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Asymptotic Regularity of Generalized Averaged Mappings in (M, K, ψ)-HR-Ćirić-Reich-Rus Contractions

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ABSTRACT. The paper investigates a new type of contraction called (M, K, ψ)-HR-Ćirić-Reich-Rus contractions, which extends the existing research on Ćirić-Reich-Rus contraction mappings. We prove fixed point theorems by enriching (M, K, ψ)-HR-Ćirić-Reich-Rus contractions with respect to the asymptotic regularity of generalized averaged mappings. These mappings are denoted as $T_\phi = (1-\phi)I + \phi T$, where I is the identity map and ϕ is a function from a normed space to real numbers.

1. INTRODUCTION AND PRELIMINARIES

Let $(X, \|\cdot\|)$ be a normed space and an mapping $T : X \to X$, for each fixed $\phi \in \Omega$, we define the generalized averaged mapping ([4]) $T_\phi : X \to X$ as follows:

$$T_\phi(x) = (1-\phi(x))x + \phi(x)T(x), \quad \forall x \in X. \tag{1.1}$$

where, $\phi \in \Omega = \{\phi : X \to \mathbb{R} : \phi(x) \neq 0, \ \forall x \in X\}$.
We would like to direct the reader's attention to the fact that the term generalized averaged mapping refers to a specific type of admissible perturbations [17, 20]. It is worth noting that the class of generalized averaged mappings includes the class of averaged mappings (a term coined in [6]) [4]. This is demonstrated by considering $\lambda \in (0,1)$ and defining $\phi(x) = \lambda$ for all $x \in X$.
Consequently, the condition (1.1) is reduced to

$$T_\lambda(x) := T_\phi(x) = (1-\lambda)x + \lambda Tx, \forall x \in X. \tag{1.2}$$

The following lemma shows that T and T_ϕ have same fixed points.

Lemma 1.1. [14] *Let $T : X \to X$ and T be a generalized averaged mapping as given in (1.1). Then, for any $\phi \in \Omega$:*

$$Fix(T) = \{x \in X : Tx = x\} = \{x \in X : T_\phi(x) = x\} = Fix(T_\phi). \tag{1.3}$$

Let $(X, \|\cdot\|)$ be a normed space. An mapping $T : X \to X$ is called:
(1) asymptotically regular ([12]) on X if for each x in X, we have

$$\left\|T^{n+1}x - T^n x\right\| \to 0 \text{ as } n \to \infty.$$

(2) nonexpansive ([14]) if

$$\|Tx - Ty\| \leq \|x - y\|$$

holds $x, y \in X$.
Banach contraction mapping ([15]) is a typical example of asymptotically regular mapping. In general, nonexpansive mapping is not an aysmptotically regular.
It is known that there is a class of mappings such that their averaged mapping for a fixed

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λ is asymptotically regular, but the mapping itself is not, , for details see [5]. For example, let $X = [0,1]$ and $T : X \to X$ be defined as $Tx = 1 - x$, for all $x \in X$. Clearly, T is not asymptotically regular, yet $T_{\frac{1}{2}}$ is an asymptotically regular mapping.

Krasnoselskii [16] proved that if K is a compact convex subset of a uniformly convex Banach space X and $T : K \to K$ is a nonexpansive mapping, then for any $x_0 \in K$, the sequence

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n) = T_{\frac{1}{2}} x_n,\ n \geq 0,$$

converges to fixed point of T.

This motivates the idea of an enrichment of given class of mappings. Indeed, an averaged mapping T_λ enriches the class of nonexpansive mappings with respect to asymptotic regularity. This suggests to impose certain contractive condition on T_λ instead of T and enrich the classes of contractive mappings in the framework of normed spaces.

Employing this approach, we recommend readers to refer to the following sources and the references therein: [1, 2, 3, 5, 7, 8, 9, 10, 11].

In 2021, Berinde and Păcurar introduced and explored the concept of enriched Ćirić-Reich-Rus contraction in their work [7], with the following result serving as the main outcome of their study.

Theorem 1.1. [7] *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be an (b, M, K)-enriched Ćirić-Reich-Rus contraction, that is an mapping satisfying:*

$$\|b(x-y) + Tx - Ty\| \leq M\|x-y\| + K\{\,\|x - Tx\| + \|y - Ty\|\,\},\ \forall\, x, y \in X, \tag{1.4}$$

where, $M, K \geq 0$, with $M + 2K < 1$. Then, T has a unique fixed point.

Clearly, any Ćirić-Reich-Rus contraction in the setting of normed spaces, as defined by Ćirić [13], Reich [18], and Rus [19], that satisfies the following condition:

$$\|Tx - Ty\| \leq M\|x-y\| + K\{\|x - Tx\| + \|y - Ty\|\},\ \forall\, x, y \in X, \tag{1.5}$$

where M and K are non-negative, with $M+2K < 1$, is indeed an $(0, M, K)$-enriched Ćirić-Reich-Rus contraction. Furthermore, it is worth noting that class of enriched Banach contractions [8] and enriched Kannan contractions [11] are specific cases of (b, M, K)-enriched Ćirić-Reich-Rus contractions.

The requirement of $M + 2K < 1$ in Theorem 1.1 is crucial to establish the existence of a fixed point for mappings that satisfy (1.4). This naturally prompts the question: What happens when we remove the condition on the constants such that $M + 2K < 1$ in Theorem 1.1? Does it still lead to the same conclusion as stated in Theorem 1.1?

In this context, Górnicki and Bisht in 2021 [14] proved the following theorem.

Theorem 1.2. [14] *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ satisfies (1.4) for all $M, K \geq 0$. Additionally, suppose there exists a $\lambda \in (0,1)$ such that T_λ is a continuous asymptotically regular mapping. Then, T has a unique fixed point, where $\lambda = \frac{1}{b+1}$.*

The following example serves as the foundation for our main concept in this paper. It illustrates a class of mappings where neither T nor T_λ exhibits asymptotic regularity for any $\lambda \in (0,1)$. However, it demonstrates the existence of a function $\phi \in \Omega$ for which T_ϕ does exhibit asymptotic regularity.

Example 1.1. Consider the mapping $T : \mathbb{R} \to \mathbb{R}$, defined as $Tx = 2x - 3$, for all $x \in \mathbb{R}$. Starting with an initial value of $x = 1$, the subsequent iterations are as follows:

$$T(1) = -1,\ T^2(1) = -5,\ T^3(1) = -13,\ T^4(1) = -29,\ \cdots$$

Notice that

$$|T^n(1) - T^{n+1}(1)| \nrightarrow 0,\ n \to \infty.$$

This demonstrates that T is not asymptotically regular.
For any $\lambda \in (0, 1)$, we obtain

$$T_\lambda(x) = x + \lambda x - 3\lambda, \quad \forall x \in \mathbb{R}.$$

For any $x = 0$ and for all $x \in \mathbb{R}$, the subsequent iterations are as follows:

$$T_\lambda(0) = -3\lambda,\ T_\lambda^2(0) = -6\lambda - 3\lambda^2,\ T_\lambda^3(0) = -3\lambda^3 - 9\lambda^2 - 9\lambda,\ \cdots$$

It's worth noting that

$$|T_\lambda^n(0) - T_\lambda^{n+1}(0)| \nrightarrow 0,\ n \to \infty.$$

Furthermore, it can be verified that for any $\lambda \in (0, 1)$, the average mapping T_λ also lacks asymptotic regularity.
On the other hand, if we take $\phi(x) = -1$, for all $x \in \mathbb{R}$. Notice that, for all $x \in \mathbb{R}$, we obtain

$$T_\phi(x) = [1 - \phi(x)]x + \phi(x)T(x) = [1 - (-1)]x + (-1)[2x - 3] = -3.$$

Clearly, T_ϕ is asymptotically regular.

Our aim in this section is to unify and extend Theorems 1.1 and 1.2 by introducing the concept of (M, K, ψ)-HR-Ćirić-Reich-Rus Contractions in Banach spaces. We prove fixed point theorems by enriching (M, K, ψ)-HR-Ćirić-Reich-Rus contractions with respect to the asymptotic regularity of generalized averaged mappings.

Fixed Point Theorems for (M, K, ψ)-HR-Ćirić-Reich-Rus Contractions

Throughout the paper, we denote $\mho$ as the set of all functions $\psi(x) : X \to \mathbb{R}$ satisfying the following property: $\psi(x) \neq -1$ for all $x \in X$. Now, we represent the following theorem.
We introduce the following idea.

Definition 1.1. Let $(X, \|\cdot\|)$ be a normed space. A mapping $T : X \to X$ is said to be HR-Ćirić-Reich-Rus type contraction if there exist $M, K \geq 0$ and $\psi \in \mho$ such that
(1.6)

$$\left\| \frac{x\psi(x) + Tx}{1 + \psi(x)} - \frac{y\psi(y) + Ty}{1 + \psi(y)} \right\| \leq M \|x - y\| + K. \left\{ \left\| \frac{1}{\psi(x) + 1}(x - Tx) \right\| + \left\| \frac{1}{\psi(y) + 1}(y - Ty) \right\| \right\},$$

for all $x, y \in X$.

To highlight the role of M, K and ψ in (1.6), we call T a (M, K, ψ)-HR-Ćirić-Reich-Rus contraction.
We begin with the following result.

Theorem 1.3. *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ a (M, K, ψ)- HR-Ćirić-Reich-Rus contraction. Assume that there exists $\phi \in \Omega$ such that T_ϕ is an continuous asymptotically regular mapping. Then, T has a unique fixed point, where $\phi(x) = \frac{1}{\psi(x)+1}$, for all $x \in X$.*

Proof. Let us denote $\phi(x) = \frac{1}{\psi(x)+1}$, $x \in X$. As given that T is (M, K, ψ)- HR-Ćirić-Reich-Rus contraction, so (1.6) becomes

$$\left\| \frac{x(\frac{1}{\phi(x)} - 1) + Tx}{\frac{1}{\phi(x)} - 1 + 1} - \frac{y(\frac{1}{\phi(y)} - 1) + Ty}{\frac{1}{\phi(y)} - 1 + 1} \right\| \leq K\left\{ \left\| \frac{x - Tx}{\frac{1}{\phi(x)} - 1 + 1} \right\| + \left\| \frac{y - Ty}{\frac{1}{\phi(y)} - 1 + 1} \right\| \right\} + M \left\| x - y \right\|,$$

$$\left\| \phi(x) \frac{(1 - \phi(x))x + \phi(x)T(x)}{\phi(x)} - \phi(y) \frac{(1 - \phi(y))y + \phi(y)T(y)}{\phi(y)} \right\| \leq K \left\| (x - Tx)\phi(x) \right\| + K \left\| (y - Ty)\phi(y) \right\| + M \left\| x - y \right\|,$$

which can be written in an equivalent form as:

$$\left\| T_\phi(x) - T_\phi(y) \right\| \leq K.\{\left\| x - T_\phi(x) \right\| + \left\| y - T_\phi(y) \right\|\} + M \left\| x - y \right\|, \quad \forall x, y \in X. \tag{1.7}$$

Define the iteration sequence by

$$x_{n+1} = (1 - \phi(x_n))x_n + \phi(x_n)Tx_n, \quad \forall n \in \mathbb{N}, \tag{1.8}$$

where, $x_1 \in X$.
By using (1.7), triangular inequality and asymptotic regularity we obtain,

$$\begin{aligned} \left\| x_{n+k} - x_n \right\| &\leq \left\| x_{n+k} - x_{n+k+1} \right\| + \left\| x_{n+k+1} - x_{n+1} \right\| + \left\| x_{n+1} - x_n \right\| \\ &\leq \left\| x_{n+k} - x_{n+k+1} \right\| + \left\| T_\phi x_{n+k} - T_\phi x_n \right\| + \left\| x_{n+1} - x_n \right\| \\ &\leq \left\| x_{n+k} - x_{n+k+1} \right\| + K\{\left\| x_{n+k} - T_\phi x_{n+k} \right\| + \left\| x_n - T_\phi x_n \right\|\} \\ &\quad + M \left\| x_{n+k} - x_n \right\| + \left\| x_{n+1} - x_n \right\| \\ &\leq \left\| x_{n+k} - x_{n+k+1} \right\| + K\{\left\| x_{n+k} - x_{n+k+1} \right\| + \left\| x_n - x_{n+1} \right\|\} \\ &\quad + M \left\| x_{n+k} - x_n \right\| + \left\| x_{n+1} - x_n \right\| \end{aligned} \tag{1.9}$$

The inequality given by (1.9) can be expressed as follows:

$$(1 - M) \left\| x_{n+k} - x_n \right\| \leq (K + 1) \left\| x_{n+k} - x_{n+k+1} \right\| + \left\| x_n - x_{n+1} \right\|,$$

Therefore, we have:

$$(1 - M) \left\| T_\phi^{n+k} x_0 - T_\phi^n x_0 \right\| \leq (K + 1).\left\{ \left\| T_\phi^{n+k} x_0 - T_\phi^{n+k+1} x_0 \right\| + \left\| T_\phi^n x_0 - T_\phi^{n+1} x_0 \right\| \right\}. \tag{1.10}$$

Since T_ϕ is asymptotically regular, we can conclude that:

$$\lim_{n\to\infty} \left\| T_\phi^{n+k} x_0 - T_\phi^{n+k+1} x_0 \right\| = 0 = \lim_{n\to\infty} \left\| T_\phi^{n+k} x_0 - T_\phi^{n+1} x_0 \right\|.$$

On taking $n \to \infty$ in (1.10), we get

$$\lim_{n\to\infty} \left\| T_\phi^{n+k} x_0 - T_\phi^n x_0 \right\| = 0.$$

The given information indicates that the $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence within the complete space X. There is an element $s^* \in X$ such that the sequence $x_n \to s^*$. This is due to the continuity of T_ϕ and the fact that $x_{n+1} = T_\phi x_n$.

Consequently, s^* is equal to $T_\phi s^*$.
Assume that t^* is another fixed point of T_ϕ. In that case, the inequality

$$\begin{aligned} 0 < \|s^* - t^*\| = \|T_\phi s^* - T_\phi t^*\| &\le M.\|s^* - t^*\| + (K+1).\{\|s^* - T_\phi s^*\| + \|t^* - T_\phi t^*\|\} \\ &= M.\|s^* - t^*\| < \|s^* - t^*\|, \end{aligned}$$

This contradiction leads to the conclusion that T_ϕ has a unique fixed point $s^* \in X$.

□

Example 1.2. Consider the mapping T defined in Example 1.1. Then, $T(3) = 3$ and

i) T does not satisfy the enriched Banach contraction. Indeed, if T would be an enriched Banach contraction, then by (Definition 2.1 of [8]), there would exist $b \in [0, \infty)$ and $\alpha \in [0, b+1)$ such that

$$(b+2)|x-y| \le \alpha|x-y|, \quad \forall x, y \in \mathbb{R},$$

for which $x = 0$ and $y = 1$, leads to the contradiction $b + 2 \le \alpha < b + 1$.

ii) T does not satisfy the enriched Kannan contraction. Indeed, if T would be an enriched Kannan contraction then, then by (Definition 2.1 of [11]), there would exist $b \in [0, \infty)$ and $\alpha \in [0, \frac{1}{2})$ such that

$$(b+2)|x-y| \le \alpha\{|-x+3| + |-y+3|\}, \quad \forall x, y \in \mathbb{R},$$

for which $x = 1$ and $y = 5$, leads to the contradiction $2 \le b + 2 \le \alpha < \frac{1}{2}$.

iii) T does not satisfy the Ćirić-Reich-Rus contraction (1.5). Indeed if T would be a Ćirić-Reich-Rus contraction then, by (1.5), there would exist $M, K \ge 0$ with $M + 2K < 1$ such that

$$2|x-y| \le M|x-y| + K\{|-x+3| + |-y+3|\}, \quad \forall x, y \in \mathbb{R},$$

for which $x = 6$ and $y = 4$, leads to the contradiction $2 \le M + 2K < 1$.

iv) T does not satisfy the (b, M, K)-enriched Ćirić-Reich-Rus contraction (1.4). Indeed if T would be a (b, M, K)-enriched Ćirić-Reich-Rus contraction then, by (1.4), there would exist $M, K \ge 0$ with $M + 2K < 1$ such that

$$(b+2)|x-y| \le M|x-y| + K\{|-x+3| + |-y+3|\}, \quad \forall x, y \in \mathbb{R},$$

for which $x = 6$ and $y = 4$, leads to the contradiction $2 \le (b+2) \le M + 2K < 1$.

v) On the other hand, if we take $\psi(x) = -2$, for all $x \in \mathbb{R}$, and for any $M, K \ge 0$, then T is a (M, K, ψ)-HR-Ćirić-Reich-Rus type contraction. Indeed, the contractive condition (1.6) holds for all $x, y \in \mathbb{R}$:

$$\begin{aligned} &\left\| \frac{x\psi(x) + Tx}{1 + \psi(x)} - \frac{y\psi(y) + Ty}{1 + \psi(y)} \right\| = \left| \frac{-2x + 2x - 3}{1 - 2} - \frac{-2y + 2y - 3}{1 - 2} \right| \\ &= 0 \le M\|x-y\| + K.\left\{ \left\| \frac{1}{\psi(x)+1}(x - Tx) \right\| + \left\| \frac{1}{\psi(y)+1}(y - Ty) \right\| \right\}. \end{aligned}$$

vi) T is continuous;

vii) It follows from Example 1.1 that T is not asymptotically regular. Additionally, for any $\lambda \in (0, 1)$, T is not asymptotically regular, but there exists $\phi \in \Omega$ such that the map $T_\phi = (1 - \phi)I + \phi T$ is asymptotically regular.

Hence, all the conditions of Theorem 1.3 are satisfied.

In the special case where $\psi(x) = 0$ for all $x \in X$, in Theorem 1.3, we deduce Theorem 2.6 of [15] in the setting of a Banach space.

Corollary 1.1. [15] *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ is a continuous asymptotically regular mapping, and if there exist $0 \leq K < \infty$ and $0 \leq M < 1$ such that for all $x, y \in X$ satisfying*

$$\|Tx - Ty\| \leq M\|x - y\| + K\{\|x - Tx\| + \|y - Ty\|\}.$$

Then, T possesses a unique fixed point.

In the particular scenario where $\psi(x) = b$ holds for all $x \in X$, as indicated in Theorem 1.3, we obtain the following result.

Corollary 1.2. *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ satisfying*

$$\|b(x - y) + Tx - Ty\| \leq (b + 1)M\|x - y\| + K\{\|x - Tx\| + \|y - Ty\|\}.$$

for all $x, y \in X$, where $b, K, M \geq 0$. Additionally, $T_{\frac{1}{b+1}}$ is a continuous asymptotically regular mapping. Then, T has a unique fixed point.

Now will provide an extension of Theorem 1.3.
Let S denote the class of those functions $\alpha : [0, \infty) \to [0, 1)$ which satisfy the simple condition : $\alpha(t_n) \to 1 \Rightarrow t_n \to 0$.
For example, $\alpha_1(t) = e^{-t}, \alpha_2(t) = \frac{1}{t+1}, t > 0$.

Theorem 1.4. *Let $(X, \|\cdot\|)$ be a Banach space. Suppose $T : X \to X$ satisfies the following condition for all $x, y \in X$,*

(1.11)
$$\left\|\frac{x\psi(x) + Tx}{\psi(x) + 1} - \frac{y\psi(y) + Ty}{\psi(y) + 1}\right\| \leq \alpha(\|x - y\|)\|x - y\| + K\left\{\left\|\frac{x - Tx}{\psi(x) + 1}\right\| + \left\|\frac{y - Ty}{\psi(y) + 1}\right\|\right\},$$

where $0 \leq k < \infty$, $\alpha \in S$ and $\psi \in \mho$. Assume that there exists $\phi \in \Omega$ such that T_ϕ is an asymptotically regular mapping. Moreover, If T is continuous, then T has a unique fixed point, where $\phi(x) = \frac{1}{\psi(x)+1}$, for all $x \in X$.

Proof. Let us denote $\phi(x) = \frac{1}{\psi(x)+1}$, $x \in X$. Then the contractive condition (1.11) becomes

$$\left\|\frac{x(\frac{1}{\phi(x)} - 1) + Tx}{\frac{1}{\phi(x)} - 1 + 1} - \frac{y(\frac{1}{\phi(y)} - 1) + Ty}{\frac{1}{\phi(y)} - 1 + 1}\right\| \leq K\left\{\left\|\frac{x - Tx}{\frac{1}{\phi(x)} - 1 + 1}\right\| + \left\|\frac{y - Ty}{\frac{1}{\phi(y)} - 1 + 1}\right\|\right\} + \alpha(\|x - y\|)\|x - y\|,$$

$$\left\|\phi(x)\frac{(1 - \phi(x))x + \phi(x)T(x)}{\phi(x)} - \phi(y)\frac{(1 - \phi(y))y + \phi(y)T(y)}{\phi(y)}\right\| \leq K\|(x - Tx)\phi(x)\| + K\|(y - Ty)\phi(y)\| + \alpha(\|x - y\|)\|x - y\|,$$

which can be written in an equivalent form as:

(1.12)
$$\|T_\phi(x) - T_\phi(y)\| \leq K\{\|x - T_\phi(x)\| + \|y - T_\phi(y)\|\} + \alpha(\|x - y\|)\|x - y\|, \quad \forall x, y \in X.$$

Define the iteration sequence $\{x_n\}$ by (1.8).

If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbb{N}$, it follows from (1.8) that x_{n_0} is a fixed point of T. On the other hand, if $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Assume that $\{x_n\}$ is not a Cauchy sequence. Then, we have

$$\limsup_{n,m\to\infty} \|x_n - x_m\| > 0. \tag{1.13}$$

By using (1.12) and triangle inequality, we obtain

$$\begin{aligned}
\|x_n - x_m\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_{m+1}\| + \|x_{m+1} - x_m\| \\
&\leq \|x_n - x_{n+1}\| + \|T_\phi(x_n) - T_\phi(x_m)\| + \|x_{m+1} - x_m\| \\
&\leq \|x_n - x_{n+1}\| + \alpha(\|x_n - x_m\|) \|x_n - x_m\| + K \|x_n - T_\phi(x_n)\| \\
&\quad + K \|x_m - T_\phi(x_m)\| + \|x_{m+1} - x_m\| \\
&\leq \alpha(\|x_n - x_m\|) \|x_n - x_m\| + \|x_n - x_{n+1}\| + \|x_{m+1} - x_m\| \\
&\quad + K\{\|x_n - x_{n+1}\| + \|x_m - x_{m+1}\|\} \\
&\leq \alpha(\|x_n - x_m\|) \|x_n - x_m\| + (K+1)\{\|x_n - x_{n+1}\| + \|x_m - x_{m+1}\|\}. \qquad (1.14)
\end{aligned}$$

The inequality (1.14), can be written as

$$\frac{\|x_n - x_m\|}{\|x_n - x_{n+1}\| + \|x_m - x_{m+1}\|} \leq \frac{K+1}{1 - \alpha(\|x_n - x_m\|)} \qquad (1.15)$$

Under the assumption $\limsup_{n,m\to\infty} \|x_n - x_m\| > 0$, asymptotic regularity of $\{x_n\}$ the inequality.

now implies

$$\lim_{n,m\to\infty} \frac{K+1}{1 - \alpha(\|x_n - x_m\|)} = +\infty,$$

for which

$$\limsup_{n,m\to\infty} \alpha(\|x_n - x_m\|) = 1, \forall x \in X.$$

But since $\alpha \in S$ this implies $\limsup_{n,m\to\infty} \|x_n - x_m\| = 0$, which is a contradiction. Thus, $\{x_n\}$ is a cauchy sequence. Because X is a Banach space $\lim_{n\to\infty} x_n = z \in X$.
Usuing the continuity of T_ϕ, we immediately obtain $Tz = z$. It is straightforward to demonstrate that T has a unique fixed point. □

In the special case where $\psi(x) = 0$ for all $x \in X$, in Theorem 1.4, we deduce Theorem 3.1 of [14].

Corollary 1.3. [14] *Let $(X, \|\cdot\|)$ be a Banach space, and $T : X \to X$ is a continuous asymptotically regular mapping. If there exists $0 \leq K < \infty$ and $\alpha \in S$ such that for all $x, y \in X$ satisfying:*

$$\|Tx - Ty\| \leq \alpha(\|x - y\|) \|x - y\| + K\{\|x - Tx\| + \|y - Ty\|\},$$

Then, T possesses a unique fixed point.

Here, we pose a question:
Does removing the asymptotic regularity condition from Theorem 1.3 lead to the same conclusion as stated in Theorem 1.3?
To investigate this, we now proceed to establish the following result.

Theorem 1.5. *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ a (M, K, ψ)- HR-Ćirić-Reich-Rus contraction. Then T has a unique fixed point, provided that $M + 2K < 1$.*

Proof. Based on the same procedure as in the proof of Theorem 1.3, we get

$$\|T_\phi(x) - T_\phi(y)\| \leq K.\{\|x - T_\phi(x)\| + \|y - T_\phi(y)\|\} + M\|x - y\|, \forall x, y \in X. \tag{1.16}$$

Utilizing the triangle inequality in (1.16), we can deduce that T_ϕ satisfies the following inequality:

$$\|T_\phi x - T_\phi y\| \leq \delta\|x - y\| + 2\delta\|y - T_\phi x\|, \quad \forall x, y \in X, \tag{1.17}$$

where, $\delta = \frac{a+b}{1-b} < 1$.

Based on the same procedure we can show that the sequence $\{x_n\}_{n=1}^{\infty}$ defined by (1.8) is a Cauchy sequence and consequently converges within the space X. Let us denote the limit as:

$$p^* = \lim_{n\to\infty} x_n.$$

Initially, we establish that p^* serves as a fixed point of T_ϕ. To do this, we have:

$$\|p^* - T_\phi p^*\| \leq \|p^* - x_{n+1}\| + \|x_{n+1} - T_\phi p^*\| = \|x_{n+1} - p^*\| + \|T_\phi x_n - T_\phi p^*\|. \tag{1.18}$$

Utilizing (1.17), we can infer that:

$$\|T_\phi x_n - T_\phi p^*\| \leq \delta\|x_n - p^*\| + 2\delta\|p^* - x_{n+1}\|,$$

Consequently, by (1.18), we arrive at the following inequality:

$$\|p^* - T_\phi p^*\| \leq (2\delta + 1)\|x_{n+1} - p^*\| + \delta\|x_n - p^*\|, n \geq 0.$$

Taking the limit as $n \to \infty$ on both sides of the above inequality, we can establish that indeed $p^* = T_\phi p^*$.

To further establish that p^* is the unique fixed point of T_ϕ, we note that, similar to the way we derived (1.17), we can obtain from (1.16) the following inequality:

$$\|T_\phi(x) - T_\phi(y)\| \leq \delta.\|x - y\| + 2\delta.\|x - T_\phi x\|, \forall x, y \in X, \tag{1.19}$$

where $\delta = \frac{a+b}{1-b}$.

Let's assume that p^* and q^* are two fixed points of T. Then, employing (1.19) with $x = p^*$ and $y = q^*$, we arrive at the following contradiction:

$$0 < \|p^* - q^*\| \leq \delta\|p^* - q^*\| < \|p^* - q^*\|,$$

This contradiction implies that $Fix(T_\phi) = p^*$, and since $Fix(T) = Fix(T_\phi)$, we have established the result.

□

In the special case where $\psi(x) = 0$ for all $x \in X$, in Theorem 1.5, we deduce the result established in 1971, by Ćirić [13], Reich [18], Rus [19].

Corollary 1.4. [13] *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$, and if there exist $M + 2K < 1$ such that for all $x, y \in X$ satisfying*

$$\|Tx - Ty\| \leq M\|x - y\| + K\{\|x - Tx\| + \|y - Ty\|\}.$$

Then, T possesses a unique fixed point.

In the particular scenario where $\psi(x) = b$ holds for all $x \in X$, as indicated in Theorem 1.5, we obtain the following result.

Corollary 1.5. *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$, and if there exist $b \in [0, \infty)$ and $M + 2K < 1$, such that for all $x, y \in X$ satisfying*

$$\|b(x-y) + Tx - Ty\| \leq (b+1)M\|x-y\| + K\{\|x-Tx\| + \|y-Ty\|\}. \tag{1.20}$$

Then, T possesses a unique fixed point.

If we take $M = 0$ in the Corollary 1.5, we obtain Theorem 2.1 of [11].

Corollary 1.6. [11] *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be (b, K)-enriched Kannan contraction, that is an mapping satisfying:*

$$\|b(x-y) + Tx - Ty\| \leq K\{\|x-Tx\| + \|y-Ty\|\} \ \forall x, y \in X,$$

where, $b \in [0, \infty)$ and $0 \leq K < 1/2$. Then, T possesses a unique fixed point.

As a special case of Corollary 1.5, we deduce the main result (Theorem 2.3) of [7].

Corollary 1.7. [7] *Let $(X, \|\cdot\|)$ be a Banach space, and let $T : X \to X$. If there exists $b \in [0, \infty)$ and $M' + 2K < 1$ such that for all $x, y \in X$ satisfying*

$$\|b(x-y) + Tx - Ty\| \leq M'\|x-y\| + K\{\|x-Tx\| + \|y-Ty\|\}, \tag{1.21}$$

then T possesses a unique fixed point.

Proof. Take $M = \frac{M'}{b+1}$. Clearly, $M + 2K < 1$. Indeed, we know that $b \geq 0$, which implies that $b + 1 \geq 1$. Therefore, $\frac{1}{b+1} \leq 1$, so

$$\frac{M'}{b+1} \leq M'.$$

Adding $2K$ to both sides, we get

$$M + 2K = \frac{M'}{b+1} + 2K \leq M' + 2K.$$

Since $M' + 2K < 1$, we obtain $M + 2K < 1$.

Hence all the conditions of Corollary 1.5 are satisfied. So for $M = \frac{M'}{b+1}$, the contractive condition (1.20) reduces to (1.21). □

As a special case of Corollary 1.5, we deduce the main result (Theorem 2.0.3) of [4].

Corollary 1.8. *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be an (ψ, a)-MR-Kannan type contraction, that is an operator satisfying:*

$$\left\|\frac{x\psi(x) + Tx}{1+\psi(x)} - \frac{y\psi(y) + Ty}{1+\psi(y)}\right\| \leq K\left\{\left\|\frac{1}{\psi(x)+1}(x-Tx)\right\| + \left\|\frac{1}{\psi(y)+1}(y-Ty)\right\|\right\},$$

for all $x, y \in X$, where $0 \leq K < \frac{1}{2}$ and $\psi \in \mho$. Then, T has a unique fixed point.

Proof. If we set $M = 0$ in Theorem 1.5, the conclusion follows. □

In order to support our Theorem 1.5, we present an example of a mapping that satisfies the (M, K, ψ)-HR-Ćirić-Reich-Rus contraction condition but does not satisfy the Banach contraction, Kannan contraction, enriched Banach contraction, enriched Kannan contraction, Ćirić-Reich-Rus contraction, or enriched Ćirić-Reich-Rus contraction conditions.

Example 1.3. Let $X = \mathbb{R}$ be endowed with the usual norm, and let $T : \mathbb{R} \to \mathbb{R}$ be defined by $Tx = 3x$, for all $x \in \mathbb{R}$. Then, $T(0) = 0$ and

i) T does not satisfy the enriched Banach contraction. Indeed, if T would be an enriched Banach contraction, then by (Definition 2.1 of [8]), there would exist $b \in [0, \infty)$ and $\alpha \in [0, b+1)$ such that

$$(b+3)|x-y| \leq \alpha|x-y|, \quad \forall x, y \in \mathbb{R},$$

for which $x = 0$ and $y = 1$, leads to the contradiction $3 \leq (b+3) \leq \alpha < b+1$.

ii) T does not satisfy the enriched Kannan contraction. Indeed, if T would be an enriched Kannan contraction then, then by (Definition 2.1 of [11]), there would exist $b \in [0, \infty)$ and $\alpha \in [0, \frac{1}{2})$ such that

$$\frac{b+3}{2}|x-y| \leq \alpha\{|x| + |y|\}, \quad \forall x, y \in \mathbb{R},$$

for which $x = 0$ and $y = 1$, leads to the contradiction $\frac{3}{2} \leq \frac{b+3}{2} \leq \alpha < \frac{1}{2}$.

iii) T does not satisfy the Ćirić-Reich-Rus contraction (1.5). Indeed if T would be a Ćirić-Reich-Rus contraction then, by (1.5), there would exist $M, K \geq 0$ with $M + 2K < 1$ such that

$$3|x-y| \leq M|x-y| + 2K\{|x| + |y|\}, \quad \forall x, y \in \mathbb{R},$$

for which $x = 0$ and $y = 1$, leads to the contradiction $3 \leq M + 2K < 1$.

iv) T does not satisfy the (b, M, K)-enriched Ćirić-Reich-Rus contraction (1.4). Indeed if T would be a (b, M, K)-enriched Ćirić-Reich-Rus contraction then, by (1.4), there would exist $M, K \geq 0$ with $M + 2K < 1$ such that

$$(b+3)|x-y| \leq M|x-y| + 2K\{|x| + |y|\}, \quad \forall x, y \in \mathbb{R},$$

for which $x = 0$ and $y = 1$, leads to the contradiction $3 \leq (b+3) \leq M + 2K < 1$.

v) On the other hand, if we take $\psi(x) = -3$, for all $x \in X$, and for any $M, K \geq 0$, then T is a (M, K, ψ)-HR-Ćirić-Reich-Rus type contraction. Indeed, the contractive condition (1.6) holds for all $x, y \in \mathbb{R}$:

$$\begin{aligned}
&\left\| \frac{x\psi(x) + Tx}{1 + \psi(x)} - \frac{y\psi(y) + Ty}{1 + \psi(y)} \right\| = \left| \frac{x(-3) + 3x}{1 + (-3)} - \frac{y(-3) + 3y}{1 + (-3)} \right| \\
&= 0 \leq M\,|x-y| + K.\{|x| + |y|\} \\
&\leq M\,\|x-y\| + K.\left\{ \left\| \frac{1}{\psi(x)+1}(x - Tx) \right\| + \left\| \frac{1}{\psi(y)+1}(y - Ty) \right\| \right\}.
\end{aligned}$$

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Competing Interests

The authors declare that they have no competing interests.

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