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Dedicated to Dr. Dan Tiba on the occasion of his 70th anniversary

On the Robustness of Polynomial Dichotomy of Discrete Nonautonomous Systems

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ABSTRACT. Starting from a characterization of polynomial dichotomy by means of admissibility, recently proved in [Dragičević, D.; Sasu, A. L.; Sasu, B. Admissibility and polynomial dichotomy of discrete nonautonomous systems. *Carpath. J. Math.* **38** (2022), 737-762.], the aim of this paper is to explore the roughness of polynomial dichotomy in the presence of perturbations and to obtain a new robustness criterion. We show that the polynomial dichotomy is robust when subjected to linear additive perturbations which are bounded by a well-chosen sequence. We emphasize that the new bounds imposed to the perturbation family improve and extend the previous approaches. Furthermore, we mention that the main result applies to discrete nonautonomous systems in Banach spaces with the only requirement that their propagators exhibit a polynomial growth.

1. INTRODUCTION

The admissibility methods represent some of the most interesting and effective tools in exploring the asymptotic behavior of dynamical systems (see [1,2,6,9,10,13–17,19–24,27–33,35,37–41,44,45,47–61,64,66–68] and the references therein). Although the foundation of these techniques had been established more than ninety years ago in the celebrated works of Perron [47] and Li [37], the notions of admissibility were introduced three decades later in the remarkable works of Massera and Schäffer (see [38, 39]) and Coffman and Schäffer (see [13]). The next notable steps in the admissibility theory were made in the monographs of Daleckiĭ and Kreĭn [15], Coppel [14] and Henry [33]. Those were succeeded by landmark works, in their majority focused on stabilities and dichotomies, which significantly contributed to the development of these methods for both nonautonomous and variational systems, among which we mention Palmer [44, 45], Chow and Leiva [10], Aulbach and Minh [1], Minh, Räbiger and Schnaubelt [41], Chicone and Latushkin [9], Pliss and Sell [48]. For detailed descriptions of the history of the admissibility theory and recent results in this topic, from various perspectives, for stability, expansiveness, dichotomy and trichotomy we refer to Dragičević, Sasu and Sasu [20, 21, 23], Dragičević, Sasu, Sasu and Singh [24], Dragičević, Zhang and Zhou [27, 28], Sasu and Sasu [56–60], Zhou, Lu and Zhang [67], Zhou and Zhang [66, 68].

An important line of studies in the asymptotic theory of dynamical systems is devoted to the analysis of the persistence of various behaviors in the presence of perturbations. Thus, diverse methods were built around the roughness topics, being afterwards extended to other classes of qualitative properties of dynamical systems (see [1, 3–6, 9–11, 14–20, 23, 25, 26, 29, 33, 34, 36, 39, 42–46, 48, 49, 53, 55, 59, 61–63, 65, 66, 68] and the references therein). In this context, it is of interest to explore both whether an asymptotic property (such as dichotomy or stability) is preserved when the initial system is perturbed (see e.g.

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Battelli and Palmer [3], Chow and Leiva [10, 11], Coppel [14], Palmer [42, 43], Pliss and Sell [48], Zhou, Lu and Zhang [65]) and to determine what kind of perturbations should be considered i.e. additive, multiplicative, multi-structured (see e.g. Battelli, Franca and Palmer [4], Dragičević [16, 17], Dragičević, Sasu and Sasu [20, 23], Dragičević, Sasu, Sasu and Şirianţu [25], Hinrichsen and Pritchard [34], Sasu and Sasu [49, 59], Sasu [53], Sasu [55], Wirth and Hinrichsen [62], Zhou and Zhang [66, 68]). Moreover, in certain cases, the method enables one to establish the largest "size" that can be allowed for a perturbation such that the perturbed system exhibits the same kind of asymptotic behavior as the initial one, which leads to the notions of stability radius or dichotomy radius or at least gives specific bounds for those radii (see Braverman and Karabash [6], Chicone and Latushkin [9], Sasu and Sasu [49, 59], Sasu [53], Wirth and Hinrichsen [62], Wirth [63]).

In two recent works we have established new characterizations for the notion of polynomial dichotomy for discrete nonautonomous systems (see Dragičević, Sasu and Sasu [22, 23]) and we have presented a description regarding the evolution of the studies on polynomial behaviors of dynamical systems and their impact in the literature over the last decades (see also [2,7,8,16,17,31,32,61] and the references therein). More precisely, in [23] we have proved that the polynomial dichotomy can be characterized in terms of some double specific admissibilities, providing a different method compared to the previous admissibility approaches to polynomial behaviours (see e.g. Dragičević [16, 17], Hai [31, 32], Silva [61]). Furthermore, we have shown that the uniform polynomial dichotomy of discrete nonautonomous systems is robust when subjected to suitable perturbations. We recall that other studies on the robustness of polynomial dichotomies have been previously done in [16, 17, 61]. We also emphasize that the method developed in [23] was distinct, being built on the admissibility criteria obtained in the same work. On the other hand, as pointed out in [59], when exploring the roughness of a dichotomy property, one of the aims is not only to deduce that it is robust under perturbations but to identify and optimize, at the same time, the suitable upper bounds for the perturbation. Thus, obtaining optimal bounds for the perturbation structures is an important goal also in the case of the polynomial dichotomies.

The aim of this paper is to give a new robustness criterion for the polynomial dichotomy of discrete nonautonomous systems. We continue and improve our method introduced in [23], providing a special bound for the perturbation that is related to the norm of the some input-output operators. The paper is organized as follows: first we recall and discuss several admissibility properties established in [23]. Next, we present a sufficient condition for the existence of a (uniform) polynomial growth for the propagator of the perturbed system. This is a natural step in the studies devoted to the robustness of a dichotomic behavior of uniform nature (see e.g. [53, 59]). After that, we deduce several properties of certain input-output operators connected to the initial system. Next, based on admissibility techniques, we prove a new sufficient condition for the robustness of the polynomial dichotomy of the initial system under additive perturbations. Furthermore, we explain how our result improves and generalizes the previous approaches and criteria. By a relevant example, we also illustrate that the new robustness theorem extends the applicability area. We emphasize that the main result applies to general discrete nonautonomous systems with the only (minimal) requirement that their propagators exhibit a polynomial growth.

2. Basic notions and preliminaries

The concept of polynomial dichotomy whose robustness is studied in this paper is the one recently explored in Dragičević, Sasu and Sasu [22, 23]. A characterization of this notion in terms of a double admissibility has been obtained in [23], as we recall in the following.

Let $\mathbb{N}=\{1,2,\ldots,n,\ldots\}$ and $\Lambda=\{(n,j)\in\mathbb{N}\times\mathbb{N}: n\geq j\}$.

Let $(X,\|\cdot\|)$ be a Banach space. The norm on the space $\mathcal{B}(X)$ of all bounded linear operators on X will be also denoted by $\|\cdot\|$.

For $r\in[1,\infty)$, consider $\ell^r(\mathbb{N},X):=\{s:\mathbb{N}\to X\ :\ \sum_{j\in\mathbb{N}}\|s(j)\|^r<\infty\}$ with the norm

$$\|s\|_r=\Big(\sum_{j\in\mathbb{N}}\|s(j)\|^r\Big)^{\frac{1}{r}}.$$

Let $\ell^\infty(\mathbb{N},X):=\{s:\mathbb{N}\to X:\sup_{j\in\mathbb{N}}\|s(j)\|<\infty\}$ with the norm

$$\|s\|_\infty=\sup_{j\in\mathbb{N}}\|s(j)\|.$$

For $r\in[1,\infty]$, let $\ell^r_0(\mathbb{N},X)$ be the space of all $s\in\ell^r(\mathbb{N},X)$ with $s(1)=0$.

Let $\{A(n)\}_{n\in\mathbb{N}}$ be an arbitrary family of bounded linear operators on X. Consider the discrete system

$$(A)\qquad x(n+1)=A(n)x(n),\quad n\in\mathbb{N}$$

and its propagator $\Phi_A=\{\Phi_A(n,j)\}_{(n,j)\in\Lambda}$, i.e.

$$\Phi_A(n,j)=\begin{cases} A(n-1)\cdots A(j), & n>j\\ I_X, & n=j\end{cases}$$

where I_X denotes the identity operator on X.

Definition 2.1. We say that (A) has a *polynomial dichotomy* if there are a sequence of projections $\{P(j)\}_{j\in\mathbb{N}}$ on X and two constants $N\geq 1,\nu>0$ such that:

- (d_1) $\Phi_A(n,j)P(j)=P(n)\Phi_A(n,j)$, for all $(n,j)\in\Lambda$;
- (d_2) $\sup_{j\in\mathbb{N}}\|P(j)\|<\infty$;
- (d_3) the restriction $\Phi_A(n,j)_{|KerP(j)}:KerP(j)\to KerP(n)$ is invertible, for all $(n,j)\in\Lambda$;
- (d_4) $\|\Phi_A(n,j)x\|\leq N\left(\dfrac{n}{j}\right)^{-\nu}\|x\|$, for all $x\in RangeP(j)$ and $(n,j)\in\Lambda$;
- (d_5) $\|\Phi_A(n,j)x\|\geq \dfrac{1}{N}\left(\dfrac{n}{j}\right)^{\nu}\|x\|$, for all $x\in KerP(j)$ and $(n,j)\in\Lambda$.

Remark 2.1. For detailed connections between the polynomial dichotomy and the notions of ordinary dichotomy and exponential dichotomy, as well as for some illustrative examples, we refer to Dragičević, Sasu and Sasu [22,23].

Let $h\in\mathbb{N}, h\geq 3$ be fixed. We define

$$(2.1)\qquad B(n):X\to X,\quad B(n)=\Phi_A(h^n,h^{n-1})$$

and we consider the discrete system

$$(B)\qquad x(n+1)=B(n)x(n),\quad n\in\mathbb{N}.$$

Then the associated propagator $\Phi_B=\{\Phi_B(n,j)\}_{(n,j)\in\Lambda}$ has the property that

$$(2.2)\qquad \Phi_B(n,j)=\Phi_A(h^{n-1},h^{j-1}),\quad\forall(n,j)\in\Lambda.$$

We associate to the systems (A) and (B) the input-output systems

$$(S_A) \qquad y(n+1) = A(n)y(n) + s(n+1), \quad n \in \mathbb{N}$$

and

$$(S_B) \qquad z(n+1) = B(n)z(n) + s(n+1), \quad n \in \mathbb{N}$$

with $s \in \ell^1(\mathbb{N}, X)$ as input sequence and $y, z \in \ell^\infty(\mathbb{N}, X)$ as output sequences.

The following two criteria obtained in [23] will play a key role in our method.

Proposition 2.1. *If (A) has a polynomial dichotomy with the projections $\{P(j)\}_{j\in\mathbb{N}}$, then the following properties hold:*

- (i) *for each $s \in \ell^1_0(\mathbb{N}, X)$ there exists a unique $y_s \in \ell^\infty(\mathbb{N}, X)$ with $y_s(1) \in Ker P(1)$ such that (y_s, s) satisfies (S_A);*
- (ii) *for each $s \in \ell^\infty_0(\mathbb{N}, X)$ there exists a unique $z_s \in \ell^\infty(\mathbb{N}, X)$ with $z_s(1) \in Ker P(1)$ such that (z_s, s) satisfies (S_B).*

Proof. This follows from the necessity part in the proof of Theorem 4.2 from [23]. □

Theorem 2.1. *(A) has a polynomial dichotomy if and only if there is a closed subspace $Y \subset X$ such that the following properties hold:*

- (i) *for each $s \in \ell^1_0(\mathbb{N}, X)$ there exists a unique $y_s \in \ell^\infty(\mathbb{N}, X)$ with $y_s(1) \in Y$ such that (y_s, s) satisfies (S_A);*
- (ii) *there is $r \in (1, \infty]$ such that for each $s \in \ell^r_0(\mathbb{N}, X)$ there exists a unique $z_s \in \ell^\infty(\mathbb{N}, X)$ with $z_s(1) \in Y$ and (z_s, s) satisfies (S_B).*

Proof. We refer to Theorem 4.2 in [23]. □

3. A ROBUSTNESS THEOREM FOR UNIFORM POLYNOMIAL DICHOTOMY

In this section we present a new robustness result which shows that the polynomial dichotomy is preserved provided that the initial nonautonomous system is subjected to suitable perturbations. The approach follows up to a point our method from [23], but it is built on new estimates and, thus, it considerably extends and improves the previous robustness results for polynomial dichotomy.

Let now $\{D(n)\}_{n\in\mathbb{N}}$ be an arbitrary family of bounded linear operators on X. Consider the perturbed system

$$(A+D) \qquad x(n+1) = (A(n) + D(n))x(n), \quad n \in \mathbb{N}$$

and then we note that the associated propagator Φ_{A+D} satisfies:

$$\Phi_{A+D}(n, j) = \Phi_A(n, j) + \sum_{k=j}^{n-1} \Phi_A(n, k+1)D(k)\Phi_{A+D}(k, j) \tag{3.1}$$

for all $(n, j) \in \Lambda, n \geq j + 1$.

For the next result we need the discrete version of the classical Gronwall's Lemma (see also [12,23,64]):

Lemma 3.1. *Let $j \in \mathbb{N}$ and $M > 0$. If $(a_n)_{n\geq j}$ and $(b_n)_{n\geq j}$ are two nonnegative sequences satisfying $a_j \leq M$ and*

$$a_n \leq M + \sum_{k=j}^{n-1} a_k b_k, \quad \forall n \geq j + 1,$$

then

$$a_n \leq M\, e^{\sum\limits_{k=j}^{n-1} b_k}, \quad \forall n \geq j+1.$$

Proposition 3.1. *Assume that Φ_A has a polynomial growth, i.e. there are $M \geq 1, \omega > 0$ such that*

$$\|\Phi_A(n,j)\| \leq M\left(\frac{n}{j}\right)^{\omega}, \quad \forall (n,j) \in \Lambda. \tag{3.2}$$

If there is $c > 0$ such that

$$\|D(n)\| \leq c \ln \frac{n+1}{n}, \quad \forall n \in \mathbb{N}, \tag{3.3}$$

then

$$\|\Phi_{A+D}(n,j)\| \leq M\left(\frac{n}{j}\right)^{\tilde{\omega}}, \quad \forall (n,j) \in \Lambda$$

where $\tilde{\omega} = \omega + cM$.

Proof. Let $j \in \mathbb{N}$. For $n \geq j$, we set

$$a_n := \left(\frac{j}{n}\right)^{\omega} \|\Phi_{A+D}(n,j)\| \quad \text{and} \quad b_n := cM \ln \frac{n+1}{n}. \tag{3.4}$$

It is obvious that $a_j = 1 \leq M$.

Using (3.1) and the hypotheses (3.2) and (3.3) we observe that

$$\|\Phi_{A+D}(n,j)\| \leq M\left(\frac{n}{j}\right)^{\omega} + \sum_{k=j}^{n-1} M\left(\frac{n}{k+1}\right)^{\omega} c \ln \frac{k+1}{k} \|\Phi_{A+D}(k,j)\|,$$

for all $n \geq j+1$. It follows that

$$\left(\frac{j}{n}\right)^{\omega} \|\Phi_{A+D}(n,j)\| \leq M + \sum_{k=j}^{n-1} \left(\frac{j}{k}\right)^{\omega} \|\Phi_{A+D}(k,j)\| \cdot cM \ln \frac{k+1}{k}, \tag{3.5}$$

for all $n \geq j+1$.

Using (3.5) and by applying Lemma 3.1 for $(a_n)_{n\geq j}$ and $(b_n)_{n\geq j}$ given by (3.4), we obtain that

$$a_n \leq M e^{\sum\limits_{k=j}^{n-1} cM \ln \frac{k+1}{k}} = M e^{cM \ln \frac{n}{j}} = M\left(\frac{n}{j}\right)^{cM}, \quad \forall n \geq j+1. \tag{3.6}$$

From (3.4) and (3.6) it yields

$$\|\Phi_{A+D}(n,j)\| \leq M\left(\frac{n}{j}\right)^{\tilde{\omega}}, \quad \forall n \geq j$$

which completes the proof. □

In all that follows we work under the hypothesis that Φ_A has *a polynomial growth*. Let $M \geq 1$ and $\omega > 0$ be such that

$$\|\Phi_A(n,j)\| \leq M\left(\frac{n}{j}\right)^{\omega}, \quad \forall (n,j) \in \Lambda. \tag{3.7}$$

Remark 3.1. From (3.7) and (2.1) it follows in particular that:

(*i*) $\|A(n)\| = \|\Phi_A(n+1,n)\| \leq M\left(\frac{n+1}{n}\right)^\omega \leq M2^\omega$, for all $n \in \mathbb{N}$;

(*ii*) $\|B(n)\| = \|\Phi_A(h^n, h^{n-1})\| \leq Mh^\omega$, for all $n \in \mathbb{N}$.

In what follows we assume that (A) has *a polynomial dichotomy* with the projections $\{P(j)\}_{j\in\mathbb{N}}$.

The main question is under what kind of perturbations the system $(A + D)$ has a polynomial dichotomy as well. A first answer was given in [23] (see Theorem 5.1 therein). In what follows we present a new (and more general) estimate for the suitable perturbations.

We take

$$Y = KerP(1) \tag{3.8}$$

and let

$$\ell_Y^\infty(\mathbb{N}, X) := \{\delta \in \ell^\infty(\mathbb{N}, X) : \delta(1) \in Y\}.$$

We note that $\ell_Y^\infty(\mathbb{N}, X)$ is a closed subspace of $\ell^\infty(\mathbb{N}, X)$.

Remark 3.2. From Proposition 2.1 we have that the following properties hold:

(*i*) for each $s \in \ell_0^1(\mathbb{N}, X)$ there exists a unique $y_s \in \ell_Y^\infty(\mathbb{N}, X)$ such that (y_s, s) satisfies (S_A);

(*ii*) for each $s \in \ell_0^\infty(\mathbb{N}, X)$ there is a unique $z_s \in \ell_Y^\infty(\mathbb{N}, X)$ such that (z_s, s) satisfies (S_B).

For every $\delta \in \ell_Y^\infty(\mathbb{N}, X)$, let

$$q_\delta : \mathbb{N} \to X, \quad q_\delta(n) = \begin{cases} \delta(n) - A(n-1)\delta(n-1), & n \geq 2 \\ 0, & n = 1 \end{cases}.$$

From Remark 3.1 (i) it follows that $q_\delta \in \ell_0^\infty(\mathbb{N}, X)$.

Let

$$D(Q) := \{\delta \in \ell_Y^\infty(\mathbb{N}, X) : \ q_\delta \in \ell_0^1(\mathbb{N}, X)\}$$

and

$$Q : D(Q) \to \ell_0^1(\mathbb{N}, X), \quad Q(\delta) = q_\delta.$$

Then Q is a closed operator and taking

$$\|\cdot\|_Q : D(Q) \to \mathbb{R}_+, \quad \|\delta\|_Q = \|\delta\|_\infty + \|q_\delta\|_1$$

we have that $(D(Q), \|\cdot\|_Q)$ is a Banach space. Furthermore, from Remark 3.2 (i) we have that Q is invertible.

Lemma 3.2. *We have that*

$$\frac{1}{\|Q^{-1}\|} \leq 1. \tag{3.9}$$

Proof. We observe that

$$\|Q(\delta)\|_1 = \|q_\delta\|_1 \leq \|\delta\|_Q, \quad \forall \delta \in D(Q).$$

This implies that $\|Q\| \leq 1$. Hence, it yields that

$$\frac{1}{\|Q^{-1}\|} \leq \|Q\| \leq 1$$

and this completes the proof. □

From Remark 3.1 (ii) it follows that we can define the operator

$$V : \ell_Y^\infty(\mathbb{N}, X) \to \ell_0^\infty(\mathbb{N}, X),\ (V(\gamma))(n) = \begin{cases} \gamma(n) - B(n-1)\gamma(n-1), & n \geq 2 \\ 0, & n = 1 \end{cases}$$

and this is a bounded linear operator. In addition, from Remark 3.2 (ii) we have that V is invertible.

Lemma 3.3. *We have that*

$$\frac{1}{\|V^{-1}\|} \leq 1 + Mh^\omega. \tag{3.10}$$

Proof. By Remark 3.1 (ii) we deduce that

$$\|(V(\gamma))(n)\| \leq \|\gamma(n)\| + \|B(n-1)\|\,\|\gamma(n-1)\| \leq (1 + Mh^\omega)\|\gamma\|_\infty$$

for all $n \in \mathbb{N}, n \geq 2$, and all $\gamma \in \ell_Y^\infty(\mathbb{N}, X)$. This implies that

$$\|V\| \leq 1 + Mh^\omega.$$

Hence it yields that

$$\frac{1}{\|V^{-1}\|} \leq \|V\| \leq 1 + Mh^\omega$$

and thus the proof is complete. □

We set

$$\alpha := \frac{1}{M^2 h^{\omega+M} (\ln h)\, \|V^{-1}\|}. \tag{3.11}$$

Lemma 3.4. *We have that* $\alpha \in (0, 1)$.

Proof. Since $h \geq 3,$ we observe that

$$h^M \ln h \geq h \ln h > e$$

and then it follows from (3.10) and (3.11) that

$$\alpha \leq \frac{1 + Mh^\omega}{M^2 h^{\omega+M} \ln h} < \frac{1 + Mh^\omega}{eM^2 h^\omega} < 1.$$

□

The central result of this paper is:

Theorem 3.1. *Assume that* $D \in \ell^1(\mathbb{N}, \mathcal{B}(X))$ *and has the property that*

$$\|D\|_1 < \frac{1}{\|Q^{-1}\|}. \tag{3.12}$$

If $c \in (0, \alpha)$ *and*

$$\|D(n)\| \leq c\, \ln \frac{n+1}{n}, \quad \forall n \in \mathbb{N} \tag{3.13}$$

then $(A + D)$ *has a polynomial dichotomy.*

Proof. We apply Theorem 2.1 for $(A + D)$ with Y given by (3.8). With this purpose we present the proof in two steps.

Step 1. Consider

$$y(n+1) = (A(n) + D(n))y(n) + s(n+1), \quad n \in \mathbb{N} \tag{S_{A+D}}$$

and we show that for each $s \in \ell_0^1(\mathbb{N}, X)$ there is a unique $y_s \in \ell_Y^\infty(\mathbb{N}, X)$ such that (y_s, s) satisfies (S_{A+D}).

For each $\delta \in D(Q)$, consider

$$w_\delta : \mathbb{N} \to X, \quad w_\delta(n) = \begin{cases} \delta(n) - (A(n-1) + D(n-1))\delta(n-1), & n \geq 2 \\ 0, & n = 1 \end{cases}.$$

Then, we observe that

$$\|w_\delta(n) - q_\delta(n)\| \leq \|D(n-1)\| \cdot \|\delta\|_\infty, \quad \forall n \geq 2 \tag{3.14}$$

and so

$$\begin{aligned} \|w_\delta(n)\| &\leq \|w_\delta(n) - q_\delta(n)\| + \|q_\delta(n)\| \\ &\leq \|D(n-1)\| \cdot \|\delta\|_\infty + \|q_\delta(n)\|, \ \forall n \geq 2. \end{aligned} \tag{3.15}$$

Since $D \in \ell^1(\mathbb{N}, \mathcal{B}(X))$ and $q_\delta \in \ell_0^1(\mathbb{N}, X)$ it yields that $w_\delta \in \ell_0^1(\mathbb{N}, X)$.

Let

$$W : (D(Q), \|\cdot\|_Q) \to \ell_0^1(\mathbb{N}, X), \quad W(\delta) = w_\delta.$$

We deduce from (3.9), (3.12) and (3.15) that

$$\|W(\delta)\|_1 \leq \|D\|_1 \|\delta\|_\infty + \|q_\delta\|_1 \leq \|\delta\|_Q, \quad \forall \delta \in D(Q).$$

It follows that W is bounded.

From (3.14) we deduce that

$$\|(W - Q)(\delta)\|_1 = \|w_\delta - q_\delta\|_1 \leq \|D\|_1 \, \|\delta\|_\infty \leq \|D\|_1 \|\delta\|_Q, \tag{3.16}$$

for all $\delta \in D(Q)$. It follows from (3.16) and (3.12) that

$$\|W - Q\| \leq \|D\|_1 < \frac{1}{\|Q^{-1}\|},$$

and thus W is invertible. Hence, it yields that for every $s \in \ell_0^1(\mathbb{N}, X)$ there is a unique $y_s \in \ell_Y^\infty(\mathbb{N}, X)$ such that (y_s, s) satisfies (S_{A+D}).

Step 2. Let

$$C(n) : X \to X, \quad C(n) = \Phi_{A+D}(h^n, h^{n-1}).$$

Consider the systems

$$x(n+1) = C(n)x(n), \quad n \in \mathbb{N} \tag{C}$$

and respectively

$$z(n+1) = C(n)z(n) + s(n+1), \quad n \in \mathbb{N} \tag{S_C}$$

with $s \in \ell^1(\mathbb{N}, X)$ and $z \in \ell^\infty(\mathbb{N}, X)$.

We show that for every $s \in \ell_0^\infty(\mathbb{N}, X)$ there exists a unique $z_s \in \ell_Y^\infty(\mathbb{N}, X)$ such that (z_s, s) satisfies (S_C).

Let $\tilde{\omega} = \omega + M$. From Lemma 3.4 we have that $\alpha < 1$, so $\tilde{\omega} > \omega + cM$. Then from (3.13), by applying Proposition 3.1 we get that

$$\|\Phi_{A+D}(n, j)\| \leq M \left(\frac{n}{j}\right)^{\tilde{\omega}}, \quad \forall n \geq j. \tag{3.17}$$

From relation (3.17) it yields that

$$\|C(n)\| = \|\Phi_{A+D}(h^n, h^{n-1})\| \leq M h^{\tilde{\omega}}, \quad \forall n \in \mathbb{N}. \tag{3.18}$$

From (3.1), (3.7), (3.13) and (3.17) we obtain

$$\begin{aligned}
\|C(n)-B(n)\| &= \|\Phi_{A+D}(h^n,h^{n-1})-\Phi_A(h^n,h^{n-1})\| \\
&\le \sum_{k=h^{n-1}}^{h^n-1} \|\Phi_A(h^n,k+1)\|\,\|D(k)\|\,\|\Phi_{A+D}(k,h^{n-1})\| \\
&\le cM^2 \sum_{k=h^{n-1}}^{h^n-1} \left(\frac{h^n}{k+1}\right)^{\omega} \ln\frac{k+1}{k}\left(\frac{k}{h^{n-1}}\right)^{\tilde\omega} \\
&\le cM^2 \sum_{k=h^{n-1}}^{h^n-1} \ln\frac{k+1}{k}\left(\frac{h^n}{k+1}\right)^{\tilde\omega}\left(\frac{k}{h^{n-1}}\right)^{\tilde\omega} \\
&< cM^2 h^{\tilde\omega} \sum_{k=h^{n-1}}^{h^n-1} \ln\frac{k+1}{k} \\
&= cM^2 h^{\tilde\omega}\ln h, \qquad \forall n\in\mathbb{N}.
\end{aligned} \tag{3.19}$$

From (3.18), it follows that $T:\ell_Y^\infty(\mathbb{N},X)\to\ell_0^\infty(\mathbb{N},X)$ defined by

$$(T(\gamma))(n)=\begin{cases}\gamma(n)-C(n-1)\gamma(n-1), & n\ge 2\\ 0, & n=1\end{cases}$$

is a well-defined. Moreover, it is a bounded linear operator.

From (3.19) we deduce that

$$\|(T-V)(\gamma)(n)\|\le\|C(n-1)-B(n-1)\|\cdot\|\gamma(n-1)\|<cM^2h^{\tilde\omega}\ln h\,\|\gamma\|_\infty, \tag{3.20}$$

for all $n\ge 2$ and all $\gamma\in\ell_Y^\infty(\mathbb{N},X)$. From (3.20) it yields

$$\|T-V\|\le cM^2h^{\tilde\omega}\ln h. \tag{3.21}$$

Since $c<\alpha$, it follows from (3.11) and (3.21) that

$$\|T-V\|<\frac{1}{\|V^{-1}\|}.$$

This implies that T is invertible. Thus, we find that for each $s\in\ell_0^\infty(\mathbb{N},X)$ there exists a unique $z_s\in\ell_Y^\infty(\mathbb{N},X)$ such that (z_s,s) satisfies (S_C).

Using the conclusions of *Step 1* and *Step 2*, from Theorem 2.1 applied for $(A+D)$ we get that $(A+D)$ has a polynomial dichotomy. □

Remark 3.3. Let us compare Theorem 3.1 with Theorem 5.1 in [23]. In order to do so, we recall that in Theorem 5.1 from [23] it is proved that $(A+D)$ has a polynomial dichotomy provided that there exist a sufficiently small $c>0$ and some $\rho>1$ such that

$$\|D(n)\|\le\frac{c}{(n+1)^\rho}\quad n\in\mathbb{N}. \tag{3.22}$$

Clearly, since

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)^\rho}<+\infty$$

we have that (3.22) implies (3.12) provided that c is sufficiently small. In addition, (3.22) implies (3.13). Hence, by Theorem 3.1 we deduce that $(A+D)$ has a polynomial dichotomy. Thus, Theorem 3.1 extends Theorem 5.1 in [23].

In addition, Theorem 3.1 gives an upper bound for the constants c under which the polynomial dichotomy of the perturbed system $(A+D)$ is preserved.

On the other hand, in order to show that Theorem 3.1 generalizes Theorem 5.1 in [23], we give an illustrative example:

Example 3.1. Let D be such that

$$\|D(n)\| \leq \begin{cases} c\ln\dfrac{n+1}{n}, & n \in \{k^2 : k \in \mathbb{N}\} \\ \dfrac{c}{(n+1)[\ln(n+1)]^2}, & n \notin \{k^2 : k \in \mathbb{N}\} \end{cases} \tag{3.23}$$

where $c > 0$.

Then, for c sufficiently small, we can apply Theorem 3.1 and thus, we deduce that $(A+D)$ has a polynomial dichotomy.

But, clearly, Theorem 5.1 in [23] is not applicable to this class of perturbations.

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