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Asymptotic solutions of differential equations with singular impulses

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ABSTRACT. The paper considers impulsive systems with singularities. The main novelty is that beside the singularity of the differential equation, the impulsive equation is a singular one. The method of boundary functions is applied to obtain the main result. Examples with simulations confirming the theoretical results are given.

1. INTRODUCTION

The interest in singularly perturbed equations is due to the fact that they are mathematical models of many application problems related to diffusion processes, chemical kinetics [19], mathematical biology [13, 17], processes in physics, fluid dynamics [10], engineering [12, 15] and many others. Different types of problems with singular perturbations are discussed in many books [6, 14, 16, 21, 23]. The problems depend on small parameter such that solutions show a nonuniform behaviour over the time axis as the parameter tends to zero. Various asymptotic methods exist to approximate solutions of singularly perturbed problems.There exist methods which enable us to construct uniform approximations with any required accuracy. The research conducted in [21, 22], led to the development of the method of boundary functions. This method is also known as the boundary layer correction method [16]. One can apply the method for solving a singularly perturbed problem if in a part of its domain the condition of the well-known Tikhonov theorem is valid. Various asymptotic methods can be found in [16]. In the present paper, we shall imply the method of boundary functions for analysis of impulsive systems. Impulse effects exist in a wide diversity of evolutionary processes in which states undergo rapid changes [2]. In many systems, in addition to singular perturbation, there also impulse actions [7, 8, 18, 9], but they are not singular there.

Let us consider the following model of a singularly perturbed differential equation,

$$\varepsilon\dot{z} = f(z, y, t),$$
$$\dot{y} = g(z, y, t),$$

where ε is a small positive real number. In the literature, results based on this system are known as theorems of Tikhonov type [4, 11, 16, 20]. In [1, 3, 5], the authors considered the system with the singularity at impact moments. The system is with singularity at impact moments as well as with the small parameter multiplied by the derivative. As a result of these articles, there was formulated a singular problem, which helps researchers to consider degenerate systems without rattling to approximate the original models and reduce complexity. The articles [7, 8] and book [6] considered impulsive systems with small parameter involved only in the differential equations, but not in the impulsive equations

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of them. We insert a small parameter into the impulse equation [1]. So the singularity concept for discontinuous dynamics is significantly extended.

Akhmet and Çağ [1, 3, 5] for the first time in the literature considered equations, when impulses are also singular, beside the differential equation. This concerns the following problem

$$\varepsilon \dot{z} = F(z),$$
$$\varepsilon \Delta z|_{t=\theta_i} = I(z, \varepsilon) \tag{1.1}$$

with $z(0, \varepsilon) = z_0$, where $z \in \mathbb{R}^m$, $t \in [0, T]$, $F(z)$ is a continuously differentiable function on D and $I(z, \varepsilon)$ is a continuous function for $(z, \varepsilon) \in D \times [0, 1]$, D is the domain $D = \{0 \leq t \leq T, \parallel z \parallel < d\}$, ${\theta_i}_{i=1}^{p}, 0 < \theta_1 < \theta_2 < ... < \theta_p < T$. If $\varepsilon = 0$ in (1.1), then one has the form

$$0 = F(z), 0 = I(z, 0). \tag{1.2}$$

It is a degenerate system, since its order is less than the order of (1.1). Consider an isolated real root $z = \varphi$ of (1.2). Moreover, for the impulse function, the following condition

$$\lim_{(z,\varepsilon)\to(\varphi,0)} \frac{I(z, \varepsilon)}{\varepsilon} = 0 \ \ or \lim_{(z,\varepsilon)\to(\varphi,0)} \frac{I(z, \varepsilon)}{\varepsilon} = I_0 \neq 0 \tag{$*$}$$

was used, which prevents impulsive function to blow up as the parameter ε decays to zero. By virtue of these conditions, two cases were shown, a singularity with one layer and a singularity with multi-layers. The subinterval, where the fast change of the solution of singular perturbed problem from the initial value to values close to the solution of unperturbed problem take place, is called a boundary layer. In the case of singularity with a single layer, the characteristic feature is the presence of a boundary layer in the neighborhood of the initial point, while in the case of singularity with multi-layers , the boundary layer is present not only in the neighborhood of the initial point, but also in the neighborhood of each discontinuity moment.

The main novelty in [1] is the extension of Tikhonov's theorem in such a way that system (1.3) has a small parameter in the impulse functions and discontinuity moments are different for each dependent variable. The peculiarity in the impulsive part of the system can be considered using perturbation theory methods. In our present study, we apply the ideas of the article [1].

More precisely, our discussion is related on the following system [1],

$$\varepsilon \dot{z} = f(z, y, t), \qquad \dot{y} = g(z, y, t), \tag{1.3a}$$

$$\varepsilon \Delta z|_{t=\theta_i} = I(z, y, \varepsilon), \ \ \Delta y|_{t=\eta_j} = J(z, y), \tag{1.3b}$$

where z, f and I are m-dimensional vector valued functions, y, g and J are n-dimensional vector valued functions, ${\theta_i}_{i=1}^{p}, 0 < \theta_1 < \theta_2 < ... < \theta_p < T$, and ${\eta_j}_{j=1}^{k}$, are distinct discontinuity moments in $(0, T)$.

The impulsive system consists of differential equations (1.3a) and impulse equations (1.3b). In [1], the authors studied the behavior of solutions of the singularly perturbed system (1.3) and considered two cases of singularity with single and multi-layer layers, which depend on the condition ($*$). By introducing a small parameter into the impulse equation, the notion of singularity for discontinuous dynamics was significantly expanded and shown that the transition to the limit for $y(t, \varepsilon)$ is uniform in the entire interval of $0 \leq t \leq T$, while the transition to the limit for $z(t, \varepsilon)$ isn't uniform in the entire interval of $0 \leq t \leq T$, but only in the subintervals $\delta \leq t \leq {\theta_i}_{i=1}^{p}$ for $\delta > 0$, outside the boundary layers.

In what follows, instead of "we construct an asymptotic series" a shorter expression is used: "we construct asymptotics". It will mean the algorithm which allows us to obtain the terms of the asymptotic series up to arbitrary natural n. We build a uniform asymptotic approximation of the solution which is valid in the entire interval $0 \le t \le T$ using the method of boundary functions. However, the theorems of article [1] do not give the order of accuracy of the asymptotic approximation $\tilde{y}(t)$ for the solution $y(t,\varepsilon)$ in $0 \le t \le T$ and that of $\tilde{z}(t)$ for $z(t,\varepsilon)$ outside the boundary layer. Our goal is to construct an approximation with higher accuracy and the complete asymptotic expansion for solutions of systems with singularly perturbed impulses.

2. Formalities of approximation

In this part of the paper, we demonstrate the algorithm for the construction of an asymptotic approximation of the solution using the method of boundary functions. Differential equations with singular perturbed impulses are considered.

Let us consider the following system

$$\begin{aligned} \varepsilon z' &= F(z,y,\varepsilon), \quad & \varepsilon \Delta z|_{t=\theta_i} &= I(z,y,\varepsilon), \\ y' &= f(z,y), & \Delta y|_{t=\theta_i} &= J(z,y) \end{aligned} \tag{2.4}$$

with initial condition

$$z(0,\varepsilon) = z^0, \quad y(0,\varepsilon) = y^0, \tag{2.5}$$

where ε is a small positive real number, z, F and I are m-dimensional functions, y, f and J are n-dimensional functions, z^0 and y^0 are assumed to be independent of ε, $\theta_i{}_{i=1}^{p}, 0 < \theta_1 < \theta_2 < \ldots < \theta_p < T$, are distinct discontinuity moments in $(0,T)$. We set $\Delta x|_{t=\theta_i} = x(\theta_i+) - x(\theta_i)$, assuming that the limits $x(\theta_i+) = \lim_{t\to\theta_i+} x(t)$ exist and $x(\theta_i-) = x(\theta_i)$.

Assume that $\varepsilon = 0$ in (2.4). Then the system reduces to the following equations

$$\begin{aligned} 0 &= F(\tilde{z},\tilde{y},0), \quad & 0 &= I(\tilde{z},\tilde{y},0), \\ \tilde{y}' &= f(\tilde{z},\tilde{y}), & \Delta \tilde{y}|_{t=\theta_i} &= J(\tilde{z},\tilde{y}), \end{aligned} \tag{2.6}$$

which we call a degenerate system due to the fact that the order is less than the order of system (2.4). Therefore, for system (2.6) the number of initial conditions to be set to less than the number of initial conditions for (2.4). We naturally insert the initial condition for y, as

$$\tilde{y}(0) = y^0, \tag{2.7}$$

and drop the initial condition for z.

To solve system (2.6), it is necessary to find $\tilde{z}$ from $0 = F(\tilde{z},\tilde{y},0)$ and $0 = I(\tilde{z},\tilde{y},0)$. Then choose one of the root $\tilde{z} = \varphi(\tilde{y}(t),t)$ such that $0 = F(\varphi(\tilde{y}(t),t),\tilde{y},0)$ and $0 = I(\varphi(\tilde{y}(t),t),\tilde{y},0)$ and substitute into (2.6) with the initial value (2.7) to obtain

$$\begin{aligned} &\tilde{y}' = f(\varphi(\tilde{y}(t),t),\tilde{y}), \qquad \Delta \tilde{y}|_{t=\theta_i} = J(\varphi(\tilde{y}(t),t),\tilde{y}), \\ &\tilde{y}(0) = y^0. \end{aligned} \tag{2.8}$$

The following conditions will be needed.

(C1) The functions $f(z,y)$ and $J(z,y)$ are infinitely differentiable in domain $H = \{(y,t) \in \overline{N} = \{0 \le t \le T, \|y\| \le c\}, \|z\| \le d\}$, $F(z,y,\varepsilon)$ and $I(z,y,\varepsilon)$ are infinitely differentiable in $H \times [0,1]$;

(C2) the algebraic equations $F(z,y,0) = 0$ and $I(z,y,0) = 0$ have a root $z = \varphi(y,t)$ in domain $\overline{N}$ such that,
1) $\varphi(y,t)$ is a continuous function in $\overline{N}$,
2) $(\varphi(y,t),y,t) \in H, (y,t) \in \overline{N}$,

3) the root $\varphi(y,t)$ is isolated in $\overline{N}$, i.e., $\exists\epsilon > 0 : F(z,y,t) \neq 0$ and $I(z,y,\varepsilon) \neq 0$, $0 <\| z-\varphi(y,t) \|< \epsilon$, $(y,t) \in \overline{N}$;

(C3) the system (2.8) has a unique solution $\tilde{y}(t)$ on $0 \le t \le T$, and $(\tilde{y}(t),t) \in \overline{N}$ for $0 \le t \le T$. Moreover, the functions $f(\varphi(\tilde{y},t),\tilde{y})$ and $J(\varphi(\tilde{y},t),\tilde{y})$ are Lipschitz with respect to $y \in \overline{N}$.

Now, setting $x = z - \varphi$ and $t = \tau\varepsilon$, we obtain the system

$$\frac{dx}{d\tau} = F(x+\varphi(y,t),y,0), \quad \tau \ge 0, \tag{2.9}$$

where y and t are considered as parameters, $x = 0$ is an isolated stationary point of (2.9) for $(y,t) \in \overline{N}$.

(C4) There exists a positive definite function $V(x,y,\tau)$ whose derivative with respect to τ along the system (2.9) is negatively defined in the domain H.

Consider adjoint system

$$\frac{d\omega}{d\tau} = F(\omega,y^0,0), \quad \tau \ge 0 \tag{2.10}$$

with initial condition

$$\omega(0) = z^0. \tag{2.11}$$

Since z^0 may be, in general, far from stationary point $\varphi(y^0,0)$, then the solution $\omega(\tau)$ of equations (2.10) and (2.11) needs not tend to $\varphi(y^0,0)$ as $\tau \to \infty$.

Assume also that the solution $\omega(\tau)$ of equations (2.10) and (2.11) satisfies the condition

(C5) $\omega(\tau) \to \varphi(y^0,0)$ as $\tau \to \infty$, and $(\omega(\tau),y^0,0)$ belong H for $\tau \ge 0$.

In this case, z^0 is said to belong to the basin of attraction of the stationary point $\omega = \varphi(y^0,0)$. By virtue of the asymptotic stability of this point all points near it will belong to its basin of attraction.

To prevent the impulse function to blow up when the parameter ε decays to zero, the following assumption is required,

(C6) $\displaystyle\lim_{(z,y,\varepsilon)\to(\varphi,y^0,0)} \frac{I(z,y,\varepsilon)}{\varepsilon} = 0.$

We will seek for a asymptotic representation of the solution $z(t,\varepsilon), y(t,\varepsilon)$ of problem (2.4)-(2.5) in the form

$$\begin{aligned} z(t,\varepsilon) &= \tilde{z}(t,\varepsilon) + \omega^{(i)}(\tau_i,\varepsilon), \quad \tau_i = \frac{t-\theta_i}{\varepsilon}, \quad i = 1,2,...,p, \\ y(t,\varepsilon) &= \tilde{y}(t,\varepsilon) + \varepsilon\nu^{(i)}(\tau_i,\varepsilon), \quad \theta_i < t \le \theta_{i+1}, \quad \theta_0 = 0, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} \tilde{z}(t,\varepsilon) &= \sum_{k=0}^{\infty} \varepsilon^k \tilde{z}_k(t), \qquad \tilde{y}(t,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \tilde{y}_k(t), \\ \omega^{(i)}(\tau_i,\varepsilon) &= \sum_{k=0}^{\infty} \varepsilon^k \omega_k^{(i)}(\tau_i), \quad \nu^{(i)}(\tau_i,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \nu_k^{(i)}(\tau_i). \end{aligned} \tag{2.13}$$

The coefficients $\omega_k^{(i)}(\tau_i)$ and $\nu_k^{(i)}(\tau_i)$ in the expansions (2.13) are called boundary functions and on them the additional condition is imposed,

$$\omega_k^{(i)}(\infty) = 0, \ \nu_k^{(i)}(\infty) = 0, i = 1,2,...,p. \tag{2.14}$$

Substituting the series (2.12) into system (2.4), we obtain the equalities

$$\begin{aligned}
&\varepsilon(\tilde{z}'(t,\varepsilon)+\frac{1}{\varepsilon}\dot{\omega}^{(i)}(\tau_i,\varepsilon))=F(\tilde{z}(t,\varepsilon),\tilde{y}(t,\varepsilon),\varepsilon)+\\
&[F(\tilde{z}(t,\varepsilon)+\omega^{(i)}(\tau_i,\varepsilon),\tilde{y}(t,\varepsilon)+\varepsilon\nu^{(i)}(\tau_i,\varepsilon),\varepsilon)-F(\tilde{z}(t,\varepsilon),\tilde{y}(t,\varepsilon),\varepsilon)],\\
&\tilde{y}'(t,\varepsilon)+\dot{\nu}^{(i)}(\tau_i,\varepsilon)=f(\tilde{z}(t,\varepsilon),\tilde{y}(t,\varepsilon))+\\
&[f(\tilde{z}(t,\varepsilon)+\omega^{(i)}(\tau_i,\varepsilon),\tilde{y}(t,\varepsilon)+\varepsilon\nu^{(i)}(\tau_i,\varepsilon))-f(\tilde{z}(t,\varepsilon),\tilde{y}(t,\varepsilon))].
\end{aligned}$$

Equalizing expressions for t and τ_i in the last equations, we get two systems

$$\begin{aligned}
\varepsilon\tilde{z}'(t,\varepsilon)&=F(\tilde{z}(t,\varepsilon),\tilde{y}(t,\varepsilon),\varepsilon),\\
\tilde{y}'(t,\varepsilon)&=f(\tilde{z}(t,\varepsilon),\tilde{y}(t,\varepsilon)),
\end{aligned}\tag{2.15}$$

and

$$\begin{aligned}
\dot{\omega}^{(i)}(\tau_i,\varepsilon)&=F(\tilde{z}(t,\varepsilon)+\omega^{(i)}(\tau_i,\varepsilon),\tilde{y}(t,\varepsilon)+\varepsilon\nu^{(i)}(\tau_i,\varepsilon),\varepsilon)-F(\tilde{z}(t,\varepsilon),\tilde{y}(t,\varepsilon),\varepsilon),\\
\dot{\nu}^{(i)}(\tau_i,\varepsilon)&=f(\tilde{z}(t,\varepsilon)+\omega^{(i)}(\tau_i,\varepsilon),\tilde{y}(t,\varepsilon)+\varepsilon\nu^{(i)}(\tau_i,\varepsilon))-f(\tilde{z}(t,\varepsilon),\tilde{y}(t,\varepsilon)).
\end{aligned}\tag{2.16}$$

Now we represent F, f, I, J in the form of power series in ε,

$$\begin{aligned}
&F(\tilde{z}(t,\varepsilon),\tilde{y}(t,\varepsilon),\varepsilon)=F(\tilde{z}_0(t)+\varepsilon\tilde{z}_1(t)+\ldots,\tilde{y}_0(t)+\varepsilon\tilde{y}_1(t)+\ldots,\varepsilon)=\\
&F(\tilde{z}_0(t),\tilde{y}_0(t),0)+\varepsilon[F_z(t)\tilde{z}_1(t)+F_y(t)\tilde{y}_1(t)+F_\varepsilon(t)]+\ldots+\\
&\varepsilon^k[F_z(t)\tilde{z}_k(t)+F_y(t)\tilde{y}_k(t)+F_k(t)]+\ldots=\tilde{F}_0(t)+\varepsilon\tilde{F}_1(t)+\ldots\varepsilon^k\tilde{F}_k(t)+\ldots,
\end{aligned}$$

where matrices $F_z(t)$,$F_y(t)$ and $F_\varepsilon(t)$ are calculated at $(\tilde{z}_0(t),\tilde{y}_0(t),0)$ and $F_k(t)$ are expressed recursively through $\tilde{z}_j(t)$ and $\tilde{y}_j(t)$ with $j<k$,

$$\begin{aligned}
&F(\tilde{z}(t,\varepsilon)+\omega^{(i)}(\tau_i,\varepsilon),\tilde{y}(t,\varepsilon)+\varepsilon\nu^{(i)}(\tau_i,\varepsilon),\varepsilon)-F(\tilde{z}(t,\varepsilon),\tilde{y}(t,\varepsilon),\varepsilon)=\\
&F(\tilde{z}_0(\theta_i+\varepsilon\tau_i)+\varepsilon\tilde{z}_1(\theta_i+\varepsilon\tau_i)+\ldots+\omega_0^{(i)}(\tau_i)+\varepsilon\omega_1^{(i)}(\tau_i)+\ldots,\\
&\tilde{y}_0(\theta_i+\varepsilon\tau_i)+\varepsilon\tilde{y}_1(\theta_i+\varepsilon\tau_i)+\ldots+\varepsilon\nu_0^{(i)}(\tau_i)+\varepsilon^2\nu_1^{(i)}(\tau_i)+\ldots,\varepsilon)-\\
&F(\tilde{z}_0(\theta_i+\varepsilon\tau_i)+\varepsilon\tilde{z}_1(\theta_i+\varepsilon\tau_i)+\ldots,\tilde{y}_0(\theta_i+\varepsilon\tau_i)+\varepsilon\tilde{y}_1(\theta_i+\varepsilon\tau_i)+\ldots,\varepsilon)=\\
&F(\tilde{z}_0(\theta_i)+\omega_0^{(i)}(\tau_i),\tilde{y}_0(\theta_i),0)-F(\tilde{z}_0(\theta_i),\tilde{y}_0(\theta_i),0)+\\
&\varepsilon[F_z(\tau_i)\omega_1^{(i)}(\tau_i)+F_y(\tau_i)\nu_0^{(i)}(\tau_i)+G_1(\tau_i)]+\ldots+\\
&\varepsilon^k[F_z(\tau_i)\omega_k^{(i)}(\tau_i)+F_y(\tau_i)\nu_{k-1}^{(i)}(\tau_i)+G_k(\tau_i)]+\ldots=\\
&\Pi_0F(\tau_i)+\varepsilon\Pi_1F(\tau_i)+\ldots+\varepsilon^k\Pi_kF(\tau_i)+\ldots,
\end{aligned}$$

where the elements $F_z(\tau_i)$ and $F_y(\tau_i)$ are calculated at $(\tilde{z}_0(\theta_i)+\omega_0^{(i)}(\tau_i),\tilde{y}_0(\theta_i),0), i=1,2,...,p$, the elements $F_z(\theta_i)$ and $F_y(\theta_i)$ are calculated at $(\tilde{z}_0(\theta_i),\tilde{y}_0(\theta_i),0)$ and $G_k(\tau_i), i=1,2,...,p$, are expressed recursively through $\omega_j^{(i)}(\tau_i)$ and $\nu_{j-1}^{(i)}(\tau_i)$ with $j<k$. Similarly one can get that

$$\begin{aligned}
&I(z(\theta_i,\varepsilon),y(\theta_i,\varepsilon),\varepsilon)=I(z(\theta_i-,\varepsilon),y(\theta_i-,\varepsilon),\varepsilon)=\\
&I(\tilde{z}(\theta_i,\varepsilon)+\omega^{(i-1)}(\frac{\theta_i-\theta_{i-1}}{\varepsilon},\varepsilon),\tilde{y}(\theta_i,\varepsilon)+\varepsilon\nu^{(i-1)}(\frac{\theta_i-\theta_{i-1}}{\varepsilon},\varepsilon),\varepsilon)=\\
&I(\tilde{z}(\theta_i,\varepsilon),\tilde{y}(\theta_i,\varepsilon),\varepsilon)=I(\tilde{z}_0(\theta_i),\tilde{y}_0(\theta_i),0)+\\
&\varepsilon[I_z(\theta_i)\tilde{z}_1(\theta_i)+I_y(\theta_i)\tilde{y}_1(\theta_i)+I_\varepsilon(\theta_i)]+\ldots+\\
&\varepsilon^k[I_z(\theta_i)\tilde{z}_k(\theta_i)+I_y(\theta_i)\tilde{y}_k(\theta_i)+I_k(\theta_i)]+\ldots=\\
&\tilde{I}_0(\theta_i)+\varepsilon\tilde{I}_1(\theta_i)+\ldots+\varepsilon^k\tilde{I}_k(\theta_i)+\ldots,
\end{aligned}\tag{2.17}$$

where the elements $I_z(\theta_i)$, $I_y(\theta_i)$ and $I_\varepsilon(\theta_i)$ calculated at the points $(\tilde{z}_0(\theta_i), \tilde{y}_0(\theta_i), 0), i = 1, 2, ..., p$, and $I_k(\theta_i)$ are expressed recursively through $\tilde{z}_j(\theta_i)$ and $\tilde{y}_j(\theta_i)$ with $j < k$. Similar expansions hold for $J(z(\theta_i, \varepsilon), y(\theta_i, \varepsilon))$.

Now the problems (2.4), (2.5) and (2.15) and (2.16) take the form

$$\varepsilon(\tilde{z}_0'(t) + \varepsilon\tilde{z}_1'(t) + \ldots + \varepsilon^k\tilde{z}_k'(t) + \ldots) = \tilde{F}_0(t) + \varepsilon\tilde{F}_1(t) + \ldots \varepsilon^k\tilde{F}_k(t) + \ldots,$$
$$\tilde{y}_0'(t) + \varepsilon\tilde{y}_1'(t) + \ldots + \varepsilon^k\tilde{y}_k'(t) + \ldots = \tilde{f}_0(t) + \varepsilon\tilde{f}_1(t) + \ldots \varepsilon^k\tilde{f}_k(t) + \ldots,$$
$$\dot{\omega}_0^{(i)}(\tau_i) + \varepsilon\dot{\omega}_1^{(i)}(\tau_i) + \ldots + \varepsilon^k\dot{\omega}_k^{(i)}(\tau_i) + \ldots = \Pi_0F(\tau_i) + \varepsilon\Pi_1F(\tau_i) + \ldots + \varepsilon^k\Pi_kF(\tau_i) + \ldots,$$
$$\dot{\nu}_0^{(i)}(\tau_i) + \varepsilon\dot{\nu}_1^{(i)}(\tau_i) + \ldots + \varepsilon^k\dot{\nu}_k^{(i)}(\tau_i) + \ldots = \Pi_0f(\tau_i) + \varepsilon\Pi_1f(\tau_i) + \ldots + \varepsilon^k\Pi_kf(\tau_i) + \ldots,$$
$$\varepsilon(\sum_{k=0}^{\infty}\varepsilon^k\Delta\tilde{z}_k|_{t=\theta_i} + \sum_{k=0}^{\infty}\varepsilon^k\omega_k^{(i)}(0)) = \tilde{I}_0(\theta_i) + \varepsilon\tilde{I}_1(\theta_i) + \ldots + \varepsilon^k\tilde{I}_k(\theta_i) + \ldots,$$
$$\sum_{k=0}^{\infty}\varepsilon^k\Delta\tilde{y}_k|_{t=\theta_i} + \varepsilon\sum_{k=0}^{\infty}\varepsilon^k\nu_k^{(i)}(0) = \tilde{J}_0(\theta_i) + \varepsilon\tilde{J}_1(\theta_i) + \ldots + \varepsilon^k\tilde{J}_k(\theta_i) + \ldots.$$

Substituting the expansion (2.13) into conditions (2.5), we obtain

$$z(0, \varepsilon) = \sum_{k=0}^{\infty}\varepsilon^k\tilde{z}_k(0) + \sum_{k=0}^{\infty}\varepsilon^k\omega_k^{(0)}(0) = z^0,$$

and

$$y(0, \varepsilon) = \sum_{k=0}^{\infty}\varepsilon^k\tilde{y}_k(0) + \varepsilon\sum_{k=0}^{\infty}\varepsilon^k\nu_k^{(0)}(0) = y^0.$$

In fact, the above expansions up to order n. We equate the coefficients according to the powers of ε. To determine the approximation of order zero $\tilde{z}_0(t), \tilde{y}_0(t), \omega_0^{(i)}(\tau_i)$ and $\nu_0^{(i)}(\tau_i), i = 1, 2, ..., p$, we obtain the systems

$$\begin{aligned} 0 &= F(\tilde{z}_0(t), \tilde{y}_0(t), 0), \\ \tilde{y}_0'(t) &= f(\tilde{z}_0(t), \tilde{y}_0(t)), \end{aligned} \tag{2.18}$$

$$\begin{aligned} \dot{\omega}_0^{(i)}(\tau_i) &= F(\tilde{z}_0(\theta_i) + \omega_0^{(i)}(\tau_i), \tilde{y}_0(\theta_i), 0) - F(\tilde{z}_0(\theta_i), \tilde{y}_0(\theta_i), 0) = \Pi_0F(\tau_i), \\ \dot{\nu}_0^{(i)}(\tau_i) &= f(\tilde{z}_0(\theta_i) + \omega_0^{(i)}(\tau_i), \tilde{y}_0(\theta_i)) - f(\tilde{z}_0(\theta_i), \tilde{y}_0(\theta_i)) = \Pi_0f(\tau_i), \end{aligned} \tag{2.19}$$

$$0 = \frac{I(\tilde{z}_0(\theta_i), \tilde{y}_0(\theta_i), 0)}{\varepsilon}, \tag{2.20}$$

$$\begin{aligned} \Delta\tilde{z}_0|_{t=\theta_i} + \omega_0^{(i)}(0) &= I_z(\theta_i)\tilde{z}_1(\theta_i) + I_y(\theta_i)\tilde{y}_1(\theta_i) + I_\varepsilon(\theta_i) = \tilde{I}_1(\tilde{z}_0(\theta_i), \tilde{y}_0(\theta_i), 0), \\ \Delta\tilde{y}_0|_{t=\theta_i} &= J(\tilde{z}_0(\theta_i), \tilde{y}_0(\theta_i)) = J_0(\tilde{z}_0(\theta_i), \tilde{y}_0(\theta_i)), \\ \tilde{z}_0(0) + \omega_0^{(0)}(0) &= z^0, \quad \tilde{y}_0(0) = y^0. \end{aligned} \tag{2.21}$$

To define the coefficients of $\varepsilon^k (k \geq 1)$, we use the equations

$$\begin{aligned} \tilde{z}_{k-1}'(t) &= F_z(t)\tilde{z}_k(t) + F_y(t)\tilde{y}_k(t) + F_k(t), \\ \tilde{y}_k'(t) &= f_z(t)\tilde{z}_k(t) + f_y(t)\tilde{y}_k(t) + f_k(t), \end{aligned} \tag{2.22}$$

$$\begin{aligned} \dot{\omega}_k^{(i)}(\tau_i) &= F_z(\tau_i)\omega_k^{(i)}(\tau_i) + F_y(\tau_i)\nu_{k-1}^{(i)}(\tau_i) + G_k(\tau_i) = \Pi_kF(\tau_i), \\ \dot{\nu}_k^{(i)}(\tau_i) &= f_z(\tau_i)\omega_k^{(i)}(\tau_i) + f_y(\tau_i)\nu_{k-1}^{(i)}(\tau_i) + Q_k(\tau_i) = \Pi_kf(\tau_i), \end{aligned}$$

$$(2.23)\qquad \begin{aligned} &\Delta\tilde{z}_k|_{t=\theta_i} + \omega_k^{(i)}(0) = I_z(\theta_i)\tilde{z}_{k+1}(\theta_i) + I_y(\theta_i)\tilde{y}_{k+1}(\theta_i) + I_{k+1}(\theta_i),\\ &\Delta\tilde{y}_k|_{t=\theta_i} + \nu_{k-1}^{(i)}(0) = J_z(\theta_i)\tilde{z}_k(\theta_i) + J_y(\theta_i)\tilde{y}_k(\theta_i) + J_k(\theta_i),\\ &\tilde{z}_k(0) + \omega_k^{(0)}(0) = 0, \ \ \tilde{y}_k(0) + \nu_{k-1}^{(0)}(0) = 0. \end{aligned}$$

Consider the interval $t \in [0, \theta_1]$. To find the coefficients of ε^0 for the approximation $\tilde{z}_0(t)$ and $\tilde{y}_0(t)$ we have the system

$$\begin{aligned} &0 = F(\tilde{z}_0(t), \tilde{y}_0(t), 0),\\ &\tilde{y}_0'(t) = f(\tilde{z}_0(t), \tilde{y}_0(t)), \ \tilde{y}_0(0) = y^0. \end{aligned}$$

In view of conditions $(C2)$ and $(C3)$ we choose $\tilde{z}_0(t) = \tilde{z}(t), \tilde{y}_0(t) = \tilde{y}(t)$.

Due to the first equation of (2.18), equation (2.19) takes the form

$$\dot{\omega}_0^{(0)}(\tau_0) = F(\tilde{z}_0(\theta_0) + \omega_0^{(0)}(\tau_0), \tilde{y}_0(\theta_0), 0).$$

From the last equation and initial condition

$$\omega_0^{(0)}(0) = z^0 - \tilde{z}_0(0)$$

one can find $\omega_0^{(0)}(\tau_0)$. In view of condition $(C5)$, $\omega_0^{(0)}(\tau_0)$ possesses the exponential estimate,

$$(2.24)\qquad \|\omega_0^{(0)}(\tau_0)\| \le c\exp(-\kappa\tau_0),$$

where c and κ are positive numbers.

It remains to solve the equation

$$\dot{\nu}_0^{(0)}(\tau_0) = f(\tilde{z}_0(\theta_0) + \omega_0^{(0)}(\tau_0), \tilde{y}_0(\theta_0)) - f(\tilde{z}_0(\theta_0), \tilde{y}_0(\theta_0)) = \Pi_0 f(\tau_0).$$

Taking into account condition (2.14), we find initial condition

$$\nu_0^{(0)}(0) = \int_\infty^0 \Pi_0 f(s)ds,$$

and obtain that

$$\nu_0^{(0)}(\tau_0) = \int_\infty^{\tau_0} \Pi_0 f(s)ds.$$

Since $\|\Pi_0 f(\tau_0)\| \le c\exp(-\kappa\tau_0)$ then it is true that

$$\|\nu_0^{(0)}(\tau_0)\| \le c\exp(-\kappa\tau_0).$$

To determine the coefficients of ε^k for the approximation $\tilde{z}_k(t)$ and $\tilde{y}_k(t)$ we apply the systems

$$\begin{aligned} &\tilde{z}_{k-1}'(t) = F_z(t)\tilde{z}_k(t) + F_y(t)\tilde{y}_k(t) + F_k(t),\\ &\tilde{y}_k'(t) = f_z(t)\tilde{z}_k(t) + f_y(t)\tilde{y}_k(t) + f_k(t), \ \ \tilde{y}_k(0) + \nu_{k-1}^{(0)}(0) = 0. \end{aligned}$$

To find $\omega_k^{(0)}(\tau_0)$ it is needed to solve the following system

$$\begin{aligned} &\dot{\omega}_k^{(0)}(\tau_0) = F_z(\tau_0)\omega_k^{(0)}(\tau_0) + F_y(\tau_0)\nu_{k-1}^{(0)}(\tau_0) + G_k(\tau_0),\\ &\omega_k^{(0)}(0) = -\tilde{z}_k(0). \end{aligned}$$

It remains to solve the equations

$$\dot{\nu}_k^{(0)}(\tau_0) = f_z(\tau_0)\omega_k^{(0)}(\tau_0) + f_y(\tau_0)\nu_{k-1}^{(0)}(\tau_0) + Q_k(\tau_0) = \Pi_k f(\tau_0).$$

Using assumption (2.14), one can find initial condition

$$\nu_k^{(0)}(0) = \int_\infty^0 \Pi_k f(s)ds,$$

and equalities

$$\nu_k^{(0)}(\tau_0) = \int_{\infty}^{\tau_0} \Pi_k f(s)ds.$$

Functions $\Pi_k F(\tau_0)$ and $\Pi_k f(\tau_0)$ admit the exponential estimates of the type (2.24). Therefore, the following inequalities are hold,

$$\|\omega_k^{(0)}(\tau_0)\| \le c\exp(-\kappa\tau_0),$$
$$\|\nu_k^{(0)}(\tau_0)\| \le c\exp(-\kappa\tau_0).$$

Now consider the next interval $t \in (\theta_i, \theta_{i+1}], i = 1, 2, ..., p$. To define the coefficients of ε^0 for the approximation $\tilde{z}_0(t)$ and $\tilde{y}_0(t)$ we utilize the systems

$$\begin{aligned} 0 =& F(\tilde{z}_0(t), \tilde{y}_0(t), 0), \quad 0 = I(\tilde{z}_0(\theta_i), \tilde{y}_0(\theta_i), 0),\\ \tilde{y}_0'(t) =& f(\tilde{z}_0(t), \tilde{y}_0(t)), \qquad \Delta\tilde{y}_0|_{t=\theta_i} = J(\tilde{z}_0(\theta_i), \tilde{y}_0(\theta_i)). \end{aligned}$$

By conditions $(C2)$ and $(C3)$, we have to choose $\tilde{z}_0(t) = \tilde{z}(t), \tilde{y}_0(t) = \tilde{y}(t)$.

According to the first equation of (2.18), the equation (2.19) can be written as

$$\dot{\omega}_0^{(i)}(\tau_i) = F(\tilde{z}_0(\theta_i) + \omega_0^{(i)}(\tau_i), \tilde{y}_0(\theta_i), 0), i = 1, 2, ..., p.$$

The last equation and initial condition

$$\omega_0^{(i)}(0) = I_z(\theta_i)\tilde{z}_1(\theta_i) + I_y(\theta_i)\tilde{y}_1(\theta_i) + I_\varepsilon(\theta_i) - \Delta\tilde{z}_0|_{t=\theta_i}, i = 1, 2, ..., p,$$

imply to find $\omega_0^{(i)}(\tau_i)$, where $\omega_0^{(i)}(0)$ may be modified as below. By differentiate both sides of the first equations of (2.18) and (2.20) we obtain that

$$\begin{aligned} \frac{\partial F}{\partial \tilde{z}_0} d\tilde{z}_0 + \frac{\partial F}{\partial \tilde{y}_0} d\tilde{y}_0 = 0 \Rightarrow \frac{\partial F}{\partial \tilde{z}_0} &= -\frac{\partial F}{\partial \tilde{y}_0}\frac{d\tilde{y}_0}{d\tilde{z}_0},\\ \frac{\partial I}{\partial \tilde{z}_0} d\tilde{z}_0 + \frac{\partial I}{\partial \tilde{y}_0} d\tilde{y}_0 = 0 \Rightarrow \frac{\partial I}{\partial \tilde{z}_0} &= -\frac{\partial I}{\partial \tilde{y}_0}\frac{d\tilde{y}_0}{d\tilde{z}_0}. \end{aligned} \tag{2.25}$$

The substitution of the first equation of (2.25) into (2.22) gives that

$$\tilde{z}_0'(\theta_i) = -\frac{\partial F}{\partial \tilde{y}_0}\frac{d\tilde{y}_0}{d\tilde{z}_0}\tilde{z}_1(\theta_i) + F_y(\theta_i)\tilde{y}_1(\theta_i) + F_1(\theta_i).$$

Hence,

$$F_y(\theta_i)[\tilde{y}_1(\theta_i) - \frac{d\tilde{y}_0}{d\tilde{z}_0}\tilde{z}_1(\theta_i)] = \tilde{z}_0'(\theta_i) - F_1(\theta_i).$$

The last equality implies that

$$\tilde{y}_1(\theta_i) - \frac{d\tilde{y}_0}{d\tilde{z}_0}\tilde{z}_1(\theta_i) = \frac{1}{F_y(\theta_i)}(\tilde{z}_0'(\theta_i) - F_1(\theta_i)). \tag{2.26}$$

Substituting the second equation of (2.25) into (2.23), we obtain

$$\omega_0^{(i)}(0) = I_y(\theta_i)[\tilde{y}_1(\theta_i) - \frac{d\tilde{y}_0}{d\tilde{z}_0}\tilde{z}_1(\theta_i)] + I_\varepsilon(\theta_i) - \Delta\tilde{z}_0|_{t=\theta_i}, i = 1, 2, ..., p.$$

Replace the square bracket with equation (2.26) to get

$$\omega_0^{(i)}(0) = \frac{I_y(\theta_i)}{F_y(\theta_i)}(\tilde{z}_0'(\theta_i) - F_1(\theta_i)) + I_\varepsilon(\theta_i) - \Delta\tilde{z}_0|_{t=\theta_i}, i = 1, 2, ..., p.$$

In view of condition $(C5)$, $\omega_0^{(i)}(\tau_i)$ possesses the exponential estimate:

$$\|\omega_0^{(i)}(\tau_i)\| \le c\exp(-\kappa\tau_i), i = 1, 2, ..., p, \tag{2.27}$$

where c and κ are positive numbers, which are various in different inequalities.

It remains to solve the equations

$$\dot{\nu}_0^{(i)}(\tau_i) = f(\tilde{z}_0(\theta_i) + \omega_0^{(i)}(\tau_i), \tilde{y}_0(\theta_i)) - f(\tilde{z}_0(\theta_i), \tilde{y}_0(\theta_i)) = \Pi_0 f(\tau_i), i = 1, 2, ..., p.$$

By virtue of condition (2.14), we find initial condition

$$\nu_0^{(i)}(0) = \int_\infty^0 \Pi_0 f(s) ds.$$

Therefore, one can obtain

$$\nu_0^{(i)}(\tau_i) = \int_\infty^{\tau_i} \Pi_0 f(s) ds.$$

As $\|\Pi_0 f(\tau_i)\| \le c \exp(-\kappa\tau_i)$, it is true that

$$\|\nu_0^{(i)}(\tau_i)\| \le c \exp(-\kappa\tau_i), i = 1, 2, ..., p.$$

To determine the coefficients of ε^k for the approximation $\tilde{z}_k(t)$ and $\tilde{y}_k(t)$ we have the systems

$$\begin{aligned} \tilde{z}'_{k-1}(t) &= F_z(t)\tilde{z}_k(t) + F_y(t)\tilde{y}_k(t) + F_k(t), \\ \tilde{y}'_k(t) &= f_z(t)\tilde{z}_k(t) + f_y(t)\tilde{y}_k(t) + f_k(t), \\ \Delta\tilde{y}_k|_{t=\theta_i} + \nu_{k-1}^{(i)}(0) &= J_z(\theta_i)\tilde{z}_k(\theta_i) + J_y(\theta_i)\tilde{y}_k(\theta_i) + J_k(\theta_i). \end{aligned}$$

To find $\omega_k^{(i)}(\tau_i), i = 1, 2, ..., p$, it is needed to solve the system

$$\begin{aligned} \dot{\omega}_k^{(i)}(\tau_i) &= F_z(\tau_i)\omega_k^{(i)}(\tau_i) + F_y(\tau_i)\nu_{k-1}^{(i)}(\tau_i) + G_k(\tau_i), \\ \omega_k^{(i)}(0) &= I_z(\theta_i)\tilde{z}_{k+1}(\theta_i) + I_y(\theta_i)\tilde{y}_{k+1}(\theta_i) + I_{k+1}(\theta_i)\Delta\tilde{z}_k|_{t=\theta_i}, \end{aligned}$$

where $\omega_k^{(i)}(0)$ can be changed as follows. Differentiating both sides of the equations (2.18) and (2.20) one can obtain that

$$\begin{aligned} \frac{\partial F}{\partial\tilde{z}_0} d\tilde{z}_0 + \frac{\partial F}{\partial\tilde{y}_0} d\tilde{y}_0 = 0 \Rightarrow \frac{\partial F}{\partial\tilde{z}_0} &= -\frac{\partial F}{\partial\tilde{y}_0}\frac{d\tilde{y}_0}{d\tilde{z}_0}, \\ \frac{\partial I}{\partial\tilde{z}_0} d\tilde{z}_0 + \frac{\partial I}{\partial\tilde{y}_0} d\tilde{y}_0 = 0 \Rightarrow \frac{\partial I}{\partial\tilde{z}_0} &= -\frac{\partial I}{\partial\tilde{y}_0}\frac{d\tilde{y}_0}{d\tilde{z}_0}. \end{aligned} \tag{2.28}$$

Substitution of the first equation of (2.28) into (2.22) implies that

$$\tilde{z}'_k(\theta_i) = -\frac{\partial F}{\partial\tilde{y}_0}\frac{d\tilde{y}_0}{d\tilde{z}_0}\tilde{z}_{k+1}(\theta_i) + F_y(\theta_i)\tilde{y}_{k+1}(\theta_i) + F_{k+1}(\theta_i).$$

Therefore,

$$F_y(\theta_i)[\tilde{y}_{k+1}(\theta_i) - \frac{d\tilde{y}_0}{d\tilde{z}_0}\tilde{z}_{k+1}(\theta_i)] = \tilde{z}'_k(\theta_i) - F_{k+1}(\theta_i).$$

The expression in the square bracket in the last relation is equal to

$$\tilde{y}_{k+1}(\theta_i) - \frac{d\tilde{y}_0}{d\tilde{z}_0}\tilde{z}_{k+1}(\theta_i) = \frac{1}{F_y(\theta_i)}(\tilde{z}'_k(\theta_i) - F_{k+1}(\theta_i)). \tag{2.29}$$

Substituting the second equation of (2.28) into (2.23), we obtain

$$\omega_k^{(i)}(0) = I_y(\theta_i)[\tilde{y}_{k+1}(\theta_i) - \frac{d\tilde{y}_0}{d\tilde{z}_0}\tilde{z}_{k+1}(\theta_i)] + I_{k+1}(\theta_i) - \Delta\tilde{z}_k|_{t=\theta_i}, i = 1, 2, ..., p.$$

Replace the square bracket with equation (2.29) to get

$$\omega_k^{(i)}(0) = \frac{I_y(\theta_i)}{F_y(\theta_i)}(\tilde{z}'_k(\theta_i) - F_{k+1}(\theta_i)) + I_{k+1}(\theta_i) - \Delta\tilde{z}_k|_{t=\theta_i}, i = 1, 2, ..., p.$$

In the end, it remains to solve the equations

$$\dot{\nu}_k^{(i)}(\tau_i) = f_z(\tau_i)\omega_k^{(i)}(\tau_i) + f_y(\tau_i)\nu_{k-1}^{(i)}(\tau_i) + Q_k(\tau_i) = \Pi_k f(\tau_i), i = 1, 2, ..., p.$$

By using condition (2.14) we get that

$$\nu_k^{(i)}(0) = \int_\infty^0 \Pi_k f(s)ds,$$

and

$$\nu_k^{(i)}(\tau_i) = \int_\infty^{\tau_i} \Pi_k f(s)ds.$$

Functions $\Pi_k F(\tau_i)$ and $\Pi_k f(\tau_i)$ possess the exponential estimate of the type (2.27). Therefore, it can be proved that the following inequalities hold,

$$\begin{aligned} &\|\omega_k^{(i)}(\tau_i)\| \le c\exp(-\kappa\tau_i), i = 1, 2, ..., p, \\ &\|\nu_k^{(i)}(\tau_i)\| \le c\exp(-\kappa\tau_i), i = 1, 2, ..., p. \end{aligned} \tag{2.30}$$

Thus, the coefficients of the expansions (2.13) are obtained at least up to order $k = n$.

3. Main results

In this section, we prove the main Theorem 3.1 and Theorem 3.2, which describe two cases respectively, singularity with one layer and singularity with multi-layers. The first one is with a single layer that occurs near $t = 0$, and the second is with multi-layers that appears near $t = 0$ and $t = \theta_{i}{}_{i=1}^{p}$. It is proved that the partial sums of series (2.12) form a sequence of uniform approximations to the solution of problem (2.4)-(2.5).

3.1. **Asymptotic expansion of singularity with a single layer.** Consider the case when convergence is not uniform near $t = 0$ since $z(0, \varepsilon) = z^0$ and $z^0 \neq \varphi$ for all $\varepsilon > 0$. The interval of nonuniform convergence is called *an initial layer*.

By virtue of condition $(C6)$ of (2.17), the following equality is valid,

$$\tilde{I}_1(\tilde{z}_0(\theta_i), \tilde{y}_0(\theta_i), 0) = 0, i = 1, 2, ..., p.$$

Then the first equation of (2.21) takes the form

$$\omega_0^{(i)}(0) = -\Delta\tilde{z}_0|_{t=\theta_i}, i = 1, 2, ..., p.$$

Substituting the last expression in (2.12) one can obtain

$$z(\theta_i+, \varepsilon) = \tilde{z}_0(\theta_i+) + \omega_0^{(i)}(0) + O(\varepsilon) = \tilde{z}_0(\theta_i) + O(\varepsilon), i = 1, 2, ..., p.$$

We can say that the region of nonuniform convergence is $O(\varepsilon)$ thick, since for $t > 0$, $\|z(t, \varepsilon) - \varphi\| = O(\varepsilon)$ can be made arbitrarily close to zero by choosing ε small enough. This implies that for enough small ε, the solution $z(t, \varepsilon)$ of the problem (2.4),(2.5) has no boundary layer phenomenon near the points $t = \theta_{i}{}_{i=1}^{p}$.

Theorem 3.1. *Let conditions $(C1) - (C5)$ and $(C6)$ are hold. Then there exist positive constants ε_0 and c such that for any $\varepsilon \in (0, \varepsilon_0]$ the problem (2.4),(2.5) has a unique solution $z(t, \varepsilon), y(t, \varepsilon)$ which satisfies the inequality*

$$\begin{aligned} &|z(t, \varepsilon) - Z_n(t, \varepsilon)| \le c\varepsilon^{n+1}, \quad 0 \le t \le T, \\ &|y(t, \varepsilon) - Y_n(t, \varepsilon)| \le c\varepsilon^{n+1}, \quad 0 \le t \le T, \end{aligned} \tag{3.31}$$

where

$$Z_n(t,\varepsilon) = \sum_{k=0}^{n} \varepsilon^k \tilde{z}_k(t) + \sum_{k=0}^{n} \varepsilon^k \omega_k^{(i)}(\tau_i),$$
$$Y_n(t,\varepsilon) = \sum_{k=0}^{n} \varepsilon^k \tilde{y}_k(t) + \varepsilon \sum_{k=0}^{n} \varepsilon^k \nu_k^{(i)}(\tau_i), i = 1, 2, ..., p.$$

Proof. By replacing the variables $z(t,\varepsilon) = u(t,\varepsilon) + Z_n(t,\varepsilon)$ and $y(t,\varepsilon) = v(t,\varepsilon) + Y_n(t,\varepsilon)$ in (2.4) and (2.5), we obtain the system

$$\begin{aligned} \varepsilon \frac{du}{dt} &= F_z u + F_y v + G_1(u,v,t,\varepsilon), \\ \frac{dv}{dt} &= f_z u + f_y v + G_2(u,v,t,\varepsilon), \\ \varepsilon \Delta u|_{t=\theta_i} &= I_z(\theta_i)u(\theta_i) + I_y(\theta_i)v(\theta_i) + Q_1(u,v,\theta_i,\varepsilon), \\ \Delta v|_{t=\theta_i} &= J_z(\theta_i)u(\theta_i) + J_y(\theta_i)v(\theta_i) + Q_2(u,v,\theta_i,\varepsilon), \end{aligned} \tag{3.32}$$

with initial condition

$$u(0,\varepsilon) = 0, \quad v(0,\varepsilon) = 0, \tag{3.33}$$

where the elements of matrices F_z, F_y, f_z and f_y are calculated at the points $(\tilde{z}_0(t) + \omega_0^{(i)}(\tau_i), \tilde{y}_0(t), 0), i = 1, 2, ..., p,$

$$\begin{aligned} G_1(u,v,t,\varepsilon) &= F(u + Z_n, v + Y_n, \varepsilon) - \varepsilon \frac{dZ_n}{dt} - F_z u - F_y v, \\ G_2(u,v,t,\varepsilon) &= f(u + Z_n, v + Y_n) - \frac{dY_n}{dt} - f_z u - f_y v, \\ Q_1(u,v,\theta_i,\varepsilon) &= I(u + Z_n^{(i-1)}, v + Y_n^{(i-1)}, \varepsilon) + \varepsilon Z_n^{(i-1)} - \varepsilon Z_n^{(i)} - I_z u - I_y v, \\ Q_2(u,v,\theta_i,\varepsilon) &= J(u + Z_n^{(i-1)}, v + Y_n^{(i-1)}) + Y_n^{(i-1)} - Y_n^{(i)} - J_z u - J_y v. \end{aligned}$$

The functions $G(u,v,t,\varepsilon)$ have the following two properties,

1) $G_1(0,0,t,\varepsilon) = O(\varepsilon^{n+1}), G_2(0,0,t,\varepsilon) = O(\varepsilon^{n+1})$.

2) For any $\varepsilon > 0$ there exist numbers $\delta = \delta(\varepsilon)$ and $\mu = \mu(\varepsilon)$ such that for $\|u_i\| \le \delta$, $\|v_i\| \le \delta,\ i = 1, 2,\ 0 < \varepsilon < \varepsilon_0$ the following inequalities hold, $\|G_i(u_1, v_1, t, \varepsilon) - G_i(u_2, v_2, t, \varepsilon)\| \le \varepsilon(\|u_2 - u_1\| + \|v_2 - v_1\|),\ i = 1, 2.$

Let us prove the property 1). For $t \in (\theta_i, \theta_{i+1}]$ we obtain that

$$\begin{aligned} G_2(0,0,t,\varepsilon) =& f(Z_n, Y_n) - \frac{dY_n}{dt} = [f(\sum_{k=0}^{n} \varepsilon^k (\tilde{z}_k(\varepsilon\tau_i) + \omega_k^{(i)}(\tau_i)), \sum_{k=0}^{n} \varepsilon^k (\tilde{y}_k(\varepsilon\tau_i) + \varepsilon\nu_k^{(i)}(\tau_i))) - \\ & f(\sum_{k=0}^{n} \varepsilon^k \tilde{z}_k(\varepsilon\tau_i), \sum_{k=0}^{n} \varepsilon^k \tilde{y}_k(\varepsilon\tau_i)) - \sum_{k=0}^{n} \varepsilon^k \dot{\nu}_k^{(i)}(\tau_i)] + \\ & [f(\sum_{k=0}^{n} \varepsilon^k \tilde{z}_k(t), \sum_{k=0}^{n} \varepsilon^k \tilde{y}_k(t)) - \sum_{k=0}^{n} \varepsilon^k \tilde{y}_k'(t)] = \\ & [\sum_{k=0}^{n} \varepsilon^k \Pi_k f(\tau_i) + O(\varepsilon^{n+1}) - \sum_{k=0}^{n} \varepsilon^k \dot{\nu}_k^{(i)}(\tau_i)] + \\ & [\sum_{k=0}^{n} \varepsilon^k \tilde{f}_k(t) + O(\varepsilon^{n+1}) - \sum_{k=0}^{n} \varepsilon^k \tilde{y}_k'(t)] = O(\varepsilon^{n+1}), \end{aligned}$$

similarly to that for functions $\tilde{y}_k(t), \nu_k^{(i)}(\tau_i), i = 1, 2, ..., p$. The second property of the functions $G_j, j = 1, 2$, follows from the mean value theorem. Actually,

$$G_i(u_1, v_1, t, \varepsilon) - G_i(u_2, v_2, t, \varepsilon) = \int_0^1 \frac{\partial}{\partial s} G_i(su_1 + (1-s)u_2, sv_1 + (1-s)v_2, t, \varepsilon) ds =$$
$$\int_0^1 \partial_u G_i(u^*(s), v^*(s), t, \varepsilon) ds \cdot (u_1 - u_2) + \int_0^1 \partial_v G_i(u^*(s), v^*(s), t, \varepsilon) ds \cdot (v_1 - v_2),$$

where $u^*(s) = su_1 + (1-s)u_2,\ v^*(s) = sv_1 + (1-s)v_2$. But

$$\partial_u G_i(u^*(s), v^*(s), t, \varepsilon) = \partial_z F(u^*(s) + Z_n, v^*(s) + Y_n, \varepsilon) - \partial_z F(Z_0, Y_0, 0),$$
$$\partial_v G_i(u^*(s), v^*(s), t, \varepsilon) = \partial_y F(u^*(s) + Z_n, v^*(s) + Y_n, \varepsilon) - \partial_y F(Z_0, Y_0, 0),$$

and

$$|u^*(s) + Z_n(t, \varepsilon) - Z_0(t)| \le |u^*(s)| + C\varepsilon,$$
$$|v^*(s) + Y_n(t, \varepsilon) - Y_0(t)| \le |v^*(s)| + C\varepsilon.$$

The continuity of the first derivatives of the functions $F(z, y, \varepsilon)$ and $f(z, y)$ implies the validity of property 2). The functions $Q_i(u, v, \theta_i, \varepsilon),\ i = 1, 2$, have the following two properties,

1*) For $0 < \varepsilon < \varepsilon_0$

$$Q_1(0, 0, \theta_i, \varepsilon) = O(\varepsilon^{n+1}), Q_2(0, 0, \theta_i, \varepsilon) = O(\varepsilon^{n+1}).$$

2*) For any $\varepsilon > 0$ there exist numbers $\delta = \delta(\varepsilon)$ and $\mu = \mu(\varepsilon)$ such that for $\|u_i\| \le \delta$, $\|v_i\| \le \delta,\ i = 1, 2,\ 0 < \varepsilon < \varepsilon_0$ the following inequalities hold,

$$\|Q_i(u_1, v_1, \theta_i, \varepsilon) - Q_i(u_2, v_2, \theta_i, \varepsilon)\| \le \varepsilon(\|u_2 - u_1\| + \|v_2 - v_1\|),\ i = 1, 2.$$

The validity of the property 1*) for $Q_1(0, 0, \theta_i, \varepsilon)$ follows from the next relation

$$Q_1(0, 0, \theta_i, \varepsilon) = I(Z_n^{(i-1)}, Y_n^{(i-1)}, \varepsilon) + \varepsilon Z_n^{(i-1)} - \varepsilon Z_n^{(i)} =$$
$$I(\sum_{k=0}^{n} \varepsilon^k (\tilde{z}_k(\theta_i) + \omega_k^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon})), \sum_{k=0}^{n} \varepsilon^k (\tilde{y}_k(\theta_i) + \varepsilon \nu_k^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon})), \varepsilon) +$$
$$\varepsilon \sum_{k=0}^{n} \varepsilon^k (\tilde{z}_k(\theta_i) + \omega_k^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon})) - \varepsilon \sum_{k=0}^{n} \varepsilon^k (\tilde{z}_k(\theta_i+) + \omega_k^{(i)}(0)) =$$
$$\sum_{k=0}^{n} \varepsilon^{k+1} \omega_k^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon}) +$$
$$[I(\sum_{k=0}^{n} \varepsilon^k (\tilde{z}_k(\theta_i) + \omega_k^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon})), \sum_{k=0}^{n} \varepsilon^k (\tilde{y}_k(\theta_i) + \varepsilon \nu_k^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon})), \varepsilon) -$$
$$I(\sum_{k=0}^{n} \varepsilon^k \tilde{z}_k(\theta_i), \sum_{k=0}^{n} \varepsilon^k \tilde{y}_k(\theta_i), \varepsilon)] + [\sum_{k=0}^{n} \varepsilon^k \tilde{I}_k(\theta_i) + O(\varepsilon^{n+1}) -$$
$$\varepsilon \sum_{k=0}^{n-1} \varepsilon^k (\Delta \tilde{z}_k|_{t=\theta_i} + \omega_k^{(i)}(0)) - \varepsilon^{n+1}(\Delta \tilde{z}_n|_{t=\theta_i} + \omega_n^{(i)}(0))] = O(\varepsilon^{n+1}).$$

The first two terms decrease exponentially as $\varepsilon \to 0$, since of the estimates (2.30) and the mean value theorem. The third term is evaluated using $O(\varepsilon^{n+1})$. The property 2*) is proved similarly to property 2). Now, we replace the impulsive system (3.32)-(3.33) by the equivalent integral equations

$$u(t,\varepsilon) = \frac{1}{\varepsilon}\int_0^t U(t,s,\varepsilon)[F_y(s,\varepsilon)v(s,\varepsilon) + G_1(u_1,v_1,t,\varepsilon)]ds + \sum_{0<\theta_i<t} U(t,\theta_i,\varepsilon)(E + \frac{I_z(\theta_i)}{\varepsilon})^{-1}(I_y(\theta_i)v(\theta_i) + Q_1(u,v,\theta_i,\varepsilon)), \tag{3.34}$$

$$v(t,\varepsilon) = \int_0^t V(t,s,\varepsilon)[f_z(s,\varepsilon)u(s,\varepsilon) + G_2(u_1,v_1,t,\varepsilon)]ds + \sum_{0<\theta_i<t} V(t,\theta_i,\varepsilon)(E + J_y(\theta_i))^{-1}(J_z(\theta_i)u(\theta_i) + Q_2(u,v,\theta_i,\varepsilon)), \tag{3.35}$$

where $U(t,s,\varepsilon)$ and $V(t,s,\varepsilon)$ are the fundamental matrices of the systems

$$\varepsilon\frac{dU}{dt} = F_z(t,\varepsilon)U,\ t \neq \theta_i,\ \varepsilon\Delta U|_{t=\theta_i} = I_z(\theta_i)U,\ U(s,s,\varepsilon) = E,$$
$$\frac{dV}{dt} = f_y(t,\varepsilon)V,\ t \neq \theta_i,\ \Delta V|_{t=\theta_i} = J_y(\theta_i)V,\ V(s,s,\varepsilon) = E.$$

For the matrix $U(t,s,\varepsilon)$ it is correct that

$$\|U(t,s,\varepsilon)\| \le c\exp(-\frac{\kappa}{\varepsilon}(t-s)),\quad 0 \le s \le t \le T.$$

Substituting the expression for $v(t,\varepsilon)$, defined by the equation (3.35), into the first equation, we obtain

$$u(t,\varepsilon) = \int_0^t K(t,s,\varepsilon)u(s,\varepsilon)ds + W_1(u,v,t,\varepsilon),$$

where K is a bounded kernel, and the operator W_1 possesses the same two properties as function $G(u,v,t,\varepsilon)$. The last equation can be replaced by the equivalent equation

$$u(t,\varepsilon) = \int_0^t R(t,s,\varepsilon)W_1(u,v,s,\varepsilon)ds + W_1(u,v,t,\varepsilon) = T_1(u,v,t,\varepsilon), \tag{3.36}$$

where R is the resolvent of the kernel K. Now, let us substitute the expression (3.36) for $u(t,\varepsilon)$ into the equation of (3.35),

$$v(t,\varepsilon) = \int_0^t V(t,s,\varepsilon)[f_z(s,\varepsilon)T_1(u,v,t,\varepsilon) + G_2(u,v,t,\varepsilon)]ds + \sum_{0<\theta_i<t} V(t,\theta_i,\varepsilon)(E + J_y(\theta_i))^{-1}(J_z(\theta_i)T_1(u,v,\theta_i,\varepsilon) + Q_2(u,v,\theta_i,\varepsilon)) = T_2(u,v,t,\varepsilon). \tag{3.37}$$

Integral operators T_1 and T_2 admit the same two properties as function $G(u,v,t,\varepsilon)$. Applying the method of successive approximations to the systems (3.36), (3.37) we find that a unique solution exists and satisfies the estimates

$$|u(t,\varepsilon)| = |z(t,\varepsilon) - Z_n(t,\varepsilon)| \le c\varepsilon^{n+1},\quad 0 \le t \le T,$$
$$|v(t,\varepsilon)| = |y(t,\varepsilon) - Y_n(t,\varepsilon)| \le c\varepsilon^{n+1},\quad 0 \le t \le T.$$

The theorem is proved. □

3.2. **Asymptotic expansion of singularity with multi-layers.** In the previous subsection, it was shown that there exists a single initial layer. Using an impulse function, the convergence can be nonuniform near several points, that is to say, that *multi-layers* emerge. These layers occur on the neighborhoods of $t = 0$ and $t = \theta_i{}_{i=1}^{p}$.

To have singularity with multi-layers, let us consider system (2.4) with the assumptions $(C1)$-$(C5)$ and additional condition

(C7) $\lim\limits_{(z,y,\varepsilon)\to(\varphi,\tilde{y},0)} \dfrac{I(z,y,\varepsilon)}{\varepsilon} = I_0 \neq 0$, and $\varphi(\tilde{y}(\theta_i),\theta_i) + I_0, i = 1,2,...,p$, are in the basin of attraction of $\varphi(\tilde{y}(t),t)$.

By the virtue of condition $(C7)$ of (2.17), the following equality is valid,

$$\tilde{I}_1(\tilde{z}_0(\theta_i),\tilde{y}_0(\theta_i),0) = I_0 \neq 0, i = 1,2,...,p.$$

Therefore, the first equation of (2.21) takes the form

$$\omega_0^{(i)}(0) = I_0 - \Delta\tilde{z}_0|_{t=\theta_i}, i = 1,2,...,p.$$

Substituting the last expression in (2.12) one can obtain

$$z(\theta_i+,\varepsilon) = \tilde{z}_0(\theta_i+) + \omega_0^{(i)}(0) + O(\varepsilon) = \tilde{z}_0(\theta_i) + I_0 + O(\varepsilon), i = 1,2,...,p.$$

By condition $(C7)$, after each impulse moment, the difference $\|z(\theta_i+,\varepsilon) - \varphi\| = I_0 + O(\varepsilon)$ is not diminish to zero as $\varepsilon \to 0$. Hence, the convergence is not uniform. Then we can say that the solution $z(t,\varepsilon)$ of the system (2.4) with the initial value (2.5) has multi-layers appearing near $t = 0$ and $t = \theta_i{}_{i=1}^{p}$.

The next theorem can be proved in the similar way as Theorem 3.1.

Theorem 3.2. *Assume that conditions $(C1)$-$(C5)$ and $(C7)$ are fulfilled. Then there exist positive constants ε_0 and c such that for any $\varepsilon \in (0,\varepsilon_0]$ problem (2.4), (2.5) has a unique solution $z(t,\varepsilon), y(t,\varepsilon)$ which satisfies the inequality*

$$\begin{aligned} |z(t,\varepsilon) - Z_n(t,\varepsilon)| &\leq c\varepsilon^{n+1}, \quad 0 \leq t \leq T, \\ |y(t,\varepsilon) - Y_n(t,\varepsilon)| &\leq c\varepsilon^{n+1}, \quad 0 \leq t \leq T, \end{aligned}$$

where

$$\begin{aligned} Z_n(t,\varepsilon) &= \sum_{k=0}^{n} \varepsilon^k \tilde{z}_k(t) + \sum_{k=0}^{n} \varepsilon^k \omega_k^{(i)}(\tau_i), \\ Y_n(t,\varepsilon) &= \sum_{k=0}^{n} \varepsilon^k \tilde{y}_k(t) + \varepsilon \sum_{k=0}^{n} \varepsilon^k \nu_k^{(i)}(\tau_i), i = 1,2,...,p. \end{aligned}$$

4. Numerical examples

Example 4.1. Consider the impulsive system with singularities

$$\begin{aligned} &\varepsilon z' = z(2-z) + \varepsilon^2 y_1, \qquad && \varepsilon\Delta z|_{t=\theta_i} = 2 - z + \varepsilon^2(y_1^2 - 2y_2), \\ &y_1' = y_2 - zy_1, && \Delta y_1|_{t=\theta_i} = 2z + y_1^2, \\ &y_2' = 2y_1 - y_2 - y_1^3, && \end{aligned} \tag{4.38}$$

and initial conditions

$$z(0,\varepsilon) = 1, \quad y_1(0,\varepsilon) = 1, \quad y_2(0,\varepsilon) = -2, \tag{4.39}$$

where $\theta_i = i/3, i = 1, 2, ...14$. Assume that $\varepsilon = 0$ in this problem. Then, the first line of (4.38) becomes $\tilde{z}(2-\tilde{z}) = 0, 2-\tilde{z} = 0$. It has the root $\tilde{z} = \varphi = 2$. One can check that

$$\frac{\partial}{\partial z}(z(2-z))|_{z=2} = 2 - 2z < 0.$$

Therefore, $\tilde{z} = 2$ is uniformly asymptotically stable. Substituting $\tilde{z} = 2$ into the second line of (4.38), one can obtain

$$\begin{aligned} &\tilde{y}_1' = \tilde{y}_2 - 2\tilde{y}_1, \quad \Delta\tilde{y}_1|_{t=\theta_i} = \tilde{y}_1^2 + 4, \\ &\tilde{y}_2' = 2\tilde{y}_1 - \tilde{y}_2 - \tilde{y}_1^3, \\ &\tilde{y}_1(0) = 1, \quad \tilde{y}_2(0) = -2. \end{aligned}$$

This system has a unique solution $\tilde{y}(t)$. Now, let us check the condition $(C6)$

$$\lim_{(z,y,\varepsilon)\to(\varphi,y^0,0)} \frac{2 - z + \varepsilon^2(y_1^2 - 2y_2)}{\varepsilon} = 0.$$

The solution $z(t,\varepsilon)$ of system (4.38) with initial value (4.39) has a single initial layer at $t = 0$. The result of simulation is seen in Figure 1 that a single layer occurs. In Figure 2 it is shown that the solution of system (4.38) with initial value (1,1,-2) tends to $z = 2$ as $\varepsilon \to 0$.

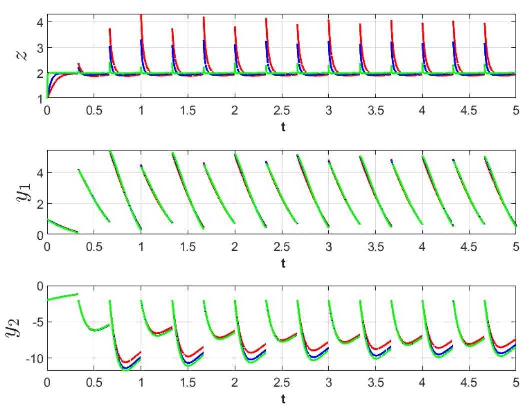

FIGURE 1. The red, blue and green lines are graphs of solutions of system (4.38) with initial values $z(0,\varepsilon) = 1$, $y_1(0,\varepsilon) = 1$ and $y_2(0,\varepsilon) = -2$ with values of $\varepsilon : 0.1, 0.05, 0.005$, respectively.

Example 4.2. Now, let us consider the following system with impulsive singularity

$$\begin{aligned} &\varepsilon z' = z - z^3, \qquad \varepsilon\Delta z|_{t=\theta_i} = (z-1)^2 - 0.5\varepsilon y + 2sin(\varepsilon), \\ &y' = yz - y^2, \qquad \Delta y|_{t=\theta_i} = 2y + z - 1, \end{aligned} \tag{4.40}$$

and initial conditions

$$z(0,\varepsilon) = 2, \quad y(0,\varepsilon) = 3, \tag{4.41}$$

where $\theta_i = i/3, i = 1, 2, ...14$. If $\varepsilon = 0$ in (4.40), then the first line becomes $\tilde{z} - \tilde{z}^3 = 0$, and $(\tilde{z} - 1)^2 = 0$. It has the root $\tilde{z} = 1$. The corresponding root $\varphi = 1$ will be uniformly asymptotically stable since

$$\frac{\partial}{\partial z}(z - z^3)|_{z=1} = 1 - 3z^2 < 0.$$

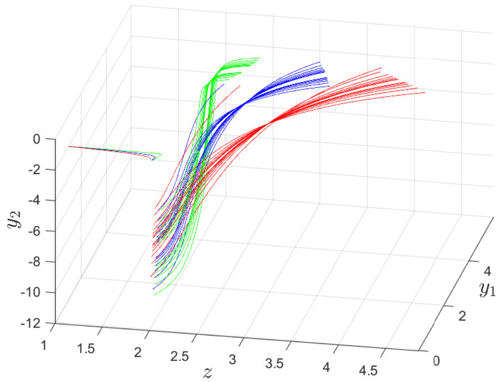

FIGURE 2. Red, blue and green lines represent solutions of system (4.38) with initial value $(1, 1, -2)$ for ε equals to $0.1, 0.05$ and 0.005, respectively.

Substituting $\tilde{z} = 1$ into the second equation of system (4.40), we obtain that

$$\tilde{y}' = \tilde{y} - \tilde{y}^2, \quad \Delta\tilde{y}|_{t=\theta_i} = 2\tilde{y},$$

with initial condition $\tilde{y}(0) = 3$. The last system admits the unique solution

$$\tilde{y}(t) = \frac{\exp(t)}{\exp(t) - \sum\limits_{k=0}^{i} \frac{2}{3^{i-k+1}} \exp(\frac{k}{3})}, i = 1, 2, ...14.$$

One can verify that condition $(C7)$ is valid

$$\lim_{(z,y,\varepsilon)\to(\varphi,\tilde{y},0)} \frac{(z-1)^2 - 0,5\varepsilon y + 2sin(\varepsilon)}{\varepsilon} = 2 - \frac{\exp(\theta_i)}{2\exp(\theta_i) - \sum\limits_{k=0}^{i} \frac{1}{3^{i-k+1}} \exp(\frac{k}{3})} \neq 0.$$

The solution $z(t, \varepsilon)$ of system (4.40) with initial value (4.41) has multi-layers near $t = 0$ and $t = \theta_i+, i = 1, 2, ...14$. Figure 3 demonstrates the solution of system (4.40) with initial value (1,1,-2), which tends to $z = 1$ as $\varepsilon \to 0$.

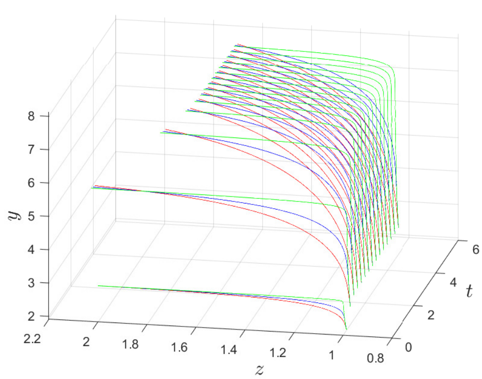

FIGURE 3. The red, blue and green curves are the behaviours of solutions of system (4.40) if ε is equal to $0.1, 0.05$ and 0.005, respectively.

5. Conclusion

In this paper, the singular impulsive differential equations of type in [1, 3, 5] is considered. The boundary function method is used to construct desired asymptotic solutions. We investigated the asymptotic expansion of solutions with any degree of accuracy. Single layer and several multi-layers are considered.

The articles [7, 8] and book [6] investigated impulse systems with small parameters, participating only in the differential equations, but not in their impulse equations. Following [1, 3], we inserted a small parameter into the impulse equation. Thus, the concept of singularity for discontinuous dynamics is greatly expanded. Illustrative examples with simulations are given to support the theoretical results.

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References

[1] Akhmet, M.; Çağ, S. Tikhonov theorem for differential equations with singular impulses. *Discontinuity, Nonlinearity, and Complexity* **7** (2018), no. 3, 291–303.

[2] Akhmet, M. *Principles of Discontinuous Dynamical Systems*. Springer, New York, 2010.

[3] Akhmet, M.; Cag, S. Chattering as a Singular Problem. *Nonlinear Dynamics* **90** (2017), no. 4, 2797–2812.

[4] Akhmet, M.; Dauylbayev, M.; Mirzakulova, A. A singularly perturbed differential equation with piecewise constant argument of generalized type. *Turkish Journal of Mathematics* **42** (2018), no. 4, 1680–1685.

[5] Akhmet, M.; Çağ, S. Bifurcation analysis of Wilson-Cowan model with singular impulses. *Discontinuity, Nonlinearity, and Complexity* **10** (2021), no. 1, 161–172.

[6] Bainov, D.; Covachev, V. *Impulsive Differential Equations with a Small Parameter*. World Scientific, 1994.

[7] Chen, W. H.; Chen, F.; Lu, X. Exponential stability of a class of singularly perturbed stochastic time-delay systems with impulse effect. *Nonlinear Analysis: Real World Applications* **11** (2010), no. 5, 3463–3478.

[8] Chen, W. H.; Wei, D.; Lu, X. Exponential stability of a class of nonlinear singularly perturbed systems with delayed impulses. *Journal of the Franklin Institute* **350** (2013), no. 9, 2678–2709.

[9] Chen, W. H.; Yuan, G.; Zheng, W. X. Robust stability of singularly perturbed impulsive systems under nonlinear perturbation. *IEEE Transactions on Automatic Control* **58** (2013), no. 1, 168–174.

[10] Damiano, E. R.; Rabbitt, R. D. A singular perturbation model of fluid dynamics in the vestibular semicircular canal and ampulla. *Journal of Fluid Mechanics* **307** (1996), 333–372.

[11] Donchev, T.; Slavov, I. Tikhonov's theorem for functional-differential inclusions. *Ann. Sofia Univ., Fac. Math and Inf.* **89** (1998), 69–78.

[12] Gondal, I. On the application of singular perturbation techniques to nuclear engineering control problems. *IEEE Transactions on Nuclear Science* **35** (1988), no. 5, 1080–1085.

[13] Hek, G. Geometric singular perturbation theory in biological practice. *Journal of Mathematical Biology* **60** (2010), 347–386.

[14] Kadalbajoo, M. K.; Patidar, K. C. Singularly perturbed problems in partial differential equations: a survey. *Applied Mathematics and Computation* **134** (2003),no. 23, 371–429.

[15] Kokotovic, P. V. Applications of singular perturbation techniques to control problems. *SIAM Review* **26** (1984), no. 4, 501–550.

[16] O'Malley, R. E. J. Singular Perturbation Methods for Ordinary Differential Equations Applied Mathematical Sciences, 89. *Springer-Verlag, New York* 1991.

[17] Owen, M. R.; Lewis, M. A. How predation can slow, stop, or reverse a prey invasion. *Bulletin of Mathematical Biology* **63** (2001), 655–684.

[18] Simeonov, P.; Bainov, D. Stability of the solutions of singularly perturbed systems with impulse effect. *Journal of Mathematical Analysis and Applications* **136** (1988), no. 2, 575–588.

[19] Segel, L. A.; Slemrod, M. The Quasi-Steady-State Assumption: A Case Study in Perturbation. *SIAM Review* **31** (1989), 446–477.

[20] Tikhonov, A. N. Systems of differential equations containing small parameters in the derivatives. *Matematicheskii sbornik* **73** (1952), no. 3, 575–586.

[21] Vasil'eva, A. B.; Butuzov, V. F. Asymptotic Expansion of Solutions of Singularly Perturbed Equations. *Nauka, Moscow*, 1973 (in russian).

[22] Vasil'eva, A.; Butuzov, V.; Kalachev, L. The Boundary Function Method for Singular Perturbation Problems. *Society for Industrial and Applied Mathematics*, 1995.

[23] Veliov, V. A generalization of the Tikhonov theorem for singularly perturbed differential inclusions. *Journal of dynamical and control systems* **3** (1997), no. 3, 291–319.

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