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# Some remarks on [Carpathian J. Math. 39 (2023), No 2, 541–551]

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ABSTRACT. We present some remarks on [Carpathian J. Math. 39 (2023), No 2, 541–551] in order to obtain a unique non trivial solution.

## 1. INTRODUCTION

In [2], the authors studied the following general functional equation

$$U(x) = pxU(h_1(x)) + (1-p)xU(h_2(x)) + p(1-x)U(h_3(x)) + (1-p)(1-x)U(h_4(x)), \tag{1.1}$$

for any $x \in [0,1]$, where $p \in [0,1]$, $U\colon [0,1] \to \mathbb{R}$ is a unknown function such that $U(0) = 0$ and $h_1, h_2, h_3, h_4\colon [0,1] \to [0,1]$ are given mappings such that

$$h_3(0) = h_4(0) = 0. \tag{1.2}$$

They considered the space $B$ of the real valued functions $U\colon [0,1] \to \mathbb{R}$ such that $U(0) = 0$ and

$$\sup_{x_1 \neq x_2} \frac{|U(x_1) - U(x_2)|}{|x_1 - x_2|} < \infty\,.$$

It is easily seen that $(B, \|\cdot\|)$ is a Banach space, where $\|\cdot\|$ is defined by

$$\|U\| = \sup_{x_1 \neq x_2} \frac{|U(x_1) - U(x_2)|}{|x_1 - x_2|}\,,$$

for any $U \in B$.

The main result of [2] is the following.

**Theorem 1.1** (Theorem 3.2 in [2])**.** *Consider the functional equation* (1.1) *with the condition* (1.2)*. Suppose that* $h_i\colon [0,1] \to [0,1]$ $(i = 1,2,3,4)$*, are Banach contraction mappings with contractive coefficients* $\alpha_i$ $(i = 1,2,3,4)$ *satisfying*

$$2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) < 1$$

*and*

$$h_1(0) = h_2(0) = 0. \tag{1.3}$$

*Then Eq.* (1.1) *has a unique solution in the space* $(B, \|\cdot\|)$.

Notice that Eq. (1.1) under condition (1.2) is satisfied by the function identically equal to zero and this function belongs to $(B, \|\cdot\|)$. By the uniqueness of the solution given by Theorem 1.1, the unique solution is the trivial solution. This is the main result of [2].

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## 2. Conclusions

In order to obtain a non trivial solution to Eq. (1.1), we consider the space $B_1$ given by

$$B_1 = \{U \in B \colon U(1) = 1\}.$$

Notice that $B_1$ is a subset of the known Banach space $H^1[0,1]$ of the Lipschitz functions, this is,

$$H^1[0,1] = \left\{U \colon [0,1] \to \mathbb{R} \colon \sup_{x_1 \neq x_2} \left\{ \frac{|U(x_1) - U(x_2)|}{|x_1 - x_2|} < \infty, \text{ for } x_1, x_2 \in [0,1] \right\} \right\},$$

where the norm is given by

$$\|U\| = |U(0)| + \sup_{x_1 \neq x_2} \left\{ \frac{|U(x_1) - U(x_2)|}{|x_1 - x_2|} \text{ for } x_1, x_2 \in [0,1] \right\}.$$

Moreover, $H^1[0,1]$ is a Banach algebra [1].

It is easily seen that $B_1$ is a closed subset of $B$ and, therefore, $(B_1, d)$ is a complete metric space, where $d$ is the distance induced by $\|\cdot\|$, this is,

$$d(U_1, U_2) = \|U_1 - U_2\| = \sup_{x_1 \neq x_2} \left\{ \frac{|(U_1 - U_2)(x_1) - (U_1 - U_2)(x_2)|}{|x_1 - x_2|},\ x_1, x_2 \in [0,1] \right\},$$

for any $U_1, U_2 \in B_1$.

Next, we present our result.

**Theorem 2.2.** *If in Theorem 1.1 we replace condition* (1.3) *by*

$$h_1(1) = h_2(1) = 1 \tag{2.4}$$

*and* $p(\alpha_1 + \alpha_3) + (1-p)(\alpha_2 + \alpha_4) < \frac{1}{2}$ *then Eq.* (1.1) *with* (1.2) *has a unique solution in* $(B_1, d)$*.*

*Proof.* We consider the operator $G$ defined on $B_1$ as

$$\begin{aligned}(GU)(x) &= pxU(h_1(x)) + (1-p)xU(h_2(x)) + p(1-x)U(h_3(x)) \\ &\quad + (1-p)(1-x)U(h_4(x)),\end{aligned}$$

for $U \in B_1$ and $x \in [0,1]$.

By condition (1.2), it is clear that $(GU)(0) = 0$ and by (2.4) we have that $(GU)(1) = 1$.

On the other hand, since $H^1[0,1]$ is a Banach algebra it is easily seen that the identity function and the composition of elements in $H^1[0,1]$ also belong to $H^1[0,1]$. Therefore, if $U \in H^1[0,1]$ then $GU \in H^1[0,1]$. Summarizing, $GU \in B_1$ and $G$ applies $B_1$ into itself.

Next, we have to prove that $G$ is a Banach contraction in $B_1$. For this, we take $U_1, U_2 \in B_1$ and, since

$$d(GU_1, GU_2) = \|GU_1 - GU_2\| = \|G(U_1 - U_2)\|,$$

we estimate $\|G(U_1 - U_2)\|$. In fact, we take $x, y \in [0,1]$ with $x \neq y$.

$$
\begin{aligned}
&\frac{|G(U_1-U_2)(x)-G(U_1-U_2)(y)|}{|x-y|}\\
&= \frac{1}{|x-y|}\,|px(U_1-U_2)(h_1(x))\\
&\quad +(1-p)x(U_1-U_2)(h_2(x))+p(1-x)(U_1-U_2)(h_3(x))\\
&\quad +(1-p)(1-x)(U_1-U_2)(h_4(x))\\
&\quad -py(U_1-U_2)(h_1(y))-(1-p)y(U_1-U_2)(h_2(y))\\
&\quad -p(1-y)(U_1-U_2)(h_3(y))-(1-p)(1-y)(U_1-U_2)(h_4(y))|\\
&\le \frac{1}{|x-y|}\Big(p|x-y||(U_1-U_2)(h_1(x))|+py|(U_1-U_2)(h_1(x))-(U_1-U_2)(h_1(y))|\\
&\quad +(1-p)|x-y||(U_1-U_2)(h_2(x))|\\
&\quad +(1-p)y|(U_1-U_2)(h_2(x))-(U_1-U_2)(h_2(y))|\\
&\quad +p|x-y||(U_1-U_2)(h_3(x))|+p(1-y)|(U_1-U_2)(h_3(x))-(U_1-U_2)(h_3(y))|\\
&\quad +(1-p)|x-y||(U_1-U_2)(h_4(x))|\\
&\quad +(1-p)(1-y)|(U_1-U_2)(h_4(x))-(U_1-U_2)(h_4(y))|\Big).
\end{aligned}
$$

Now, as $(U_1-U_2)(0)=(U_1-U_2)(1)=0$ we obtain that

$$
\begin{aligned}
&\frac{|G(U_1-U_2)(x)-G(U_1-U_2)(y)|}{|x-y|}\\
&\le p\,\frac{|(U_1-U_2)(h_1(x))-(U_1-U_2)(1))|}{|h_1(x)-1|}\,|h_1(x)-1|\\
&\quad +(1-p)\,\frac{|(U_1-U_2)(h_2(x))-(U_1-U_2)(1))|}{|h_2(x)-1|}\,|h_2(x)-1|\\
&\quad +p\,\frac{|(U_1-U_2)(h_3(x))-(U_1-U_2)(0))|}{|h_3(x)|}\,|h_3(x)|\\
&\quad +(1-p)\,\frac{|(U_1-U_2)(h_4(x))-(U_1-U_2)(0))|}{|h_4(x)|}\,|h_4(x)|\\
&\quad +\frac{p}{|x-y|}\,\|U_1-U_2\|\,|h_1(x)-h_1(y)|+\frac{1-p}{|x-y|}\,\|U_1-U_2\|\,|h_2(x)-h_2(y)|\\
&\quad +\frac{p}{|x-y|}\,\|U_1-U_2\|\,|h_3(x)-h_3(y)|+\frac{1-p}{|x-y|}\,\|U_1-U_2\|\,|h_4(x)-h_4(y)|\\
&\le p\,\|U_1-U_2\|\,|h_1(x)-h_1(1)|+(1-p)\,\|U_1-U_2\|\,|h_2(x)-h_2(1)|\\
&\quad +p\,\|U_1-U_2\|\,|h_3(x)-h_3(0)|+(1-p)\,\|U_1-U_2\|\,|h_4(x)-h_4(0)|\\
&\quad +p\,\|U_1-U_2\|\,\alpha_1+(1-p)\,\|U_1-U_2\|\,\alpha_2+p\,\|U_1-U_2\|\,\alpha_3\\
&\quad +(1-p)\,\|U_1-U_2\|\,\alpha_4\\
&\le p\,\|U_1-U_2\|\,\alpha_1|x-1|+(1-p)\,\|U_1-U_2\|\,\alpha_2|x-1|\\
&\quad +p\,\|U_1-U_2\|\,\alpha_3|x|+(1-p)\,\|U_1-U_2\|\,\alpha_4|x|\\
&\quad +p\,\|U_1-U_2\|\,\alpha_1+(1-p)\,\|U_1-U_2\|\,\alpha_2+p\,\|U_1-U_2\|\,\alpha_3\\
&\quad +(1-p)\,\|U_1-U_2\|\,\alpha_4\\
&\le 2\Big(p(\alpha_1+\alpha_3)+(1-p)(\alpha_2+\alpha_4)\Big)\|U_1-U_2\|.
\end{aligned}
$$

Finally, taking into account our assumption, we obtain that the operator $G$ is a contraction in $(B_1, \|\cdot\|)$. Therefore, by the Banach's contraction principle, Eq. (1.1) has a unique solution in this space. □

**Remark 2.1.** Since the solution $U^\star$ to Eq. (1.1) given by Theorem 2.2 belongs to $(B_1, \|\cdot\|)$ we have that $U^\star(1) = 1$ and, therefore, $U^\star$ is not the trivial solution.

Finally, we present an example illustrating our result.

**Example 2.1.** Consider the following functional equation

$$\begin{aligned} U(x) =& \frac{1}{3}\, x\, U\left(\frac{1}{5}\, x + \frac{4}{5}\right) + \frac{2}{3}\, x\, U\left(\frac{1}{7}\, x + \frac{6}{7}\right) + \frac{1}{3}\,(1-x)\, U\left(\frac{1}{8}\, x\right) \\ & + \frac{2}{3}\,(1-x)\, U\left(\frac{1}{9}\, x\right). \end{aligned} \tag{2.5}$$

Eq. (2.5) is a particular case of Eq. (1.1) with $p = \frac{1}{3}$, $h_1(x) = \frac{1}{5}\, x + \frac{4}{5}$, $h_2(x) = \frac{1}{7}\, x + \frac{6}{7}$, $h_3(x) = \frac{1}{8}\, x$, $h_4(x) = \frac{1}{9}\, x$.

Moreover, it is clear that $h_1(1) = h_2(1) = 1$, $h_3(0) = h_4(0) = 0$ and $h_i$ are contractions of $[0, 1]$ into itself with constants $\alpha_1 = 1/5$, $\alpha_2 = 1/7$, $\alpha_3 = 1/8$ and $\alpha_4 = 1/9$.

Since

$$p(\alpha_1 + \alpha_3) + (1-p)(\alpha_2 + \alpha_4) = \frac{1}{3}\left(\frac{1}{5} + \frac{1}{8}\right) + \frac{2}{3}\left(\frac{1}{7} + \frac{1}{9}\right) < \frac{1}{2},$$

Theorem 2.2 says us that Eq. (2.5) has a unique nontrivial solution.

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