(t) - \frac{\delta({}_HI_{1^+}^{2-\gamma_1}u)(1)(\log t)^{\gamma_1-1}}{\Gamma(\gamma_1)} - \frac{({}_HI_{1^+}^{2-\gamma_1}u)(1)(\log t)^{\gamma_1-2}}{\Gamma(\gamma_1-1)} + k_1\,{}_HI_{1^+}^{\alpha_1}\,{}_HI_{1^+}^{1-\alpha_1}u(t) = {}_HI_{1^+}^{\alpha_1}h_1(t),$$

and

$$v(t) - \frac{\delta({}_HI_{1^+}^{2-\gamma_2}v)(1)(\log t)^{\gamma_2-1}}{\Gamma(\gamma_2)} - \frac{({}_HI_{1^+}^{2-\gamma_2}v)(1)(\log t)^{\gamma_2-2}}{\Gamma(\gamma_2-1)} + k_2\,{}_HI_{1^+}^{\alpha_2}\,{}_HI_{1^+}^{1-\alpha_2}v(t) = {}_HI_{1^+}^{\alpha_2}h_2(t),$$

which can written as

$$u(t) = c_0(\log t)^{\gamma_1-1} + c_1(\log t)^{\gamma_1-2} - k_1\int_1^t \frac{u(s)}{s}ds + \frac{1}{\Gamma(\alpha_1)}\int_1^t \frac{h_1(s)}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}ds, \tag{3.8}$$

and

$$v(t) = d_0(\log t)^{\gamma_2-1} + d_1(\log t)^{\gamma_2-2} - k_2\int_1^t \frac{v(s)}{s}ds + \frac{1}{\Gamma(\alpha_2)}\int_1^t \frac{h_2(s)}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_2-1}ds, \tag{3.9}$$

where c_0, c_1, d_0, d_1 are arbitrary constants. Using the first boundary conditions ($u(1) = 0, v(1) = 0$) in (3.8), (3.9) yields $c_1 = 0, d_1 = 0$ since $\gamma_i \in [\alpha_i, 2], i = 1, 2$. In consequence, equations (3.8) and (3.9) takes the form:

$$u(t) = c_0(\log t)^{\gamma_1-1} - k_1\int_1^t \frac{u(s)}{s}ds + \frac{1}{\Gamma(\alpha_1)}\int_1^t \frac{h_1(s)}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}ds \tag{3.10}$$

$$v(t) = d_0(\log t)^{\gamma_2-1} - k_2\int_1^t \frac{v(s)}{s}ds + \frac{1}{\Gamma(\alpha_2)}\int_1^t \frac{h_2(s)}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_2-1}ds. \tag{3.11}$$

Next, the second boundary conditions of (1.2) together with (3.10) and (3.11) yields the system

$$c_0 + d_0A = J_1, \quad c_0B + d_0 = J_2, \tag{3.12}$$

where J_1, J_2 are defined as follows,

$$\begin{aligned}
J_1 &= k_1\int_1^e \frac{u(s)}{s}ds - \frac{1}{\Gamma(\alpha_1)}\int_1^e \frac{h_1(s)}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_1-1}ds \\
&\quad +\lambda\bigg(-k_2\int_1^\theta \frac{v(s)}{s}ds + \frac{1}{\Gamma(\alpha_2)}\int_1^\theta \frac{h_2(s)}{s}\Big(\log\frac{\theta}{s}\Big)^{\alpha_2-1}ds\bigg), \\
J_2 &= k_2\int_1^e \frac{v(s)}{s}ds - \frac{1}{\Gamma(\alpha_2)}\int_1^e \frac{h_2(s)}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_2-1}ds \\
&\quad +\mu\bigg(-k_1\int_1^\eta \frac{u(s)}{s}ds + \frac{1}{\Gamma(\alpha_1)}\int_1^\eta \frac{h_1(s)}{s}\Big(\log\frac{\eta}{s}\Big)^{\alpha_1-1}ds\bigg).
\end{aligned}$$

Solving the system (3.12), we get

$$c_0 = \frac{J_1 - AJ_2}{\Delta} \quad \text{and} \quad d_0 = \frac{J_2 - BJ_1}{\Delta}.$$

Substituting c_0 and d_0 back in equations (3.10), (3.11), we get the integral equation (3.6) and (3.7). The converse of this proof follows by direct computation. This completes the proof. □

Let us introduce the Banach space $X = C([1,e],\mathbb{R})$ endowed with the norm defined by $\|u\| = \sup\{|u(t)| : t \in [1,e]\}$. The product space $X \times X$ equipped with the norm $\|(u,v)\| = \|u\| + \|v\|$ is also a Banach space. In view of Lemma 3.1, we define an operator $\mathcal{T} : X \times X \to X \times X$ by

$$\mathcal{T}(u,v)(t) = (\mathcal{T}_1(u,v)(t), \mathcal{T}_2(u,v)(t)), \tag{3.13}$$

where

$$\begin{aligned}
\mathcal{T}_1(u,v)(t) &= \frac{1}{\Delta}\Bigg\{\Bigg[k_1\int_1^e \frac{u(s)}{s}ds - \frac{1}{\Gamma(\alpha_1)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{f(s,u(s),v(s))}{s}ds \\
&\quad +\lambda\Bigg(-k_2\int_1^\theta \frac{v(s)}{s}ds + \frac{1}{\Gamma(\alpha_2)}\int_1^\theta \left(\log\frac{\theta}{s}\right)^{\alpha_2-1}\frac{g(s,u(s),v(s))}{s}ds\Bigg)\Bigg] \\
&\quad -A\Bigg[k_2\int_1^e \frac{v(s)}{s}ds - \frac{1}{\Gamma(\alpha_2)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_2-1}\frac{g(s,u(s),v(s))}{s}ds \\
&\quad +\mu\Bigg(-k_1\int_1^\eta \frac{u(s)}{s}ds + \frac{1}{\Gamma(\alpha_1)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha_1-1}\frac{f(s,u(s),v(s))}{s}ds\Bigg)\Bigg]\Bigg\} \\
&\quad \cdot(\log t)^{\gamma_1-1} - k_1\int_1^t \frac{u(s)}{s}ds + \frac{1}{\Gamma(\alpha_1)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{f(s,u(s),v(s))}{s}ds
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{T}_2(u,v)(t) &= \frac{1}{\Delta}\Bigg\{\Bigg[k_2\int_1^e \frac{v(s)}{s}ds - \frac{1}{\Gamma(\alpha_2)}\int_1^e \frac{g(s,u(s),v(s))}{s}\left(\log\frac{e}{s}\right)^{\alpha_2-1}ds \\
&\quad +\mu\Bigg(-k_1\int_1^\eta \frac{u(s)}{s}ds + \frac{1}{\Gamma(\alpha_1)}\int_1^\eta \frac{f(s,u(s),v(s))}{s}\left(\log\frac{\eta}{s}\right)^{\alpha_1-1}ds\Bigg)\Bigg] \\
&\quad -B\Bigg[k_1\int_1^e \frac{u(s)}{s}ds - \frac{1}{\Gamma(\alpha_1)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{f(s,u(s),v(s))}{s}ds \\
&\quad +\lambda\Bigg(-k_2\int_1^\theta \frac{v(s)}{s}ds + \frac{1}{\Gamma(\alpha_2)}\int_1^\theta \left(\log\frac{\theta}{s}\right)^{\alpha_2-1}\frac{g(s,u(s),v(s))}{s}ds\Bigg)\Bigg]\Bigg\} \\
&\quad \cdot(\log t)^{\gamma_2-1} - k_2\int_1^t \frac{v(s)}{s}ds + \frac{1}{\Gamma(\alpha_2)}\int_1^t \frac{g(s,u(s),v(s))}{s}\left(\log\frac{t}{s}\right)^{\alpha_2-1}ds.
\end{aligned}$$

We will use the following notations in the proof :

$$M_1 = \frac{1+|\mu\lambda|+|\Delta|}{|\Delta|},\ W_1 = \frac{2|\lambda|}{|\Delta|},\ W_2 = \frac{2|\mu|}{|\Delta|}.$$

3.2. **Existence result via Leray-Schauder alternative.** First, we will prove the existence result based on Leray-Schauder alternative ([17]).

Theorem 3.2. *Assume that:*

(H_1) *$f, g : [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions for which there exist real constants $m_i, n_i \geq 0, (i = 1, 2)$ and $m_0 > 0$, $n_0 > 0$, such that for all $t \in [1, e]$, $x_i \in \mathbb{R}$, $i = 1, 2$,*

$$|f(t, x_1, x_2)| \leq m_0 + m_1|x_1| + m_2|x_2|,$$
$$|g(t, x_1, x_2)| \leq n_0 + n_1|x_1| + n_2|x_2|.$$

(H_2) *$Q_1 < 1, Q_2 < 1$, where*

$$Q_1 := k_1(M_1 + W_2) + \frac{m_1(M_1 + W_2)}{\Gamma(\alpha_1 + 1)} + \frac{n_1(W_1 + M_1)}{\Gamma(\alpha_2 + 1)},$$
$$Q_2 := k_2(W_1 + M_1) + \frac{m_2(M_1 + W_2)}{\Gamma(\alpha_1 + 1)} + \frac{n_2(W_1 + M_1)}{\Gamma(\alpha_2 + 1)}.$$

Then, the system (1.1)-(1.2) has at least one solution on $[1, e]$.

Proof. We will prove that $\mathcal{T}$, defined by (3.13), has a fixed point. We divide the proof into two steps.

Step I : We show that the operator $\mathcal{T} : X \times X \to X \times X$, defined by (3.13), is completely continuous.

First we note that the continuity of the functions f and g implies that the operator $\mathcal{T}$ is continuous.

Now, we show that $\mathcal{T}$ is compact. Let $\Omega = \{(u, v) \in X \times X : \|(u, v)\| \leq \rho\}$ be a bounded subset of $X \times X$. By (H_1) for $(u, v) \in \Omega$ we have

$$\begin{aligned} |f(t, u(t), v(t)| &\leq m_0 + m_1|u(t)| + m_2|v(t)| \\ &\leq m_0 + (m_1 + m_2)\|(u, v)\| \\ &\leq m_0 + (m_1 + m_2)\rho := L_1, \end{aligned}$$

and similarly $|g(t, u(t), v(t)| \leq n_0 + (n_1 + n_2)\rho := L_2$.

Then, we have:

$$\begin{aligned} |\mathcal{T}_1(u, v)(t)| &\leq \frac{1}{|\Delta|}\Bigg\{\Bigg[k_1 \int_1^e \frac{|u(s)|}{s} ds + \frac{1}{\Gamma(\alpha_1)} \int_1^e \frac{|f(s, u(s), v(s))|}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_1 - 1} ds \\ &\quad + |\lambda|\Big(k_2 \int_1^\theta \frac{|v(s)|}{s} ds + \frac{1}{\Gamma(\alpha_2)} \int_1^\theta \frac{|g(s, u(s), v(s))|}{s}\Big(\log\frac{\theta}{s}\Big)^{\alpha_2 - 1} ds\Big)\Bigg] \\ &\quad + |\lambda|\Bigg[k_2 \int_1^e \frac{|v(s)|}{s} ds + \frac{1}{\Gamma(\alpha_2)} \int_1^e \frac{|g(s, u(s), v(s))|}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_2 - 1} ds \\ &\quad + |\mu|\Big(k_1 \int_1^\eta \frac{|u(s)|}{s} ds + \frac{1}{\Gamma(\alpha_1)} \int_1^\eta \frac{|f(s, u(s), v(s))|}{s}\Big(\log\frac{\eta}{s}\Big)^{\alpha_1 - 1} ds\Big)\Bigg]\Bigg\} \\ &\quad + k_1 \int_1^t \frac{|u(s)|}{s} ds + \frac{1}{\Gamma(\alpha_1)} \int_1^t \frac{|f(s, u(s), v(s))|}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_1 - 1} ds \\ &\leq \frac{1}{|\Delta|}\Bigg\{\Bigg[k_1\|u\| + \frac{L_1}{\Gamma(\alpha_1 + 1)} + |\lambda|\Big(k_2\|v\| + \frac{L_2}{\Gamma(\alpha_2 + 1)}\Big)\Bigg] \\ &\quad + |\lambda|\Bigg[k_2\|v\| + \frac{L_2}{\Gamma(\alpha_2 + 1)} + |\mu|\Big(k_1\|u\| + \frac{L_1}{\Gamma(\alpha_1 + 1)}\Big)\Bigg]\Bigg\} + k_1\|u\| + \frac{L_1}{\Gamma(\alpha_1 + 1)} \\ &\leq \frac{1}{|\Delta|}\Bigg\{\Bigg[k_1\rho + \frac{L_1}{\Gamma(\alpha_1 + 1)} + |\lambda|\Big(k_2\rho + \frac{L_2}{\Gamma(\alpha_2 + 1)}\Big)\Bigg] \\ &\quad + |\lambda|\Bigg[k_2\rho + \frac{L_2}{\Gamma(\alpha_2 + 1)} + |\mu|\Big(k_1\rho + \frac{L_1}{\Gamma(\alpha_1 + 1)}\Big)\Bigg]\Bigg\} + k_1\rho + \frac{L_1}{\Gamma(\alpha_1 + 1)}, \end{aligned}$$

which on taking the norm for $t \in [1,e]$, yields

$$\begin{aligned}\|\mathcal{T}_1(u,v)\| &\le \frac{1}{|\Delta|}\bigg\{\bigg[k_1\rho+\frac{L_1}{\Gamma(\alpha_1+1)}+|\lambda|\bigg(k_2\rho+\frac{L_2}{\Gamma(\alpha_2+1)}\bigg)\bigg]\\ &\quad+|\lambda|\bigg[k_2\rho+\frac{L_2}{\Gamma(\alpha_2+1)}+|\mu|\bigg(k_1\rho+\frac{L_1}{\Gamma(\alpha_1+1)}\bigg)\bigg]\bigg\}+k_1\rho+\frac{L_1}{\Gamma(\alpha_1+1)}\\ &= M_1k_1\rho+\frac{M_1L_1}{\Gamma(\alpha_1+1)}+W_1k_2\rho+\frac{W_1L_2}{\Gamma(\alpha_2+1)}.\end{aligned}$$

Similarly, we have

$$\|\mathcal{T}_2(u,v)\| \le M_1k_2\rho+\frac{M_1L_2}{\Gamma(\alpha_2+1)}+W_2k_1\rho+\frac{W_2L_1}{\Gamma(\alpha_1+1)}.$$

Hence

$$\|\mathcal{T}(u,v)\| \le k_1(M_1+W_2)\rho+k_2(W_1+M_1)\rho+\frac{L_1(M_1+W_2)}{\Gamma(\alpha_1+1)}+\frac{L_2(W_1+M_1)}{\Gamma(\alpha_2+1)} := P,$$

which proves that $\mathcal{T}$ is uniformly bounded.

Finally, we show that $\mathcal{T}$ is equicontinuous. Let $t, t_0 \in [1,e]$ with $t_0 < t$. Then we have

$$\begin{aligned}&|\mathcal{T}_1(u,v)(t)-\mathcal{T}_1(u,v)(t_0)|\\ \le\ &\frac{1}{|\Delta|}\bigg\{\bigg[k_1\int_1^e\frac{|u(s)|}{s}ds+\frac{1}{\Gamma(\alpha_1)}\int_1^e\frac{|f(s,u(s),v(s))|}{s}\bigg(\log\frac{e}{s}\bigg)^{\alpha_1-1}ds\\ &+|\lambda|\bigg(k_2\int_1^\theta\frac{|v(s)|}{s}ds+\frac{1}{\Gamma(\alpha_2)}\int_1^\theta\frac{|g(s,u(s),v(s))|}{s}\bigg(\log\frac{\theta}{s}\bigg)^{\alpha_2-1}ds\bigg)\bigg]\\ &+|\lambda|\bigg[k_2\int_1^e\frac{|v(s)|}{s}ds+\frac{1}{\Gamma(\alpha_2)}\int_1^e\bigg(\log\frac{e}{s}\bigg)^{\alpha_2-1}\frac{|g(s,u(s),v(s))|}{s}ds\\ &+|\mu|\bigg(k_1\int_1^\eta\frac{|u(s)|}{s}ds+\frac{1}{\Gamma(\alpha_1)}\int_1^\eta\frac{|f(s,u(s),v(s))|}{s}\bigg(\log\frac{\eta}{s}\bigg)^{\alpha_1-1}ds\bigg)\bigg]\bigg\}\\ &\cdot\bigg[(\log t)^{\gamma_1-1}-(\log t_0)^{\gamma_1-1}\bigg]+k_1\int_{t_0}^t\frac{|u(s)|}{s}ds+\frac{1}{\Gamma(\alpha_1)}\bigg|\int_1^{t_0}\frac{|f(s,u(s),v(s))|}{s}\\ &\bigg(\bigg(\log\frac{t}{s}\bigg)^{\alpha_1-1}-\bigg(\log\frac{t_0}{s}\bigg)^{\alpha_1-1}\bigg)ds+\int_{t_0}^t\frac{|f(s,u(s),v(s))|}{s}\bigg(\log\frac{t}{s}\bigg)^{\alpha_1-1}ds\bigg|\\ \le\ &\frac{1}{|\Delta|}\bigg\{\bigg[k_1\|u\|\int_1^e\frac{1}{s}ds+\frac{L_1}{\Gamma(\alpha_1)}\int_1^e\frac{1}{s}\bigg(\log\frac{e}{s}\bigg)^{\alpha_1-1}ds\\ &+|\lambda|\bigg(k_2\|v\|\int_1^\theta\frac{1}{s}ds+\frac{L_2}{\Gamma(\alpha_2)}\int_1^\theta\frac{1}{s}\bigg(\log\frac{\theta}{s}\bigg)^{\alpha_2-1}ds\bigg)\bigg]\\ &+|\lambda|\bigg[k_2\|v\|\int_1^e\frac{1}{s}ds+\frac{L_2}{\Gamma(\alpha_2)}\int_1^e\frac{1}{s}\bigg(\log\frac{e}{s}\bigg)^{\alpha_2-1}ds\\ &+|\mu|\bigg(k_1\|u\|\int_1^\eta\frac{1}{s}ds+\frac{L_1}{\Gamma(\alpha_1)}\int_1^\eta\frac{1}{s}\bigg(\log\frac{\eta}{s}\bigg)^{\alpha_1-1}ds\bigg)\bigg]\bigg\}\bigg[(\log t)^{\gamma_1-1}-(\log t_0)^{\gamma_1-1}\bigg]\\ &+k_1\|u\|\int_{t_0}^t\frac{1}{s}ds+\frac{L_1}{\Gamma(\alpha_1)}\bigg|\int_1^{t_0}\frac{1}{s}\bigg[\bigg(\log\frac{t}{s}\bigg)^{\alpha_1-1}ds-\bigg(\log\frac{t_0}{s}\bigg)^{\alpha_1-1}\bigg]ds\\ &+\int_{t_0}^t\frac{1}{s}\bigg(\log\frac{t}{s}\bigg)^{\alpha_1-1}ds\bigg|\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{|\Delta|}\Bigg\{\Bigg[k_1\rho\int_1^e \frac{1}{s}ds+\frac{L_1}{\Gamma(\alpha_1)}\int_1^e \frac{1}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_1-1}ds\\
&+|\lambda|\Big(k_2\rho\int_1^\theta \frac{1}{s}ds+\frac{L_2}{\Gamma(\alpha_2)}\int_1^\theta \frac{1}{s}\Big(\log\frac{\theta}{s}\Big)^{\alpha_2-1}ds\Big)\Bigg]\\
&+|\lambda|\Bigg[k_2\rho\int_1^e \frac{1}{s}ds+\frac{L_2}{\Gamma(\alpha_2)}\int_1^e \frac{1}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_2-1}ds\\
&+|\mu|\Big(k_1\rho\int_1^\eta \frac{1}{s}ds+\frac{L_1}{\Gamma(\alpha_1)}\int_1^\eta \frac{1}{s}\Big(\log\frac{\eta}{s}\Big)^{\alpha_1-1}ds\Big)\Bigg]\Bigg\}\Big[(\log t)^{\gamma_1-1}-(\log t_0)^{\gamma_1-1}\Big]\\
&+k_1\rho\int_{t_0}^t \frac{1}{s}ds+\frac{L_1}{\Gamma(\alpha_1)}\Bigg|\int_1^{t_0}\frac{1}{s}\Bigg[\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}ds-\Big(\log\frac{t_0}{s}\Big)^{\alpha_1-1}\Bigg]ds\\
&+\int_{t_0}^t \frac{1}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}ds\Bigg|,
\end{aligned}$$

which tends to zero as $t \to t_0$ and is independed of u, v. Similarly we can prove that $|\mathcal{T}_2(u,v)(t) - \mathcal{T}_2(u,v)(t_0)| \to 0$ as $t \to t_0$. Hence $\mathcal{T}_1$ and $\mathcal{T}_2$ are equicontinuous and so the operator $\mathcal{T}$ is equicontinuous. Therefore, by Arzelá-Ascoli's theorem, we deduce that the operator $\mathcal{T}$ is completely continuous.

Step II : We show that the set $\Phi=\{(u,v) \in X \times X \mid (u,v) = \kappa\mathcal{T}(u,v),\ 0 \leq \kappa \leq 1\}$ is bounded.

Let $(u,v) \in \Phi$, then $(u,v) = \kappa\mathcal{T}(u,v)$. For any $t \in [1,e]$, we have $u(t) = \kappa\mathcal{T}_1(u,v)(t)$, $v(t) = \kappa\mathcal{T}_2(u,v)(t)$. Then, in view of the assumption (H_1), we obtain

$$\begin{aligned}
|u(t)| &\leq |\mathcal{T}_1(u,v)(t)|\\
&\leq \frac{1}{|\Delta|}\Bigg\{\Bigg[k_1\int_1^e \frac{|u(s)|}{s}ds+\frac{1}{\Gamma(\alpha_1)}\int_1^e \frac{|f(s,u(s),v(s))|}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_1-1}ds\\
&+|\lambda|\Big(k_2\int_1^\theta \frac{|v(s)|}{s}ds+\frac{1}{\Gamma(\alpha_2)}\int_1^\theta \frac{|g(s,u(s),v(s))|}{s}\Big(\log\frac{\theta}{s}\Big)^{\alpha_2-1}ds\Big)\Bigg]\\
&+|\lambda|\Bigg[k_2\int_1^e \frac{|v(s)|}{s}ds+\frac{1}{\Gamma(\alpha_2)}\int_1^e \Big(\log\frac{e}{s}\Big)^{\alpha_2-1}\frac{|g(s,u(s),v(s))|}{s}ds\\
&+|\mu|\Big(k_1\int_1^\eta \frac{|u(s)|}{s}ds+\frac{1}{\Gamma(\alpha_1)}\int_1^\eta \frac{|f(s,u(s),v(s))|}{s}\Big(\log\frac{\eta}{s}\Big)^{\alpha_1-1}ds\Big)\Bigg]\Bigg\}\\
&+k_1\int_1^t \frac{|u(s)|}{s}ds+\frac{1}{\Gamma(\alpha_1)}\int_1^t \frac{|f(s,u(s),v(s))|}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}ds\\
&\leq \frac{1}{|\Delta|}\Bigg\{\Bigg[k_1\|u\|\int_1^e \frac{1}{s}ds+\frac{m_0+m_1\|u\|+m_2\|v\|}{\Gamma(\alpha_1)}\int_1^e \frac{1}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_1-1}ds\\
&+|\lambda|\Big(k_2\|v\|\int_1^\theta \frac{1}{s}ds+\frac{n_0+n_1\|u\|+n_2\|v\|}{\Gamma(\alpha_2)}\int_1^\theta \frac{1}{s}\Big(\log\frac{\theta}{s}\Big)^{\alpha_2-1}ds\Big)\Bigg]\\
&+|\lambda|\Bigg[k_2\|v\|\int_1^e \frac{1}{s}ds+\frac{n_0+n_1\|u\|+n_2\|v\|}{\Gamma(\alpha_2)}\int_1^e \frac{1}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_2-1}ds\\
&+|\mu|\Big(k_1\|u\|\int_1^\eta \frac{1}{s}ds+\frac{m_0+m_1\|u\|+m_2\|v\|}{\Gamma(\alpha_1)}\int_1^\eta \frac{1}{s}\Big(\log\frac{\eta}{s}\Big)^{\alpha_1-1}ds\Big)\Bigg]\Bigg\}\\
&+k_1\|u\|\int_1^e \frac{1}{s}ds+\frac{m_0+m_1\|u\|+m_2\|v\|}{\Gamma(\alpha_1)}\int_1^e \frac{1}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{|\Delta|}\left\{\left[k_1\|u\| + \frac{m_0+m_1\|u\|+m_2\|v\|}{\Gamma(\alpha_1+1)} + |\lambda|\left(k_2\|v\| + \frac{n_0+n_1\|u\|+n_2\|v\|}{\Gamma(\alpha_2+1)}\right)\right]\right.\\
&\quad \left. +|\lambda|\left[k_2\|v\| + \frac{n_0+n_1\|u\|+n_2\|v\|}{\Gamma(\alpha_2+1)} + |\mu|\left(k_1\|u\| + \frac{m_0+m_1\|u\|+m_2\|v\|}{\Gamma(\alpha_1+1)}\right)\right]\right\}\\
&\quad +k_1\|u\| + \frac{m_0+m_1\|u\|+m_2\|v\|}{\Gamma(\alpha_1+1)}\\
&= M_1k_1\|u\| + M_1\frac{m_0+m_1\|u\|+m_2\|v\|}{\Gamma(\alpha_1+1)} + W_1k_2\|v\| + W_1\frac{n_0+n_1\|u\|+n_2\|v\|}{\Gamma(\alpha_2+1)},
\end{aligned}$$

which on taking maximum for $t \in [1, e]$, yields

(3.14)

$$\|u\| \leq M_1k_1\|u\| + \frac{M_1(m_0+m_1\|u\|+m_2\|v\|)}{\Gamma(\alpha_1+1)} + W_1k_2\|v\| + \frac{W_1(n_0+n_1\|u\|+n_2\|v\|)}{\Gamma(\alpha_2+1)}.$$

In the same way, one can obtain

(3.15)

$$\|v\| \leq M_1k_2\|v\| + \frac{M_1(n_0+n_1\|u\|+n_2\|v\|)}{\Gamma(\alpha_2+1)} + W_2k_1\|u\| + \frac{W_2(m_0+m_1\|u\|+m_2\|v\|)}{\Gamma(\alpha_1+1)}.$$

From (3.14) and (3.15), we have

$$\begin{aligned}
\|(u,v)\| =& \|u\| + \|v\|\\
\leq& \frac{m_0(M_1+W_2)}{\Gamma(\alpha_1+1)} + \frac{n_0(W_1+M_1)}{\Gamma(\alpha_2+1)} + \|u\|\left(k_1(M_1+W_2) + \frac{m_1(M_1+W_2)}{\Gamma(\alpha_1+1)}\right.\\
&\left. + \frac{n_1(W_1+M_1)}{\Gamma(\alpha_2+1)}\right) + \|v\|\left(k_2(W_1+M_1) + \frac{m_2(M_1+W_2)}{\Gamma(\alpha_1+1)} + \frac{n_2(W_1+M_1)}{\Gamma(\alpha_2+1)}\right)
\end{aligned}$$

and consequently

$$\|(u,v)\| \leq \frac{\dfrac{m_0(M_1+W_2)}{\Gamma(\alpha_1+1)} + \dfrac{n_0(W_1+M_1)}{\Gamma(\alpha_2+1)}}{Q_0},$$

where Q_0 is defined by

$$Q_0 = \min\{1 - Q_1, 1 - Q_2\}.$$

Therefore the set Φ is bounded. By Leray-Schauder alternative, we get that the operator $\mathcal{T}$ has at least one fixed point. Therefore, the problem (1.1)-(1.2) has at least one solution on $[1, e]$. □

3.3. **Existence and uniqueness result via Banach's fixed point theorem.** Next, we prove an existence and uniqueness result based on Banach's contraction mapping principle ([14]).

Theorem 3.3. *Assume that $f, g : [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy the following condition:*

(H_3) *There exist positive constants $L, \bar{L}$, such that for all $t \in [1, e]$, $u_i, v_i \in \mathbb{R}$, $i = 1, 2$,*

$$\begin{aligned}
|f(t,u_1,u_2) - f(t,v_1,v_2)| &\leq L(|u_1-v_1| + |u_2-v_2|),\\
|g(t,u_1,u_2) - g(t,v_1,v_2)| &\leq \bar{L}(|u_1-v_1| + |u_2-v_2|).
\end{aligned}$$

Then the system (1.1)-(1.2) has a unique solution on $[1, e]$, provided that

$$\varepsilon := \left[k_1(M_1+W_2) + k_2(W_1+M_1) + \frac{L(M_1+W_2)}{\Gamma(\alpha_1+1)} + \frac{\bar{L}(W_1+M_1)}{\Gamma(\alpha_2+1)}\right] < 1. \tag{3.16}$$

Proof. We will use Banach fixed point theorem to prove that $\mathcal{T}$, defined by (3.13), has a unique fixed point. Fixing $N_1 = \max_{t\in[1,e]} |f(t,0,0)| < \infty$, $N_2 = \max_{t\in[1,e]} |g(t,0,0)| < \infty$ and using the assumption (H_3), we obtain

$$\begin{aligned}
|f(t,u(t),v(t))| &= |f(t,u(t),v(t)) - f(t,0,0) + f(t,0,0)| \\
&\le L(\|u\| + \|v\|) + N_1 = L\|(u,v)\| + N_1, \\
|g(t,u(t),v(t))| &= |g(t,u(t),v(t)) - g(t,0,0) + g(t,0,0)| \\
&\le \bar{L}(\|u\| + \|v\|) + N_2 = \bar{L}\|(u,v)\| + N_2. \qquad (3.17)
\end{aligned}$$

We consider $B_r = \{(u,v) \in X \times X : \|(u,v)\| \le R\}$ with

$$R \ge \frac{\dfrac{N_1(M_1+W_2)}{\Gamma(\alpha_1+1)} + \dfrac{N_2(W_1+M_1)}{\Gamma(\alpha_2+1)}}{1 - \left[k_1(M_1+W_2) + k_2(W_1+M_1) + \dfrac{L(M_1+W_2)}{\Gamma(\alpha_1+1)} + \dfrac{\bar{L}(W_1+M_1)}{\Gamma(\alpha_2+1)}\right]}.$$

We divide the proof into two steps:

Step I : First we show that $\mathcal{T}(B_r) \subset B_R$. Let $(u,v) \in B_R$. Then, using (3.17), we obtain

$$\begin{aligned}
\big|\mathcal{T}_1(u,v)(t)\big| \;\le\;& \frac{1}{|\Delta|}\Bigg\{\Bigg[k_1\int_1^e \frac{|u(s)|}{s}ds + \frac{1}{\Gamma(\alpha_1)}\int_1^e \frac{|f(s,u(s),v(s))|}{s}\left(\log\frac{e}{s}\right)^{\alpha_1-1} ds \\
&+|\lambda|\left(k_2\int_1^\theta \frac{|v(s)|}{s}ds + \frac{1}{\Gamma(\alpha_2)}\int_1^\theta \frac{|g(s,u(s),v(s))|}{s}\left(\log\frac{\theta}{s}\right)^{\alpha_2-1} ds\right)\Bigg] \\
&+|\lambda|\Bigg[k_2\int_1^e \frac{|v(s)|}{s}ds + \frac{1}{\Gamma(\alpha_2)}\int_1^e \frac{|g(s,u(s),v(s))|}{s}\left(\log\frac{e}{s}\right)^{\alpha_2-1} ds \\
&+|\mu|\left(k_1\int_1^\eta \frac{|u(s)|}{s}ds + \frac{1}{\Gamma(\alpha_1)}\int_1^\eta \frac{|f(s,u(s),v(s))|}{s}\left(\log\frac{\eta}{s}\right)^{\alpha_1-1} ds\right)\Bigg]\Bigg\} \\
&+k_1\int_1^t \frac{|u(s)|}{s}ds + \frac{1}{\Gamma(\alpha_1)}\int_1^t \frac{|f(s,u(s),v(s))|}{s}\left(\log\frac{t}{s}\right)^{\alpha_1-1} ds \\
\le\;& \frac{1}{|\Delta|}\Bigg\{\Bigg[k_1\|u\| + \frac{LR+N_1}{\Gamma(\alpha_1+1)} + |\lambda|\left(k_2\|v\| + \frac{\bar{L}R+N_2}{\Gamma(\alpha_2+1)}\right)\Bigg] \\
&+|\lambda|\Bigg[k_2\|v\| + \frac{\bar{L}R+N_2}{\Gamma(\alpha_2+1)} + |\mu|\left(k_1\|u\| + \frac{LR+N_1}{\Gamma(\alpha_1+1)}\right)\Bigg]\Bigg\} + k_1\|u\| + \frac{LR+N_1}{\Gamma(\alpha_1+1)} \\
=\;& \frac{1}{|\Delta|}\left(k_1\|u\| + \frac{LR+N_1}{\Gamma(\alpha_1+1)}\right)\Big[1 + |\mu\lambda| + |\Delta|\Big] + \frac{1}{|\Delta|}\left(k_2\|v\| + \frac{\bar{L}R+N_2}{\Gamma(\alpha_2+1)}\right)2|\lambda| \\
=\;& M_1k_1\|u\| + M_1\frac{LR+N_1}{\Gamma(\alpha_1+1)} + W_1k_2\|v\| + W_1\frac{\bar{L}R+N_2}{\Gamma(\alpha_2+1)},
\end{aligned}$$

which on taking the norm for $t \in [1,e]$, yields

$$\|\mathcal{T}_1(u,v)\| \le M_1k_1\|u\| + M_1\frac{LR+N_1}{\Gamma(\alpha_1+1)} + W_1k_2\|v\| + W_1\frac{\bar{L}R+N_2}{\Gamma(\alpha_2+1)}.$$

In the same way, one has

$$\|\mathcal{T}_2(u,v)\| \le M_1k_2\|v\| + M_1\frac{\bar{L}R+N_2}{\Gamma(\alpha_2+1)} + W_2k_1\|u\| + W_2\frac{LR+N_1}{\Gamma(\alpha_1+1)}.$$

Hence

$$\|\mathcal{T}(u,v)(t)\| \le R\left[k_1(M_1+W_2) + k_2(W_1+M_1) + \frac{L(M_1+W_2)}{\Gamma(\alpha_1+1)} + \frac{\bar{L}(W_1+M_1)}{\Gamma(\alpha_2+1)}\right]$$

$$+\frac{N_1(M_1+W_2)}{\Gamma(\alpha_1+1)}+\frac{N_2(W_1+M_1)}{\Gamma(\alpha_2+1)}\le R.$$

Thus $\|\mathcal{T}(u,v)\|\le R$, that is, $\mathcal{T}(u,v)\in B_R$. Hence $\mathcal{T}(B_R)\subset B_R$.

Step II : We show that the operator $\mathcal{T}$ is a contraction.

Let $(u_2,v_2),(u_1,v_1)\in X\times X$. Then, for any $t\in[1,e]$, we have

$$\begin{aligned}
&|\mathcal{T}_1(u_2,v_2)(t)-\mathcal{T}_1(u_1,v_1)(t)|\\
\le\ & \frac{1}{|\Delta|}\Bigg\{\Bigg[k_1\int_1^e\frac{|u_2(s)-u_1(s)|}{s}ds+\frac{1}{\Gamma(\alpha_1)}\int_1^e\frac{|f(s,u_2(s),v_2(s))-f(s,u_1(s),v_1(s))|}{s}\left(\log\frac{e}{s}\right)^{\alpha_1-1}ds\\
&+|\lambda|\Bigg(k_2\int_1^\theta\frac{|v_2(s)-v_1(s)|}{s}ds+\frac{1}{\Gamma(\alpha_2)}\int_1^\theta\frac{|g(s,u_2(s),v_2(s))-g(s,u_1(s),v_1(s))|}{s}\left(\log\frac{\theta}{s}\right)^{\alpha_2-1}ds\Bigg)\Bigg]\\
&+|\lambda|\Bigg[k_2\int_1^e\frac{|v_2(s)-v_1(s)|}{s}ds+\frac{1}{\Gamma(\alpha_2)}\int_1^e\frac{|g(s,u_2(s),v_2(s))-g(s,u_1(s),v_1(s))|}{s}\left(\log\frac{e}{s}\right)^{\alpha_2-1}ds\\
&+|\mu|\Bigg(k_1\int_1^\eta\frac{|u_2(s)-u_1(s)|}{s}ds+\frac{1}{\Gamma(\alpha_1)}\int_1^\eta\frac{|f(s,u_2(s),v_2(s))-f(s,u_1(s),v_1(s))|}{s}\left(\log\frac{\eta}{s}\right)^{\alpha_1-1}ds\Bigg)\Bigg]\Bigg\}\\
&+k_1\int_1^t\frac{|u_2(s)-u_1(s)|}{s}ds+\frac{1}{\Gamma(\alpha_1)}\int_1^t\frac{|f(s,u_2(s),v_2(s))-f(s,u_1(s),v_1(s))|}{s}\left(\log\frac{t}{s}\right)^{\alpha_1-1}ds\\
\le\ & \frac{1}{|\Delta|}\Bigg\{\Bigg[k_1\|u_2-u_1\|+\frac{L(\|u_2-u_1\|+\|v_2-v_1\|)}{\Gamma(\alpha_1+1)}+|\lambda|\Bigg(k_2\|v_2-v_1\|+\frac{\bar{L}(\|u_2-u_1\|+\|v_2-v_1\|)}{\Gamma(\alpha_2+1)}\Bigg)\Bigg]\\
&+|\lambda|\Bigg[k_2\|v_2-v_1\|+\frac{\bar{L}(\|u_2-u_1\|+\|v_2-v_1\|)}{\Gamma(\alpha_2+1)}+|\mu|\Bigg(k_1\|u_2-u_1\|+\frac{L(\|u_2-u_1\|+\|v_2-v_1\|)}{\Gamma(\alpha_1+1)}\Bigg)\Bigg]\Bigg\}\\
&+k_1\|u_2-u_1\|+\frac{L(\|u_2-u_1\|+\|v_2-v_1\|)}{\Gamma(\alpha_1+1)}\\
=\ & M_1\Bigg(k_1\|u_2-u_1\|+\frac{L(\|u_2-u_1\|+\|v_2-v_1\|)}{\Gamma(\alpha_1+1)}\Bigg)+W_1\Bigg(k_2\|v_2-v_1\|+\frac{\bar{L}(\|u_2-u_1\|+\|v_2-v_1\|)}{\Gamma(\alpha_2+1)}\Bigg)\\
\le\ & \Bigg[M_1k_1+\frac{M_1L}{\Gamma(\alpha_1+1)}+W_1k_2+\frac{W_1\bar{L}}{\Gamma(\alpha_2+1)}\Bigg](\|u_2-u_1\|+\|v_2-v_1\|),
\end{aligned}$$

which on taking the norm for $t\in[1,e]$, yields

(3.18)

$$\|\mathcal{T}_1(u_2,v_2)-\mathcal{T}_1(u_1,v_1)\|\le\Bigg[M_1k_1+\frac{M_1L}{\Gamma(\alpha_1+1)}+W_1k_2+\frac{W_1\bar{L}}{\Gamma(\alpha_2+1)}\Bigg](\|u_2-u_1\|+\|v_2-v_1\|).$$

Similarly,

(3.19)

$$\|\mathcal{T}_2(u_2,v_2)-\mathcal{T}_2(u_1,v_1)\|\le\Bigg[M_1k_2+\frac{M_1\bar{L}}{\Gamma(\alpha_2+1)}+W_2k_1+\frac{W_2L}{\Gamma(\alpha_1+1)}\Bigg](\|u_2-u_1\|+\|v_2-v_1\|).$$

It follows from (3.18) and (3.19) that $\|\mathcal{T}(u_2,v_2)-\mathcal{T}(u_1,v_1)\|\le\varepsilon(\|u_2-u_1\|+\|v_2-v_1\|)$. By (3.16), it shows that the operator $\mathcal{T}$ is a contraction. Hence, operator $\mathcal{T}$ has a unique fixed point by Banach contraction mapping principle. Therefore, the system (1.1)-(1.2) has a unique solution on $[1,e]$. □

4. EXAMPLES

In this section, we give two examples to illustrate our main results.

Example 4.1. Consider the following system

$$\begin{cases} \left({}_HD^{\frac{3}{2},\frac{1}{2}} + \frac{1}{5}{}_HD^{\frac{1}{2},\frac{1}{2}}\right)u(t) = \frac{|u(t)|}{(\log t + 50)} + \frac{|v(t)|}{(t+9)^2(1+|v(t)|)} + \frac{1}{(t+7)^2}, \ t \in [1,e], \\ \left({}_HD^{2,\frac{1}{3}} + \frac{1}{10}{}_HD^{1,\frac{1}{3}}\right)v(t) = \frac{|u(t)|}{5\sqrt{(t+99)}} + \frac{\sin(\pi v(t))}{70\pi} + \frac{1}{65}, \ t \in [1,e], \\ u(1) = 0, \ u(e) = -2v\left(\frac{3}{2}\right), \ v(1) = 0, \ v(e) = \frac{4}{9}u(2). \end{cases} \tag{4.20}$$

Here

$\alpha_1 = \frac{3}{2}, \ \alpha_2 = 2, \ \beta_1 = \frac{1}{2}, \ \beta_2 = \frac{1}{3}, \ \gamma_1 = \frac{7}{4}, \ \gamma_2 = \frac{7}{3}, \ k_1 = \frac{1}{5}, \ k_2 = \frac{1}{10}, \ \lambda = -2, \ \mu = \frac{4}{9},$
$\theta = \frac{3}{2}, \ \eta = 2, \ A = 0.600, \ B = -0.337, \ \Delta = 1.202, \ M_1 = 2.570, \ W_1 = 3.326, \ W_2 = 0.739.$

We see that (H_1) holds, since

$$|f(t,u,v)| \leq \frac{1}{64} + \frac{|u(t)|}{50} + \frac{|v(t)|}{100} \ \text{ and } \ |g(t,u,v)| \leq \frac{1}{65} + \frac{|u(t)|}{50} + \frac{|v(t)|}{70},$$

with

$$m_0 = \frac{1}{64}, \ m_1 = \frac{1}{50}, \ m_2 = \frac{1}{100}, \ n_0 = \frac{1}{65}, \ n_1 = \frac{1}{50}, \ n_2 = \frac{1}{70}.$$

In addition, $Q_1 \approx 0.770$ and $Q_2 \approx 0.656$. Thus, the hypotheses of Theorem 3.2 are satisfied and hence the system (4.20) has at least one solution on $[1,e]$.

Example 4.2. Consider the following Hilfer-Hadamard system

$$\begin{cases} \left({}_HD^{\frac{3}{2},1} + \frac{2}{51}{}_HD^{\frac{1}{2},1}\right)v(t) = \frac{\sin(u(t))}{(3+t)^3} + \frac{|v(t)|}{(9+t)^2(1+|v(t)|)} + \frac{\log t}{100}, \ t \in [1,e] \\ \left({}_HD^{\frac{5}{4},\frac{1}{2}} + \frac{3}{23}{}_HD^{\frac{1}{4},\frac{1}{2}}\right)u(t) = \frac{(2+\log t)|u(t)|}{120} + \frac{|v(t)|}{\sqrt{99+t}(6+|v(t)|)} + \frac{1}{50+t^3}, \ t \in [1,e], \\ u(1) = 0, \ u(e) = -\frac{5}{9}v(2), \ v(1) = 0, \ v(e) = 5u\left(\frac{5}{3}\right). \end{cases} \tag{4.21}$$

Here
$\alpha_1 = \frac{3}{2}, \ \alpha_2 = \frac{5}{4}, \ \beta_1 = 1, \ \beta_2 = \frac{1}{2}, \ \gamma_1 = 2, \ \gamma_2 = \frac{13}{8}, \ k_1 = \frac{2}{51}, \ k_2 = \frac{3}{23}, \ \lambda = -\frac{5}{9}, \ \mu = 5,$
$\theta = 2, \ \eta = \frac{5}{3}, \ A = 0.441, \ B = -2.554, \ \Delta = 2.128, \ M_1 = 2.774, \ W_1 = 0.522, \ W_2 = 4.698.$

Note that (H_3) holds, because

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \leq \frac{1}{64}(|u_1 - v_1| + |u_2 - v_2|)$$

and

$$|g(t,u_1,u_2) - g(t,v_1,v_2)| \leq \frac{1}{40}(|u_1 - v_1| + |u_2 - v_2|),$$

with $L = \frac{1}{64}, \ \bar{L} = \frac{1}{40}$. Therefore, we have

$$\varepsilon := \left[k_1(M_1 + W_2) + k_2(W_1 + M_1) + \frac{L(M_1 + W_2)}{\Gamma(\alpha_1 + 1)} + \frac{\bar{L}(W_1 + M_1)}{\Gamma(\alpha_2 + 1)}\right] \approx 0.883 < 1.$$

Thus, all the conditions of Theorem 3.3 are satisfied and therefore the system (4.21) has a unique solution on $[1,e]$.

5. Conclusions

In this paper, we conducted research on the existence and uniqueness of solutions for a system of Hilfer-Hadamard sequential fractional differential equations with three-point boundary conditions. Firstly, via a linear variant of the given problem, we have converted the nonlinear problem into a fixed point problem. Once the fixed point operator were available, the existence result was established using the Leray-Schauder alternative, while the Banach contraction principle is applied to achieve the existence and uniqueness result. Additionally, we provide examples that illustrate the obtained results. Our results are new in the given configuration and enrich the literature on coupled systems involving Hilfer-Hadamard fractional derivatives of order in $(1, 2]$.

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