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Asymptotically $\alpha-$hemicontractive mappings in Hilbert spaces and a new algorithm for solving associated split common fixed point problem

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ABSTRACT. We introduce a novel class of asymptotically $\alpha-$hemicontractive mappings and demonstrate its relationship with the existing related families of mappings. We establish certain interesting properties of the fixed point set of the new class of mappings. Furthermore, we propose and investigate a new iterative algorithm for solving split common fixed point problem for the new class of mappings. In particular, weak and strong convergence theorems for solving split common fixed point problem for our new class of mappings in Hilbert spaces are proved. Moreover, using our method, we require no prior knowledge of norm of the transfer operator. The results presented in the paper extend and improve the results of Censor and Segal [Censor, Y.; Segal, A. The split common fixed point problem for directed operators. *J. Convex Anal.* **16** (2009), no. 2, 587–600.], Moudafi [Moudafi, A. The split common fixed-point problem for demicontractive mappings. *Inverse Problems* **26** (2010), no. 5:055007.; Moudafi, A. A note on the split common fixed-point problem for quasi-nonexpansive operators. *Nonlinear Anal.* **74** (2011), no. 12, 4083–4087.], Chima and Osilike [Chima, E. E.; Osilike, M. O. Split common fixed point problem for class of asymptotically hemicontractive mappings. *J. Nigerian Math. Soc.* **38** (2019), no. 3, 363–390.], Fan *et al* [Fan, Q.; Peng, J.; He, H. Weak and strong convergence theorems for the split common fixed point problem with demicontractive operators. *Optimization* **70** (2021), no. 5-6, 1409–1423.] and host of other related results in literature.

1. INTRODUCTION

The split common fixed point problem $(SCFPP)$ in finite-dimensional spaces was first introduced by Censor and Segal [13] as a generalisation of the split feasibility problem (SFP). The (SFP) has received a robust attention due to its extensive applications in various disciplines such as image restoration, computer tomograph, and radiation therapy treatment planning see [9, 10, 11] and references therein. In order to formulate (SCFPP) in Hilbert spaces, H_1 and H_2, Censor and Segal [13] assumed that T and U are directed mappings with nonempty fixed point sets, $F(T)$ and $F(U)$ respectively so that $T : H_1 \to H_1$ and $U : H_2 \to H_2$. While, $A : H_1 \to H_2$ is a bounded linear operator. Then, as presented in Censor and Segal [13], the split common fixed point problem $(SCFPP)$ is formulated as follows:

$$Find\ an\ element\ x^* \in F(T)\ such\ that\ Ax^* \in F(U). \tag{1.1}$$

The split feasibility problem (SFP), introduced by Censor and Elfving [12], is to find a point

$$x \in C\ such\ that\ Ax \in Q. \tag{1.2}$$

Where C and Q are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively and $A : H_1 \to H_2$ is a bounded linear operator. Clearly, $(SCFPP)$ reduces

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to (SFP) whenever fixed point sets, $F(T)$ and $F(S)$ in (1.1) are closed and convex. Many researchers have studied SFP as well as its generalisations. In his own contribution, Xu [36] stated that x^* is a solution of the SFP (1.2) if and only if x^* is a solution of the fixed point equation

$$P_C[I - \gamma A^*(I - P_Q)A]x^* = x^*. \tag{1.3}$$

This suggests that we can use fixed point algorithms to solve SFP as well as its generalisation, $(SCFPP)$. On their part, at the point of introducing $(SCFPP)$, Censor and Segal [13] studied, in finite-dimensional spaces, the convergence of the following algorithm for directed operators U and T:

$$x_{n+1} = U(x_n + \gamma A^t(T - I)Ax_n), n \geq 1. \tag{1.4}$$

Where $\gamma \in (0, \frac{2}{\lambda})$, with λ being the largest eigenvalue of the matrix A^tA (A^t stands for matrix transposition) and they proved that the sequence $\{x_n\}$ weakly converges to a solution of the SCFPP (1.1) (if any).

Some improvements have been obtained, which mainly focus on the extension of the family of the operators U and T as in [2, 16, 19, 23, 24, 26] and the ease with which the associated algorithm is implemented as in [14, 21, 35, 37].

We also notice that the choice of the stepsize γ in the above algorithm (1.4) actually depends on the operator norm, $\|A\|$ of A, the calculation of which is not always an easy task. In order to overcome the constraint of computing or estimating the operator norm, many researchers have developed substitute algorithms for that of Censor and Segal [13]. In particular, Cui and Wang [17] proposed the following algorithm:

$$x_{n+1} = U_\lambda[x_n - \rho_n A^*(I - T)Ax_n],\ n \geq 0. \tag{1.5}$$

Where $U_\lambda = (1-\lambda)I + \lambda U,\ \lambda \in (0, 1-\kappa)$, U is a κ-demicontractive operator with $\kappa < 1$ and T is a τ-demicontractive operator with $\tau < 1$. Weak convergence of the algorithm (1.5) was proved in Cui and Wang [17] with the step size ρ_n being chosen in the following way, which is not dependent on the operator norm $\|A\|$ of A:

$$\rho_n = \begin{cases} (I-\tau)||(I-T)Ax_n||^2, & Ax_n \neq T(Ax_n); \\ 0, & otherwise. \end{cases} \tag{1.6}$$

In our novel contribution which is dual in purpose, we extend the work of Censor and Segal [13] to the class of asymptotically $\alpha-$hemicontractive mappings without requiring computation or estimation of operator norm. Our new family of mappings is shown to be a generalisation of asymptotically hemicontractive mappings and all its subclasses while our new iterative scheme is shown to converge weakly and strongly to the solution of (1.1) under mild conditions.

2. Preliminaries and notations

Here, we recall some relevant definitions and lemmas which will be needed in the proof of our main results. In what shall follow, we denote strong and weak convergence by "$\rightarrow$" and "$\rightharpoonup$" respectively, the fixed point set of a mapping T by $F(T)$ and the solution set of (1.1) by Γ.

Definition 2.1. [Demiclosedness principle] Let H be a real Hilbert space and $T : H \rightarrow H$ be a mapping, then $(I - T)$ is said to be demiclosed at zero (see for example Browder [7]) if for any sequence, $\{x_n\} \subset H$ with $x_n \rightharpoonup x^*$ and $(I - T)x_n \rightarrow 0$, we have $x^* = Tx^*$.

Definition 2.2. Let H be a real Hilbert space and $T : H \rightarrow H$ be a mapping, then a single valued mapping $T : H \rightarrow H$ is said to be *semicompact* (see for example Petryshyn [32]) if for any bounded sequence $\{x_n\} \subset H$ with $\|(I-T)x_n\| \to 0$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to a point $p \in H$.

Definition 2.3. Let H be a real Hilbert space with $C \subset H$ being nonempty then a mapping $T : C \to C$ is said to be nonexpansive, see Browder [6], if

$$\|Tx - Ty\| \leq \|x - y\| \;\forall\, x, y \in C \;and\; all\; n \geq 1. \tag{2.7}$$

Definition 2.4. Let $(X, \|.\|)$ be a linear normed space. A mapping $T : X \to X$ is said to be an enriched nonexpansive mapping (see Berinde [4]) if there exists $b \in [0, \infty)$ such that,

$$\|b(x-y) + Tx - Ty\| \leq (b+1)\|x - y\| \;\forall\, x, y \in C \;and\; n \geq 1. \tag{2.8}$$

Definition 2.5. Let H be a real Hilbert space with $C \subset H$ being nonempty then a mapping $T : C \to C$ is said to be asymptotically nonexpansive, see Goebel and Kirk [20], if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n\|x - y\| \;\forall\, x, y \in C \;and\; all\; n \geq 1. \tag{2.9}$$

Definition 2.6. Let H be a real Hilbert space and C be a nonempty subset of H. A mapping $T : C \to C$ is said to be uniformly $L-$Lipschitzian, if there exists a constant $L \geq 0$ such that for all $(x, y) \in C \times C$,

$$\|T^n x - T^n y\| \leq L\|x - y\|. \tag{2.10}$$

Definition 2.7. Let H be a real Hilbert space with $C \subset H$ being nonempty then a mapping $T : C \to C$ is said to be κ-strictly pseudo-contractive, see Browder and Petryshyn [8] and references therein, if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \;\leq\; \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2 \;\forall x, y \in C. \tag{2.11}$$

Definition 2.8. Let H be a real Hilbert space with $C \subset H$ being nonempty then a mapping $T : C \to C$ is said to be α-demi-contractive, see Măruşter and Măruşter [22] if $F(T) \neq \emptyset$ and there exist $\alpha \geq 1$ and a constant $\kappa \in [0, 1)$ such that

$$\|Tx - \alpha p\|^2 \;\leq\; \|x - \alpha p\|^2 + k\|x - Tx\|^2 \;\forall (x, p) \in C \times F(T). \tag{2.12}$$

Clearly, (2.12) is equivalent to

$$\langle x - Tx, x - \alpha p\rangle \geq \lambda\|x - Tx\|^2 \;\forall (x, p) \in C \times F(T) \;where\; \lambda = \frac{1-\kappa}{2} > 0. \tag{2.13}$$

Definition 2.9. Let H be a real Hilbert space with $C \subset H$ being nonempty then a mapping $T : C \to C$ is said to be pseudo-contractive, see Browder and Petryshyn [8], if

$$\|Tx - Ty\|^2 \;\leq\; \|x - y\|^2 + \|x - Tx - (y - Ty)\|^2 \;\forall x, y \in C. \tag{2.14}$$

It is demonstrated in Rhoades [34] that the class of κ-strictly pseudo-contractive mappings is a proper subclass of pseudo-contractive mappings.

Definition 2.10. Let H be a real Hilbert space with $C \subset H$ being nonempty then a mapping $T : C \to C$ is said to be hemi-contractive, see Naimpally and Singh [25] and the references therein, if $F(T) \neq \emptyset$ and

$$\|Tx - p\|^2 \;\leq\; \|x - p\|^2 + \|x - Tx\|^2 \;\forall (x, p) \in C \times F(T). \tag{2.15}$$

Clearly, (2.15) is equivalent to

$$\langle x - Tx, x - p\rangle \geq 0 \;\forall (x, p) \in C \times F(T). \tag{2.16}$$

Definition 2.11. Let H be a real Hilbert space with $K \subset H$ being nonempty then a mapping $T : K \to K$ is said to be a generalised pseudo-contraction, see Berinde [3], if for all $x, y \in K$ there exists a constant $r > 0$ such that

$$\langle Tx - Ty, x - y\rangle \;\le\; r\|x - y\|^2 \;\forall x, y \in K. \tag{2.17}$$

Equivalently, a mapping $T : K \to K$ is said to be a generalised pseudo-contraction, if for all $x, y \in K$ there exists a constant $r > 0$ such that

$$\|Tx - Ty\|^2 \;\le\; r^2\|x - y\|^2 + \|Tx - Ty - r(x - y)\|^2 \tag{2.18}$$

Remark 2.1. It is obvious that if $r = 1$ in (2.18) then, the class of generalised pseudo-contractions coincided with the class of pseudo-contractive mappings. Thus, the class of pseudo-contractive mappings together with all its subfamilies including family of κ-strictly pseudo-contractive mappings is a proper subclass of generalised pseudo-contractions.

It can easily be shown that the class of enriched nonexpansive mappings is a subclass of generalised pseudo-contractions. For an arbitrary enriched nonexpansive mapping, T with the associated $b > 0$, we have that

$$\|b(x - y) + Tx - Ty\|^2 \le (b + 1)\|x - y\|^2.$$

This in turn implies that $b\|x - y\|^2 + \|Tx - Ty\|^2 + 2b\langle Tx - Ty, x - y\rangle \le (b + 1)\|x - y\|^2$. Hence,

$$\langle Tx - Ty, x - y\rangle \le \frac{1}{2b}\|x - y\|^2$$

which is a generalised pseudo-contraction with $r = \frac{1}{2b}$. The case of enriched nonexpansive mappings, T with the associated $b = 0$ coincides with nonexpansive mappings which is a subclass of generalised pseudo-contractions.

Definition 2.12. Let H be a real Hilbert space with $C \subset H$ being nonempty then a mapping $T : C \to C$ is said to be $\alpha-$hemicontractive, see Osilike and Onah [30], if $F(T) \neq \emptyset$ and there exist $\alpha \ge 1$ such that

$$\|Tx - \alpha p\|^2 \le \|x - \alpha p\|^2 + \|x - Tx\|^2 \;\forall (x, p) \in C \times F(T). \tag{2.19}$$

Equivalently, a mapping $T : C \to C$ is said to be $\alpha-$hemicontractive if $F(T) \neq \emptyset$ and there exist $\alpha \ge 1$ such that

$$\langle x - Tx, x - \alpha p\rangle \ge 0 \;\forall (x, p) \in C \times F(T). \tag{2.20}$$

For additional information, you may see also [1, 2, 18, 26, 27, 28]

Definition 2.13. Let H be a real Hilbert space with $C \subset H$ being nonempty then a mapping $T : C \to C$ is said to be asymptotically hemicontractive, see Qihou [33], if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - p\|^2 \le \kappa_n\|x - p\|^2 + \|x - T^n x\|^2 \;\forall (x, p) \in C \times F(T). \tag{2.21}$$

Equivalently, a mapping $T : C \to C$ is said to be asymptotically hemicontractive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that any of the following equivalent equations are satisfied:

$$\langle T^n x - x, x - p\rangle \le \frac{\kappa_n - 1}{2}\|x - p\|^2 \;\forall (x, p) \in C \times F(T) \text{ or} \tag{2.22}$$

$$\langle x - T^n x, x - p\rangle \ge -\frac{\kappa_n - 1}{2}\|x - p\|^2 \;\forall (x, p) \in C \times F(T). \tag{2.23}$$

Definition 2.14. Opial property, see Opial [29]. A Banach space E is said to have the Opial property if for any sequence $\{x_n\}$ with $x_n \rightharpoonup x^*$, we have

$$\liminf_{n\to\infty} \|x_n - x^*\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.24}$$

for all $y \in E$ with $y \neq x^*$. It is known that each Hilbert space possess the Opial property.

We shall need the following well-known results in Hilbert spaces (see for example Chidume [15]):

Lemma 2.1. *Let H be a real Hilbert space. Then,for all $x, y \in H$, we have that*

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2.$$

Lemma 2.2. *Let H be a real Hilbert space. Then, for all $x, y \in H$ and for all $t \in [0, 1]$, we have*

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2.$$

Lemma 2.3. *(Osilike and Igbokwe [31]) Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 + c_n)a_n + b_n,\ \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then the limit $\lim_{n\to\infty} a_n$ exists.

To the best of our knowledge, nothing has been reported about the class of asymptotically $\alpha-$hemicontractive mappings. Thus, the following questions arise naturally.
question 1: Can one introduce the concept of asymptotically $\alpha-$hemicontractive mappings using terminologies that are in conformity with the existing standard?
question 2: Can one display illustrative examples to show the relationship existing among the new class of mappings and already existing related families of mappings?
question 3: Can one establish the solution of split common fixed point problem for the new class of mappings even when prior knowledge of operator norm is not required?
question 4: Is there any numerical example to demonstrate such solutions established?
Inspired and motivated by the above innovations as well as the above questions raised, we introduce in this paper, a new family of mappings, asymptotically $\alpha-$hemicontractive mappings which is more general than the class of asymptotically hemicontractive mappings in Hilbert space. Illustrative examples given here show that our new class of mappings is independent on the closely related class of asymptotically hemicontractive mappings when $\alpha > 1$.

3. Main results

Definition 3.15. Let H be a real Hilbert space with $C \subset H$ being nonempty then a mapping $T : C \to C$ is said to be asymptotically $\alpha-$hemicontractive if $F(T) \neq \emptyset$ and there exist $\alpha \geq 1$ and a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - \alpha p\|^2 \leq k_n\|x - \alpha p\|^2 + \|x - T^n x\|^2 \ \forall (x, p) \in C \times F(T). \tag{3.25}$$

Equivalently, a mapping $T : C \subset H \to C$ is said to be asymptotically $\alpha-$hemicontractive if $F(T) \neq \emptyset$ and there exist $\alpha \geq 1$ and a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that any of the following equivalent equations are satisfied:

$$\langle T^n x - x, x - \alpha p\rangle \leq \frac{\kappa_n - 1}{2}\|x - \alpha p\|^2 \ \forall (x, p) \in C \times F(T);\ \textit{or} \tag{3.26}$$

$$\langle x - T^n x, x - \alpha p\rangle \geq -\frac{\kappa_n - 1}{2}\|x - \alpha p\|^2 \ \forall (x, p) \in C \times F(T). \tag{3.27}$$

Remark 3.2. From, (2.7), (2.13) and (3.25), it can easily be deduced that, for $\alpha = 1$, the class of asymptotically quasi nonexpansive mappings is a proper subclass of the class of asymptotically $\alpha-$hemicontractive mappings which, in this case, coincides with the class of asymptotically hemicontractive mappings. However, if $\alpha > 1$, then the class of asymptotically $\alpha-$hemicontractive mappings and the class of asymptotically hemicontractive mappings are independent. Below are illustrative examples for verification of their independence.

Example 3.1. Here is an example of an asymptotically hemicontractive mapping which is not asymptotically $\alpha-$hemicontractive for any $\alpha > 1$. Let $C = \left[\frac{1}{2}, 2\right]$; $C_1 = \left[\frac{1}{2}, 1\right]$ and $C_2 = [1, 2]$. Define $T : C \to C$ by

$$Tx = \frac{1}{x} \tag{3.28}$$

with $F(T) = \{1\}$. Clearly, $\frac{1}{2} \leq T^n x \leq 2 \; \forall \, x \in \left[\frac{1}{2}, 2\right]$. Thus, T is a self map.

Case 1 Suppose that $x \in \left[\frac{1}{2}, 1\right]$, then, we show that T is asymptotically hemicontractive as follows. If n is odd, then, $T^n x = \frac{1}{x} \in [1, 2]$. Hence,

$$\begin{aligned} x - T^n x &= x - \frac{1}{x} \\ &= \frac{(x-1)(x+1)}{x} \\ &\leq 0. \end{aligned} \tag{3.29}$$

$$While, \; x - p = x - 1 \leq 0.$$

Thus, for $\{\kappa_n\} = \{1\}$, we have,

$$\langle x - T^n x, x - p \rangle \geq 0 \geq -\frac{\kappa_n - 1}{2} \|x - p\|^2.$$

If n is even then, $T^n x = x \in \left[\frac{1}{2}, 1\right]$ and $x - T^n x = 0$. Hence, for $\kappa_n = \{1\}$, we have

$$\langle x - T^n x, x - p \rangle = \frac{\kappa_n - 1}{2} \|x - p\|^2.$$

Case 2: If $x \in [1, 2]$ and n is odd. Then, we show that T is asymptotically hemicontractive as follows. $T^n x = \frac{1}{x} \in \left[\frac{1}{2}, 1\right]$. Hence,

$$\begin{aligned} x - T^n x &= x - \frac{1}{x} \\ &= \frac{(x-1)(x+1)}{x} \\ &\geq 0. \end{aligned} \tag{3.30}$$

$$While, \; x - p = x - 1 \geq 0.$$

Thus, for $\{\kappa_n\} = \{1\}$, we have,

$$\langle x - T^n x, x - p \rangle \geq 0 \geq -\frac{\kappa_n - 1}{2} \|x - p\|^2.$$

If n is even then, $T^n x = x \in \left[\frac{1}{2}, 1\right]$ and $x - T^n x = 0$. Hence, for $\{\kappa_n\} = \{1\}$, we have

$$\langle x - T^n x, x - p \rangle = \frac{\kappa_n - 1}{2} \|x - p\|^2.$$

Therefore, T is asymptotically hemicontractive indeed.

Next, we show that T is not asymptotically $\alpha-$hemicontractive when $\alpha > 1$. Suppose by contradiction that there exist $\kappa_n \subseteq [1, \infty)$ and $\alpha > 1$ for which T is asymptotically

$\alpha-$hemicontractive. Then, choose $\epsilon = \frac{1}{4(\alpha-1)}$ so that $\kappa_n \to 1 \; as \; n \to \infty$ implies that there exists $n^* \in \mathbb{N}$ such that

$$|\kappa_n - 1| < \epsilon \, \forall n \geq n^*.$$

Thus, for $x = 1 + \frac{\alpha-1}{\alpha} \in \left[\frac{1}{2}, 2\right]$, $1 < x < 2$ is satisfied.
Then, for all odd numbers greater than or equal to n^*, we have that

$$\begin{aligned}\langle T^n x - x, x - \alpha p\rangle &= \left\langle \frac{-[(\alpha-1)(3\alpha-1)]}{\alpha(2\alpha-1)}, \frac{-(\alpha-1)^2}{\alpha} \right\rangle \\ &= \frac{(\alpha-1)^3(3\alpha-1)}{\alpha^2(2\alpha-1)} \\ &> \frac{\epsilon}{2}\|x-\alpha p\|^2 \\ &> \frac{\kappa_n - 1}{2}\|x - \alpha p\|^2.\end{aligned}$$

Thus , we have that

$$\langle T^n x - x, x - \alpha p\rangle > \frac{\kappa_n - 1}{2}\|x - \alpha p\|^2. \tag{3.31}$$

Therefore,from above inequality and from (3.26), the fact that T is not asymptotically $\alpha-$hemicontractive is established.

Remark 3.3. Example (3.1) was also shown to be a generalised pseudocontraction in Berinde [3], enriched nonexpansive in Berinde [4] and strictly pseudocontractive in Berinde and Berinde [5].

Example 3.2. Here is an example of an asymptotically $\alpha-$hemicontractive mapping which is not asymptotically hemicontractive. Let $C = [1, 2]$. Define $T : C \to C$ by

$$Tx = x + (x-1)^2(2-x) \tag{3.32}$$

with $F(T) = \{1, 2\}$. Clearly, $(x-1)^2 \leq 1 \, \forall x \in C, Tx \geq x, \; \forall x \in C$ and $\{T^n x\}$ is monotone increasing. Hence,

$$\begin{aligned}1 &\leq T^n x \\ &= x + (x-1)^2(2-x) \\ &< x + (2-x) \\ &= 2.\end{aligned}$$

Thus, T is a self map and so $T^n x \in C \, \forall x \in C$ and $\forall n \geq 1$.
Suppose that $\alpha = 2$ then $\forall p \in F(T)$, we have that $x - \alpha p \leq 0$. From the definition of T, it follows that $x \leq Tx$ which in turn implies that $x \leq T^n x$ and $x - T^n x \leq 0$. Thus, for $\kappa_n = \{1\}$, we have that

$$\langle x - T^n x, x - \alpha p\rangle \geq 0 \geq -\frac{\kappa_n - 1}{2}\|x - \alpha p\|^2$$

is satisfied $\forall (x, p) \in C \times F(T)$ and so T is asymptotically $\alpha-$hemicontractive.
However, T is not asymptotically hemicontractive because for arbitrary $\kappa_n \subset [1, \infty)$ with $\kappa_n \to 1 \; as \; n \to \infty$. Choose, $\epsilon = 0.01$ so that $\kappa_n \to 1 \; as \; n \to \infty$ implies that there exists $n^* \in \mathbb{N}$ such that

$$|\kappa_n - 1| < \epsilon \, \forall n \geq n^*.$$

Thus, for $x = 1.9 \in [1, 2]$ $p = 1$ then, from the definition of T, it follows that $x \leq Tx \leq T^n x$. Hence, for all odd numbers greater than or equal to n^*, we have that

$$\begin{aligned} \langle T^n x - x, x - p\rangle &\geq \langle Tx - x, x - p\rangle \\ &= \langle 0.081, 0.9\rangle \\ &= 0.0729 \\ &> 0.00405 \\ &= \frac{\epsilon}{2}(0.81) \\ &> \frac{\kappa_n - 1}{2}\|x - p\|^2. \end{aligned}$$

Thus, we have that

$$\langle T^n x - x, x - p\rangle > \frac{\kappa_n - 1}{2}\|x - p\|^2. \tag{3.33}$$

Therefore, from (3.33) and from (2.14), the fact that T is not asymptotically hemicontractive is established.

Example 3.3. Here is an example of an asymptotically $\alpha-$hemicontractive mapping which is not α-hemicontractive. Let $X = \ell_2$. For, each $\bar{x} = (x_1, x_2, x_3, \cdots) \in X$, define $T : X \to X$ by

$$T(x_1, x_2, x_3, \cdots) = (0, \rho_1 x_1, \rho_2 x_2, \rho_3 x_3, \cdots). \tag{3.34}$$

where $\{\rho_k\}_{k\geq 1}$ is a sequence of real numbers defined for each $k \in \mathbb{N}$ by $\rho_k = \begin{cases} 2, & k = 1 \\ 1 - \frac{1}{k^2}, & k > 1. \end{cases}$

From (3.34),

$$\begin{aligned} &T(x_1, x_2, x_3, \cdots) = (0, 2x_1, \rho_2 x_2, \rho_3 x_3, \cdots), \\ &T^2(x_1, x_2, x_3, \cdots) = (0, 0, 2\rho_2 x_1, \rho_2\rho_3 x_2, \rho_3\rho_4 x_3, \cdots), \\ &T^3(x_1, x_2, x_3, \cdots) = (0, 0, 0, 2\rho_2\rho_3 x_1, \rho_2\rho_3\rho_4 x_2, \rho_3\rho_4\rho_5 x_3, \cdots)\ etc. \end{aligned}$$

Clearly, $F(T) = \{(0, 0, 0, 0, \cdots)\}$. Since, $1 - \frac{1}{k^2} = \frac{(k-1)(k+1)}{k^2}$, then

$$\prod_{k=1}^{n} \rho_k = 2 \times \frac{(1)(3)}{2^2} \times \frac{(2)(4)}{3^2} \times \frac{(3)(5)}{4^2} \times \cdots \times \frac{(n-2)(n)}{(n-1)^2} \times \frac{(n-1)(n+1)}{n^2}.$$

Indeed, by induction, it can be established that $\prod_{k=1}^{n} \rho_k = \frac{n+1}{n} = 1 + \frac{1}{n}$ as follows.

For base of induction, $n = 1$, we have

$$\prod_{k=1}^{1} \rho_k = \rho_1 = 2 = 1 + \frac{1}{1} = 1 + \frac{1}{n}.$$

Similarly, for $n = 2$, we have

$$\prod_{k=1}^{2} \rho_k = \rho_1 \times \rho_2 = 2 \times \left(1 - \frac{1}{2^2}\right) = \frac{6}{4} = 1 + \frac{1}{2} = 1 + \frac{1}{n}.$$

For inductive hypothesis, we assume that for $n = j$ with $j > 1$ being an integer, we have

$$\prod_{k=1}^{j} \rho_k = \rho_1 \times \rho_2 \times \rho_3 \times \rho_4 \times \cdots \times \rho_j = 1 + \frac{1}{j}.$$

Finally, it remains to show that for $n = j+1$, we have

$$\prod_{k=1}^{j+1} \rho_k = \rho_1 \times \rho_2 \times \rho_3 \times \rho_4 \times \cdots \times \rho_j \times \rho_{j+1} = 1 + \frac{1}{j+1}.$$

On applying the inductive hypothesis to the above equation, we have

$$\begin{aligned}\prod_{k=1}^{j+1} \rho_k &= \rho_1 \times \rho_2 \times \rho_3 \times \rho_4 \times \cdots \times \rho_j \times \rho_{j+1} \\ &= \left(1 + \frac{1}{j}\right) \times \left(1 - \frac{1}{(j+1)^2}\right) \\ &= \left(\frac{j+1}{j}\right) \times \left(\frac{j^2+2j}{(j+1)^2}\right) \\ &= \left(\frac{j+2}{j+1}\right) \\ &= 1 + \frac{1}{j+1}.\end{aligned}$$

Therefore, $\prod\limits_{k=1}^{n} \rho_k = 1 + \frac{1}{n}\ \forall n \in \mathbb{N}$. Thus, For arbitrary $n \in \mathbb{N}$, and $\bar{x}, \bar{y} \in \ell_2$ we have that

$$\begin{aligned}\|T^n\bar{x} - T^n\bar{y}\|^2 &\leq \left(\prod_{k=1}^{n} \rho_k\right)^2 \|\bar{x} - \bar{y}\|^2 \\ &= \left(1 + \frac{1}{n}\right)^2 \|\bar{x} - \bar{y}\|^2 \\ &= \left(\kappa_n \|\bar{x} - \bar{y}\|\right)^2.\end{aligned}$$

This implies that T is asymptotically nonexpansive with $\kappa_n = \left(1 + \frac{1}{n}\right)$. Since, $F(T) \neq \emptyset$, it follows that T is asymptotically quasi-nonexpansive as well as asymptotically α-hemicontractive.

However, for $\bar{x} = (1, 1, \frac{1}{2}, 0, 0, 0, \cdots) \in \ell_2$, we have, for an arbitrary $\alpha \geq 1$, that

$$\begin{aligned}\langle \bar{x} - T\bar{x}, \bar{x} - \alpha p\rangle &= \langle (1, 1, \frac{1}{2}, 0, 0, 0, \cdots) - (0, 2, \frac{3}{4}, \frac{4}{9}, 0, 0, \cdots), (1, 1, \frac{1}{2}, 0, 0, 0, \cdots) \\ &\quad -(0, 0, 0, \cdots)\rangle \\ &= \left\langle (1, -1, -\frac{1}{4}, -\frac{4}{9}, 0, 0, \cdots), (1, 1, \frac{1}{2}, 0, 0, 0, \cdots) \right\rangle \\ &= 1 - 1 - \frac{1}{8} \\ &= -\frac{1}{8} \\ &< 0.\end{aligned}$$

Therefore, T is not α-hemicontractive.

Example 3.4. Here is an example of an asymptotically $\alpha-$hemicontractive mapping which is not a generalised pseudocontraction. It is neither enriched nonexpansive nor κ-strictly pseudocontractive. In fact, it is not even hemicontractive. Let $X = \mathbb{R} \to \mathbb{R}$. For, each

$x \in X$, define $T : X \to X$ by

$$Tx = \begin{cases} \frac{1}{8-16x}, & x \in \left(-\infty, \frac{1}{2}\right) \\ x, & x \in \left[\frac{1}{2}, \infty\right) \end{cases} \tag{3.35}$$

Clearly, $F(T) = \{\frac{1}{4}\} \cup \left[\frac{1}{2}, \infty\right)$. To prove that T is asymptotically $2-$hemicontractive, we consider the following cases.
Case 1: Suppose that $x \in \left(-\infty, \frac{1}{2}\right)$. Then, $x \leq Tx \leq T^n x \;\forall\; n \in \mathbb{N}$. Thus, $x - T^n x \leq x - Tx = \frac{(4x-1)^2}{16(x-\frac{1}{2})} \leq 0$. While, for arbitrary $p \in F(T)$ and with $\alpha = 2$, we have that $x - \alpha p \leq 0$. Therefore, (3.27) is satisfied as shown below.

$$\langle x - T^n x, x - \alpha p\rangle \geq 0 \geq -\frac{\kappa_n - 1}{2}\|x - \alpha p\|^2$$

Case 2: Suppose that $x \in \left[\frac{1}{2}, \infty\right)$. Then, $x = Tx = T^n x \;\forall\; n \in \mathbb{N}$. Thus, $x - T^n x = x - Tx = 0$. For arbitrary $p \in F(T)$ and with $\alpha = 2$, we have that (3.27) is satisfied as shown below.

$$\langle x - T^n x, x - \alpha p\rangle = 0 \geq -\frac{\kappa_n - 1}{2}\|x - \alpha p\|^2$$

Therefore, T is asymptotically $\alpha-$hemicontractive indeed with $\alpha = 2$.
Next, we show that T is not a generalised pseudocontraction. Following the method used in Rhoades [33], assume for contradiction purpose that T is a generalised pseudocontraction. Then, it suffices to show that $\exists r > 0$ for which (2.17) is satisfied $\forall\; x, y \in \mathbb{R}$. In particular, take $y = 0$ and $x \in \mathbb{R}$ such that $\frac{1}{2} - \frac{1}{8r} < x < \frac{1}{2}$. Hence,

$$\begin{aligned} \langle Tx - Ty, x - y\rangle &= \left\langle \frac{1}{8-16x} - \frac{1}{8}, x - 0\right\rangle \\ &= \frac{1}{8}\left\langle \frac{2x}{1-2x}, x\right\rangle \\ &= \frac{1}{4(1-2x)}(x^2) \\ &> r\|x - y\|^2. \end{aligned}$$

It follows that T is not a generalised pseudocontraction as well as its subclasses including enriched nonexpansive, pseudocontractive and strictly pseudocontractive mappings.
T is not hemicontractive for if $p = \frac{1}{4}$ and $x = \frac{9}{32}$, then

$$\begin{aligned} \langle x - Tx, x - p\rangle &= \left\langle \frac{(4x-1)^2}{16(x-\frac{1}{2})}, x - \frac{1}{4}\right\rangle \\ &= \left\langle -\frac{1}{224}, \frac{1}{32}\right\rangle \\ &< 0. \end{aligned}$$

Remark 3.4. From example (3.1) and example (3.4), it can be deduced that the class of asymptotically $\alpha-$hemicontractive mappings is independent on the class of generalised pseudocontractions together with all its subclasses including enriched nonexpansive and (strictly) pseudocontractive mappings.

Lemma 3.4. *Let C be a closed and convex subset of a real Hilbert space. Let $T : C \to C$ be an arbitrary* uniformly $L-$Lipschitzean asymptotically α-hemicontractive *for some $\alpha \geq 1$. Then, for a mapping $G_n : C \to C$ defined by*

$$G_n(x) = T^n[(1-\beta)x + \beta T^n x], \tag{3.36}$$

where β is a positive constants satisfying, $\beta \in \left(0, \frac{1}{\sqrt{\frac{(k+1)^2}{4} + L^2 + \frac{(k+1)}{2}}}\right)$ *and* $k = \sup_{n\geqslant 1}\{k_n\}$,

then, the following hold

(i) $F(T) = F(G_n)$;

(ii) $\alpha p \in F(T)$ *provided that* $p \in F(T)$ *and* $\alpha p \in D(T)$;

Proof. For (i), we recall that convexity of C guarantees that $G_n x \in C$, $\forall x \in C$. Hence, proof of (i) suffices to show that:

(a) $F(T) \subseteq F(G_n)$ and

(b) $F(G_n) \subseteq F(T)$.

To show that $F(T) \subseteq F(G_n)$, let $p \in F(T)$ be arbitrary. Then, $Tp = p$. Hence,

$$G_n(p) = T^n[(1-\beta)p + \beta T^n p] = p \tag{3.37}$$

This implies that $p \in F(G_n)$ and so $F(T) \subseteq F(G_n)$, as required in (a). Similarly, to show that $F(G_n) \subseteq F(T)$, let $p \in F(G_n)$ be arbitrary. Then, $G_n(p) = T^n[(1-\beta)p + \beta T^n p] = p$. Hence,

$$\begin{aligned} \|p - T^n p\|^2 &= \|G_n(p) - T^n p\|^2 \\ &= \|T^n[(1-\beta)p + \beta T^n p] - T^n p\|^2 \\ &\leq L^2\beta^2\|T^n p - p\|^2 \\ &< (1-2\beta)\|T^n p - p\|^2. \end{aligned} \tag{3.38}$$

This implies that $2\beta\|T^n p - p\|^2 \leq 0$ and so $\|T^n p - p\| = 0\ \forall n \geq 1$. This, in turn, implies that $Tp = p$ which means that $p \in F(T)$. Consequently, $F(G_n) \subseteq F(T)$, as required in (b). Therefore, $F(T) = F(G_n)$ which completes the proof of Lemma (3.4)(i).

For the proof of Lemma (3.4)(ii), it suffices to show that: $\|G_n(\alpha p) - \alpha p\| = 0$, where p is an arbitrary fixed point of T and α is the same as in the definition of T. We have that;

$$\begin{aligned} \|G_n(\alpha p) - \alpha p\|^2 &= \|T^n[(1-\beta)\alpha p + \beta T^n(\alpha p)] - \alpha p\|^2 \\ &\leq \kappa_n\|(1-\beta)\alpha p + \beta T^n(\alpha p) - \alpha p\|^2 \\ &\quad + \|(1-\beta)\alpha p + \beta T^n(\alpha p) - G_n(\alpha p)\|^2 \\ &= \kappa_n\beta^2\|T^n(\alpha p) - \alpha p\|^2 \\ &\quad + \|(1-\beta)(\alpha p - G_n(\alpha p)) + \beta(T^n(\alpha p) - G_n(\alpha p))\|^2 \\ &= \kappa_n\beta^2\|T^n(\alpha p) - \alpha p\|^2 + (1-\beta)\|\alpha p - G_n(\alpha p)\|^2 \\ &\quad + \beta\|T^n(\alpha p) - G_n(\alpha p)\|^2 - \beta(1-\beta)\|T^n(\alpha p) - \alpha p\|^2 \\ &= \beta[\beta(\kappa_n+1) - 1]\|T^n(\alpha p) - \alpha p\|^2 + (1-\beta)\|\alpha p - G_n(\alpha p)\|^2 \\ &\quad + \beta\|T^n(\alpha p) - T^n[(1-\beta)\alpha p + \beta T^n(\alpha p)]\|^2 \\ &\leq \beta[\beta(\kappa_n+1) - 1]\|T^n(\alpha p) - \alpha p\|^2 + (1-\beta)\|\alpha p - G_n(\alpha p)\|^2 \\ &\quad + \beta L^2\|\alpha p - (1-\beta)\alpha p - \beta T^n(\alpha p)]\|^2 \\ &= \beta[\beta(\kappa_n+1) - 1]\|T^n(\alpha p) - \alpha p\|^2 \\ &\quad + (1-\beta)\|\alpha p - G_n(\alpha p)\|^2 + \beta^3 L^2\|T^n(\alpha p) - \alpha p]\|^2 \\ &= (1-\beta)\|\alpha p - G_n(\alpha p)\|^2 - \beta[1 - \beta(\kappa_n+1) - \beta^2 L^2]\|T^n(\alpha p) - \alpha p\|^2 \\ &\leq (1-\beta)\|\alpha p - G_n(\alpha p)\|^2. \end{aligned} \tag{3.39}$$

Hence, $\beta\|\alpha p - G_n(\alpha p)\|^2 \leq 0$ and so $\|\alpha p - G_n(\alpha p)\| = 0$. Therefore, $G_n\alpha p = \alpha p$.

Consequently, from result of Lemma (3.4)(i), it is obvious that for any arbitrary $p \in F(T)$, $\alpha p \in F(T)$, provided that $\alpha p \in D(T)$, completing the proof of Lemma (3.4)(ii). $\square$

Lemma 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a uniformly $L-$ Lipschitzian asymptotically $\alpha-$hemicontractive mapping. Then, the following hold:*

(i) $F(T)$ is closed;
(ii) The line segment joining αp_1 and αp_2 lies in $F(T)$ $\forall p_1, p_2 \in F(T)$ and $\alpha \geq 1$.

Proof. For (i) above, define $\{p_n\}_{n\geq 1} \subseteq F(T)$ such that $p_n \to p$. We prove that $p \in F(T)$.

$$\begin{aligned} \|T^n p - p\| &= \|T^n p - T^n p_n + T^n p_n - p\| \\ &\leq \|T^n p - T^n p_n\| + \|T^n p_n - p\| \\ &\leq L\|p_n - p\| + \|p_n - p\| \\ &= (1+L)\|p_n - p\| \\ &\to 0 \; as \; n \to \infty. \end{aligned} \tag{3.40}$$

Thus, $p \in F(T)$ and $F(T)$ is closed.
Next, for (ii), let $p_1, p_2 \in F(T)$ be arbitrary. It follows from Lemma (3.4) that $\alpha p_1, \alpha p_2 \in F(T)$ also. Define

$$p = \lambda \alpha p_1 + (1-\lambda)\alpha p_2 \tag{3.41}$$

for an arbitrary $\lambda \in [0,1]$. We show that $p \in F(T)$. To this end, we set $G_n(x) := T^n[(1-\beta)x + \beta T^n x]$, where $\beta \in \left(0, \frac{1}{\sqrt{\frac{(k+1)^2}{4} + L^2} + \frac{(k+1)}{2}}\right)$ and $k = \sup_{n\geqslant 1}\{k_n\}$.
Clearly, from Lemma (3.4) and (3.41), we have that

$$\begin{cases} G_n(\alpha p_1) = \alpha p_1; \; G_n(\alpha p_2) = \alpha p_2; \\ \|p - \alpha p_1\| = (1-\lambda)\|\alpha p_1 - \alpha p_2\| \text{ and } \|p - \alpha p_2\| = \lambda\|\alpha p_1 - \alpha p_2\|. \end{cases} \tag{3.42}$$

Furthermore,

$$\begin{aligned} \|G_n p - p\|^2 &= \|p - G_n p\|^2 \\ &= \|\lambda \alpha p_1 + (1-\lambda)\alpha p_2 - G_n p\|^2 \\ &= \|\lambda(\alpha p_1 - G_n p) + (1-\lambda)(\alpha p_2 - G_n p)\|^2 \\ &= \lambda\|\alpha p_1 - G_n p\|^2 + (1-\lambda)\|\alpha p_2 - G_n p\|^2 - \lambda(1-\lambda)\|\alpha p_1 - \alpha p_2\|^2. \end{aligned} \tag{3.43}$$

Moreover,

$$\begin{aligned} \|G_n p - \alpha p_1\|^2 &= \|T^n[(1-\beta)p + \beta T^n p] - \alpha p_1\|^2 \\ &\leq \kappa_n\|(1-\beta)p + \beta T^n p - \alpha p_1\|^2 + \|(1-\beta)p + \beta T^n p - G_n p\|^2 \\ &= \kappa_n\|(1-\beta)(p - \alpha p_1) + \beta(T^n p - \alpha p_1)\|^2 \\ &\quad + \|(1-\beta)(p - G_n p) + \beta(T^n p - G_n p)\|^2 \\ &= \kappa_n(1-\beta)\|p - \alpha p_1\|^2 + \beta\kappa_n\|T^n p - \alpha p_1\|^2 - \beta(1-\beta)\kappa_n\|T^n p - p\|^2 \\ &\quad + (1-\beta)\|p - G_n p\|^2 + \beta\|T^n p - T^n[(1-\beta)p + \beta T^n p]\|^2 \\ &\quad - \beta(1-\beta)\|T^n p - p\|^2 \\ &\leq \kappa_n(1-\beta)\|p - \alpha p_1\|^2 + \beta\kappa_n[\kappa_n\|p - \alpha p_1\|^2 + \|T^n p - p\|^2] \\ &\quad - \beta(1-\beta)\kappa_n\|T^n p - p\|^2 + (1-\beta)\|p - G_n p\|^2 + \beta^3 L^2\|T^n p - p\|^2 \\ &\quad - \beta(1-\beta)\|T^n p - p\|^2 \\ &= \kappa_n[1 + \beta(\kappa_n - 1)]\|p - \alpha p_1\|^2 + (1-\beta)\|p - G_n p\|^2 \\ &\quad - \beta[1 - \beta(\kappa_n + 1) - \beta^2 L^2]\|T^n p - p\|^2. \end{aligned}$$

Thus,

$$\|G_np-\alpha p_1\|^2 \leq \kappa_n[1+\beta(\kappa_n-1)]\|p-\alpha p_1\|^2+(1-\beta)\|p-G_np\|^2. \tag{3.44}$$

Similarly,

$$\|G_np-\alpha p_2\|^2 \leq \kappa_n[1+\beta(\kappa_n-1)]\|p-\alpha p_2\|^2+(1-\beta)\|p-G_np\|^2. \tag{3.45}$$

Substituting (3.44), (3.45) and (3.42) in (3.43), we have

$$\begin{aligned}\|G_np-p\|^2 &\leq \lambda[\kappa_n[1+\beta(\kappa_n-1)]\|p-\alpha p_1\|^2+(1-\beta)\|p-G_np\|^2]\\ &\quad +(1-\lambda)[\kappa_n[1+\beta(\kappa_n-1)]\|p-\alpha p_2\|^2+(1-\beta)\|p-G_np\|^2]\\ &\quad -\lambda(1-\lambda)\|\alpha p_1-\alpha p_2\|^2\\ &= \lambda\kappa_n[1+\beta(\kappa_n-1)](1-\lambda)^2\|\alpha p_1-\alpha p_2\|^2+\lambda(1-\beta)\|p-G_np\|^2\\ &\quad +(1-\lambda)\kappa_n[1+\beta(\kappa_n-1)]\lambda^2\|\alpha p_1-\alpha p_2\|^2\\ &\quad +(1-\beta)(1-\lambda)\|p-G_np\|^2-\lambda(1-\lambda)\|\alpha p_1-\alpha p_2\|^2\\ &= \lambda(1-\lambda)(\kappa_n-1)(1+\beta\kappa_n)\|\alpha p_1-\alpha p_2\|^2+(1-\beta)\|p-G_np\|^2.\end{aligned}$$

Thus,

$$\beta\|G_np-p\|^2 \leq \lambda(1-\lambda)(\kappa_n-1)(1+\beta\kappa_n)\|\alpha p_1-\alpha p_2\|^2. \tag{3.46}$$

Since $k_n \to 1$ as $n\to\infty$, we obtain from (3.46) that

$$\lim_{n\to\infty}\|G_np-p\|=0. \tag{3.47}$$

Also,

$$\begin{aligned}\|T^np-p\| &= \|T^np-G_np+G_np-p\|\\ &\leq \|T^np-G_np\|+\|G_np-p\|\\ &= \|T^np-T^n[(1-\beta)p+\beta T^np]\|+\|G_np-p\|\\ &\leq L\|p-[(1-\beta)p+\beta T^np]\|+\|G_np-p\|\\ &\leq L\beta\|T^np-p\|+\|G_np-p\|.\end{aligned}$$

Thus, $(1-L\beta)\|T^np-p\|\leq\|G_np-p\|$, which implies that

$$\lim_{n\to\infty}\|T^np-p\|=0. \tag{3.48}$$

Therefore, $T^np\to p$ as $n\to\infty$. This in turn implies that

$$p=\lim_{n\to\infty}T^np=T\lim_{n\to\infty}(T^{n-1}p)=Tp. \tag{3.49}$$

Hence, $p\in F(T)$. □

Theorem 3.1. *Suppose H_1 and H_2 are real Hilbert spaces with $A: H_1\to H_2$ being a bounded linear operator. While, $T: H_1\to H_1$ as well as $S: H_2\to H_2$ are uniformly Lipschitzian asymptotically $\alpha-$hemicontractive mapping with the respective sequences $\{c_n\}\subset[1,+\infty)$ and $\{d_n\}\subset[1,+\infty)$ such that $\sum_{n=1}^{\infty}(c_n^2-1)<\infty$ and $\sum_{n=1}^{\infty}(d_n^2-1)<\infty$. Let the Uniformly Lipschitzian constants of T and S be L_1 and L_2 respectively with $L=Max\{L_1,L_2\}$ and let $\kappa_n=Max\{c_n,d_n\}$. Suppose in addition that $I-T$ and $I-S$ are demiclosed at the origin and $\Gamma=\{u\in F(T): Au\in F(S)\}\neq\emptyset$. Then, for arbitrary $u_0\in H_1$ the iterative scheme defined for all $n\in\mathbb{N}$ by*

$$\begin{cases} x_n=u_n+\gamma_nA^*[e_nI+(1-e_n)S^n[(1-b_n)I+b_nS^n]-I]Au_n\\ u_{n+1}=(1-a_n)x_n+a_nT^n[(1-\beta_n)x_n+\beta_nT^nx_n]\end{cases} \tag{3.50}$$

where the following conditions are satisfied:

(i) the step size, $\gamma_n > 0$ *is chosen in such a way that for* $D_n = e_n I + (1-e_n)S^n[(1-b_n)I + b_n S^n]$ *we have for small enough* $\epsilon > 0$ *that*

$$\gamma_n = \begin{cases} t_n \ if \ u_n \notin \Gamma \\ \epsilon, \ otherwise \end{cases} \tag{3.51}$$

Where $t_n \in \left(\epsilon, \frac{(1-e_n)||(D_n - I)Au_n||^2}{||A^*(D_n-I)Au_n||^2}\right)$.

(ii) $\{a_n\}$ *and* $\{\beta_n\}$ *are real sequences in* $[0,1]$ *satisfying* $0 < a \leq a_n \leq \beta_n \leq \beta < 1$ *with* $\beta \in \left(0, \frac{1}{\sqrt{\frac{(k+1)^2}{4} + L^2} + \frac{(k+1)}{2}}\right)$ *and* $0 < e \leq 1 - e_n < b_n \leq b < 1$ *with* $b \in \left(0, \frac{1}{\sqrt{\frac{(k+1)^2}{4} + L^2} + \frac{(k+1)}{2}}\right)$.

While, $k = \sup_{n \geqslant 1} \{k_n\}$.

Then, the sequence $\{u_n\}$ *generated by* (3.50) *converges weakly to a solution of problem* (1.1).

Proof. Firstly, we prove that $\lim_{n\to\infty} \|u_n - p\|$ exists for any $p \in \Gamma$. To this end and for purpose of simplicity, let $G_i = T^i[(1-\beta_n)I + \beta_n T^i]$ and $h_i = S^i[(1-b_n)I + b_n S^i]$. From the assumption, $\Gamma \neq \emptyset$, let $p \in \Gamma$ be arbitrary then $p \in F(T)$ for consistency and $Ap \in F(S)$ which by lemma(3.4) together with linearity of A implies that $\alpha p \in F(T)$ and $A(\alpha p) = \alpha Ap \in F(S)$. Thus, $\alpha p \in \Gamma$ also. Assuming that $u_1 \notin \Gamma$, then, from (3.50) (a) and Lemma (2.2), we have:

$$\begin{aligned}
\|u_{n+1} - \alpha p\|^2 &= \|(1-a_n)x_n + a_n G_n x_n - \alpha p\|^2 \\
&= \|(1-a_n)(x_n - \alpha p) + a_n(G_n x_n - \alpha p)\|^2 \\
&= (1-a_n)\|x_n - \alpha p\|^2 + a_n\|G_n x_n - \alpha p\|^2 - a_n(1-a_n)\|x_n - G_n x_n\|^2 \\
&= (1-a_n)\|x_n - \alpha p\|^2 + a_n\|T^n[(1-\beta_n)I + \beta_n T^n]x_n - \alpha p\|^2 \\
&\quad -a_n(1-a_n)\|x_n - G_n x_n\|^2 \\
&\leq (1-a_n)\|x_n - \alpha p\|^2 + a_n\kappa_n\|[(1-\beta_n)x_n + \beta_n T^n x_n] - \alpha p\|^2 \\
&\quad +a_n\|[(1-\beta_n)x_n + \beta_n T^n x_n - G_n x_n\|^2] - a_n(1-a_n)\|x_n - G_n x_n\|^2 \\
&= (1-a_n)\|x_n - \alpha p\|^2 + a_n\kappa_n\|(1-\beta_n)(x_n - \alpha p) + \beta_n(T^n x_n - \alpha p)\|^2 \\
&\quad +a_n\|(1-\beta_n)(x_n - G_n x_n) + \beta_n(T^n x_n - G_n x_n)\|^2 \\
&\quad -a_n(1-a_n)\|x_n - G_n x_n\|^2 \\
&= (1-a_n)\|x_n - \alpha p\|^2 + a_n\kappa_n(1-\beta_n)\|x_n - \alpha p\|^2 + a_n\beta_n\kappa_n\|T^n x_n - \alpha p\|^2 \\
&\quad -a_n\kappa_n\beta_n(1-\beta_n)\|x_n - T^n x_n\|^2 + a_n(1-\beta_n)\|x_n - G_n x_n\|^2 \\
&\quad +a_n\beta_n\|T^n x_n - G_n x_n\|^2 - a_n\beta_n(1-\beta_n)\|x_n - T^n x_n\|^2 \\
&\quad -a_n(1-\alpha_n)\|x_n - G_n x_n\|^2 \\
&\leq (1-a_n)\|x_n - \alpha p\|^2 + a_n\kappa_n(1-\beta_n)\|x_n - \alpha p\|^2 + a_n\beta_n\kappa_n^2\|x_n - \alpha p\|^2 \\
&\quad +a_n\beta_n\kappa_n\|x_n - T^n x_n\|^2 - a_n\kappa_n\beta_n(1-\beta_n)\|x_n - T^n x_n\|^2 \\
&\quad +a_n(1-\beta_n)\|x_n - G_n x_n\|^2 + a_n\beta_n^3 L^2\|x_n - T^n x_n\|^2 \\
&\quad -a_n\beta_n(1-\beta_n)\|x_n - T^n x_n\|^2 - a_n(1-a_n)\|x_n - G_n x_n\|^2 \\
&= [1 + a_n(\kappa_n - 1)(1 + \beta_n\kappa_n)]\|x_n - \alpha p\|^2 \\
&\quad -a_n\beta_n[1 - \beta_n(\kappa_n + 1) - \beta_n^2 L^2]\|x_n - T^n x_n\|^2 \\
&\quad -a_n(\beta_n - a_n)\|x_n - G_n x_n\|^2. \qquad (3.52)
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
\|h_n(Au_n) - A(\alpha p)\|^2 &\leq \kappa_n\|(1-b_n)Au_n + b_nS^nAu_n - A(\alpha p)\|^2 \\
&\quad +\|(1-b_n)Au_n + b_nS^nAu_n - h_n(Au_n)\|^2 \\
&= \kappa_n\|(1-b_n)(Au_n - A(\alpha p)) + b_n(S^nAu_n - A(\alpha p))\|^2 \\
&\quad +\|(1-b_n)[Au_n - h_n(Au_n)] + b_n[S^nAu_n - h_n(Au_n)]\|^2 \\
&= \kappa_n(1-b_n)\|Au_n - A(\alpha p)\|^2 + \kappa_nb_n\|S^nAu_n - A(\alpha p)\|^2 \\
&\quad -\kappa_nb_n(1-b_n)\|Au_n - S^nAu_n\|^2 + (1-b_n)\|Au_n - h_n(Au_n)\|^2 \\
&\quad +b_n\|S^nAu_n - h_n(Au_n)\|^2 - b_n(1-b_n)\|Au_n - S^nAu_n\|^2 \\
&\leq \kappa_n(1-b_n)\|Au_n - A(\alpha p)\|^2 + b_n\kappa_n[\kappa_n\|Au_n - A(\alpha p)\|^2 \\
&\quad +\|Au_n - S^nAu_n\|^2] - b_n(1-b_n)(1+\kappa_n)\|Au_n - S^nAu_n\|^2 \\
&\quad +(1-b_n)\|Au_n - h_n(Au_n)\|^2 \\
&\quad +b_n\|S^nAu_n - S^n[(1-b_n)I + b_nS^n]Au_n\|^2 \\
&= \kappa_n[1 + b_n(\kappa_n - 1)]\|Au_n - A(\alpha p)\|^2 + b_n\kappa_n\|Au_n - S^nAu_n\|^2 \\
&\quad -b_n(1-b_n)(1+\kappa_n)\|Au_n - S^nAu_n\|^2 \\
&\quad +(1-b_n)\|Au_n - h_n(Au_n)\|^2 \\
&\quad +b_nL^2\|Au_n - [(1-b_n)Au_n + b_nS^nAu_n]\|^2 \\
&= \kappa_n[1 + b_n(\kappa_n - 1)]\|Au_n - A(\alpha p)\|^2 + (1-b_n)\|Au_n - h_n(Au_n)\|^2 \\
&\quad -b_n[1 - b_n(1+\kappa_n) - b_n^2L^2]\|Au_n - S^nAu_n\|^2. \qquad (3.53)
\end{aligned}
$$

By applying the above result, we have that

$$
\begin{aligned}
\|D_n(Au_n) - A(\alpha p)\|^2 &= \|e_nAu_n + (1-e_n)h_n(Au_n) - A(\alpha p)\|^2 \\
&= \|e_n(Au_n - A(\alpha p)) + (1-e_n)[h_n(Au_n) - A(\alpha p)]\|^2 \\
&= e_n\|Au_n - A(\alpha p)\|^2 + (1-e_n)\|h_n(Au_n) - A(\alpha p)\|^2 \\
&\quad -e_n(1-e_n)\|Au_n - h_n(Au_n)\|^2 \\
&\leq e_n\|Au_n - A(\alpha p)\|^2 \\
&\quad +(1-e_n)\{\kappa_n[1 + b_n(\kappa_n - 1)]\|Au_n - A(\alpha p)\|^2 \\
&\quad +(1-b_n)\|Au_n - h_n(Au_n)\|^2 \\
&\quad -b_n[1 - b_n(1+\kappa_n) - b_n^2L^2]\|Au_n - S^nAu_n\|^2\} \\
&\quad -e_n(1-e_n)\|Au_n - h_n(Au_n)\|^2 \\
&= \{e_n + (1-e_n)\kappa_n[1 + b_n(\kappa_n - 1)]\}\|Au_n - A(\alpha p)\|^2 \\
&\quad +(1-e_n)[1 - b_n - e_n]\|Au_n - h_n(Au_n)\|^2 \\
&\quad -(1-e_n)b_n[1 - b_n(1+\kappa_n) - b_n^2L^2]\|Au_n - S^nAu_n\|^2 \\
&= \{1 + (1-e_n)(\kappa_n - 1)(1 + b_n\kappa_n)\}\|Au_n - A(\alpha p)\|^2 \\
&\quad -(1-e_n)[b_n + e_n - 1]\|Au_n - h_n(Au_n)\|^2 \\
&\quad -(1-e_n)b_n[1 - b_n(1+\kappa_n) - b_n^2L^2]\|Au_n - S^nAu_n\|^2. \qquad (3.54)
\end{aligned}
$$

$$\begin{aligned}
\|x_n - \alpha p\|^2 &= \|u_n + \gamma_n A^*[e_n I + (1-e_n)S^n[(1-b_n)I + b_n S^n] - I]Au_n - \alpha p\|^2 \\
&= \|(u_n - \alpha p) + \gamma_n A^*[(D_n - I)Au_n]\|^2 \\
&= \|u_n - \alpha p\|^2 + \gamma_n^2\|A^*[(D_n - I)Au_n]\|^2 \\
&\quad +2\gamma_n\langle A^*[(D_n - I)Au_n], u_n - \alpha p\rangle \\
&= \|u_n - \alpha p\|^2 + \gamma_n^2\|A^*[(D_n - I)Au_n]\|^2 \\
&\quad +2\gamma_n\langle (D_n - I)Au_n, Au_n - A(\alpha p)\rangle \\
&= \|u_n - \alpha p\|^2 + \gamma_n(1-e_n)\|(D_n - I)Au_n\|^2 - \gamma_n\|(D_n - I)Au_n\|^2 \\
&\quad +\gamma_n(\|D_n(Au_n) - A\alpha p\|^2 - \|Au_n - A\alpha p\|^2). \\
&\le \|u_n - \alpha p\|^2 - e_n\gamma_n\|(D_n - I)Au_n\|^2 + \gamma_n(\|D_n(Au_n) - A\alpha p\|^2 \\
&\quad -\|Au_n - A\alpha p\|^2) \\
&\le \|u_n - \alpha p\|^2 - e_n\gamma_n\|(D_n - I)Au_n\|^2 \\
&\quad +\gamma_n\{1 + (1-e_n)(\kappa_n - 1)(1 + b_n\kappa_n)\}\|Au_n - A(\alpha p)\|^2 \\
&\quad -(1-e_n)[b_n + e_n - 1]\|Au_n - h_n(Au_n)\|^2 \\
&\quad -(1-e_n)b_n[1 - b_n(1+\kappa_n) - b_n^2L^2]\|Au_n - S^nAu_n\|^2 \\
&\quad -\|Au_n - A\alpha p\|^2). \\
&= \|u_n - \alpha p\|^2 - e_n\gamma_n\|(D_n - I)Au_n\|^2 \\
&\quad +\gamma_n(1-e_n)(\kappa_n - 1)(1 + b_n\kappa_n)\|Au_n - A(\alpha p)\|^2 \\
&\quad -(1-e_n)[b_n + e_n - 1]\|Au_n - h_n(Au_n)\|^2 \\
&\quad -(1-e_n)b_n[1 - b_n(1+\kappa_n) - b_n^2L^2]\|Au_n - S^nAu_n\|^2 \\
&\le \|u_n - \alpha p\|^2 - e_n\gamma_n\|(D_n - I)Au_n\|^2 \\
&\quad +\gamma_n(1-e_n)(\kappa_n - 1)(1 + b_n\kappa_n)\|A\|^2\|u_n - \alpha p\|^2 \\
&\quad -(1-e_n)[b_n + e_n - 1]\|Au_n - h_n(Au_n)\|^2 \\
&\quad -(1-e_n)b_n[1 - b_n(1+\kappa_n) - b_n^2L^2]\|Au_n - S^nAu_n\|^2 \\
&= (1 + \gamma_n(1-e_n)(\kappa_n - 1)(1 + b_n\kappa_n)\|A\|^2)\|u_n - \alpha p\|^2 \\
&\quad -(1-e_n)[b_n + e_n - 1]\|Au_n - h_n(Au_n)\|^2 \\
&\quad -(1-e_n)b_n[1 - b_n(1+\kappa_n) - b_n^2L^2]\|Au_n - S^nAu_n\|^2 \\
&\quad -e_n\gamma_n\|(D_n - I)Au_n\|^2 \\
&\le (1 + (\kappa_n - 1)(1 + b_n\kappa_n))\|u_n - \alpha p\|^2 \\
&\quad -(1-e_n)[b_n + e_n - 1]\|Au_n - h_n(Au_n)\|^2 \\
&\quad -(1-e_n)b_n[1 - b_n(1+\kappa_n) - b_n^2L^2]\|Au_n - S^nAu_n\|^2 \\
&\quad -e_n\gamma_n\|(D_n - I)Au_n\|^2. \qquad (3.55)
\end{aligned}$$

Setting $r_n = (\kappa_n - 1)(1 + b_n\kappa_n)$ and $\delta_n = (\kappa_n - 1)(1 + \beta_n\kappa_n)$, then, from (3.52) and (3.55), we have

$$\begin{aligned}
\|u_{n+1} - \alpha p\|^2 &\le (1 + r_n)(1 + \delta_n)\|u_n - \alpha p\|^2 \\
&\quad -(1 + r_n)(1 - e_n)[b_n + e_n - 1]\|Au_n - h_n(Au_n)\|^2 \\
&\quad -(1 + r_n)(1 - e_n)b_n[1 - b_n(1+\kappa_n) - b_n^2L^2]\|Au_n - S^nAu_n\|^2 \\
&\quad -(1 + r_n)e_n\gamma_n\|(D_n - I)Au_n\|^2 \\
&\quad -a_n\beta_n[1 - \beta_n(\kappa_n + 1) - \beta_n^2L^2]\|x_n - T^nx_n\|^2 \\
&\quad -a_n(\beta_n - a_n)\|x_n - G_nx_n\|^2. \qquad (3.56)
\end{aligned}$$

Consequently, we have that

$$\|u_{n+1}-\alpha p\|^2 \leq [1+(\delta_n+r_n+r_n\delta_n)]\|u_n-\alpha p\|^2. \tag{3.57}$$

From the assumption that $\sum\limits_{n=1}^{\infty}(\kappa_n^2-1)<\infty$, we have that

$$\begin{aligned}
\sum_{n=1}^{\infty}(\delta_n+r_n+r_n\delta_n) &= \sum_{n=1}^{\infty}\delta_n+\sum_{n=1}^{\infty}r_n+\sum_{n=1}^{\infty}\delta_n r_n \\
&< \sum_{n=1}^{\infty}[(\kappa_n-1)(1+\kappa_n)]+\sum_{n=1}^{\infty}[(\kappa_n-1)(1+\kappa_n)] \\
&\quad +\sum_{n=1}^{\infty}[(\kappa_n-1)(1+\kappa_n)]^2 \\
&= \sum_{n=1}^{\infty}(\kappa_n^2-1)+\sum_{n=1}^{\infty}(\kappa_n^2-1)+\sum_{n=1}^{\infty}(\kappa_n^2-1)^2 \\
&< \infty.
\end{aligned}$$

Hence, applying Lemma (2.3) on (3.57), we have that $\{\|u_n-\alpha p\|\}$ converges. Thus, $\{u_n\}$, $\{x_n\}$ and $\{\|u_n-\alpha p\|\}$ are bounded.
By linearity of A, $\{\|Au_n-A(\alpha p)\|\}=\{\|A(u_n-\alpha p)\|\}$ is also convergent. Thus, (3.56) yields that

$$\begin{aligned}
\lim_{n\to\infty}\|Au_n-S^nAu_n\| &= \lim_{n\to\infty}\|(D_n-I)Au_n\| \\
&= \lim_{n\to\infty}\|x_n-T^nx_n\| \\
&= \lim_{n\to\infty}\|x_n-G_nx_n\| \\
&= \lim_{n\to\infty}\|Au_n-h_n(Au_n)\|=0. \qquad (3.58)
\end{aligned}$$

From the condition on γ_n, we have that

$$\epsilon<\gamma_n<\frac{(1-e_n)||(D_n-I)Au_n||^2}{||A^*(D_n-I)Au_n||^2}.$$

This implies that $\gamma_n||A^*(D_n-I)Au_n||^2<(1-e_n)||(D_n-I)Au_n||^2$ which in turn implies that

$$\gamma_n||A^*(D_n-I)Au_n||^2<(1-e_n)||(D_n-I)Au_n||^2\to 0.$$

Thus,

$$\lim_{n\to\infty}||A^*(D_n-I)Au_n||=0. \tag{3.59}$$

Again, from (3.50) and (3.59), we have that

$$\begin{aligned}
\|x_n-u_n\| &= \|\gamma_nA^*[e_nI+(1-e_n)S^n[(1-b_n)I+b_nS^n]-I]Au_n\| \\
&= \gamma_n\|A^*[(D_n-I)Au_n]\| \\
&\to 0.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.60}$$

This, in turn, implies that

$$\lim_{n\to\infty}\|x_n\|=\lim_{n\to\infty}\|u_n\| \; and \; \lim_{n\to\infty}\|x_n-\alpha p\|=\lim_{n\to\infty}\|u_n-\alpha p\|. \tag{3.61}$$

Hence,

$$\begin{aligned}\|u_n - T^n u_n\| &= \|u_n - x_n + x_n - T^n x_n + T^n x_n - T^n u_n\| \\ &\leq \|u_n - x_n\| + \|x_n - T^n x_n\| + \|T^n x_n - T^n u_n\| \\ &\leq \|u_n - x_n\| + \|x_n - T^n x_n\| + L\|x_n - u_n\| \\ &= (1+L)\|u_n - x_n\| + \|x_n - T^n x_n\| \\ &\to 0.\end{aligned}$$

This means that

$$\lim_{n\to\infty} \|u_n - T^n u_n\| = 0. \tag{3.62}$$

Next, we show that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

From (3.50), (3.58) and (3.60)

$$\begin{aligned}\|u_{n+1} - u_n\| &= \|(1-a_n)x_n + a_n T^n[(1-\beta_n)x_n + \beta_n T^n x_n] - u_n\| \\ &= \|(1-a_n)x_n + a_n G_n x_n - u_n\| \\ &= \|x_n - u_n + a_n(G_n x_n - x_n)\| \\ &\leq \|x_n - u_n\| + a_n\|G_n x_n - x_n\| \\ &\to 0.\end{aligned}$$

Hence,

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{3.63}$$

Similarly, from (3.50), (3.59) and (3.63) we have that if $y_n = \|x_{n+1} - x_n\|$, then

$$\begin{aligned}y_n &= \|x_{n+1} - x_n\| \\ &= \|[u_{n+1} + \gamma_{n+1}A^*[D_{n+1}(Au_{n+1}) - Au_{n+1}]\|] - [u_n + \gamma_n A^*[D_n(Au_n) - Au_n]]\| \\ &= \|(u_{n+1} - u_n) + \gamma_{n+1}A^*[D_{n+1}(Au_{n+1}) - Au_{n+1}] + (-\gamma_n A^*[D_n(Au_n) - Au_n]\|) \\ &\leq \|u_{n+1} - u_n\| + \|\gamma_{n+1}A^*[D_{n+1}(Au_{n+1}) - Au_{n+1}]\| + \|\gamma_n A^*[D_n(Au_n) - Au_n]\|) \\ &\to 0.\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.64}$$

Consequently,

$$\begin{aligned}\|u_n - Tu_n\| &\leq \|u_n - u_{n+1}\| + \|u_{n+1} - T^{n+1}u_{n+1}\| + \|T^{n+1}u_{n+1} - T^{n+1}u_n\| \\ &\quad + \|T^{n+1}u_n - Tu_n\| \\ &\leq \|u_n - u_{n+1}\| + \|u_{n+1} - T^{n+1}u_{n+1}\| + L\|u_{n+1} - u_n\| \\ &\quad + L\|T^n u_n - u_n\| \to 0 \text{ as } n \to \infty.\end{aligned}$$

While,

$$\begin{aligned}\|Au_n - S(Au_n)\| &\leq \|Au_n - Au_{n+1}\| + \|Au_{n+1} - S^{n+1}Au_{n+1}\| \\ &\quad + \|S^{n+1}Au_{n+1} - S^{n+1}Au_n\| + \|S^{n+1}Au_n - S(Au_n)\| \\ &\leq \|Au_n - Au_{n+1}\| + \|Au_{n+1} - S^{n+1}Au_{n+1}\| \\ &\quad + L\|Au_{n+1} - Au_n\| + L\|S^n Au_n - Au_n\| \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\|u_n - Tu_n\| = \lim_{n\to\infty}\|Au_n - S(Au_n)\| = 0. \tag{3.65}$$

Consequently, for any subsequence $\{u_{n_j}\} \subseteq \{u_n\}$, we have that

$$\lim_{j\to\infty}\|u_{n_j} - Tu_{n_j}\| = \lim_{n\to\infty}\|Au_{n_j} - S(Au_{n_j})\| = 0. \tag{3.66}$$

Finally, we prove that $u_n \rightharpoonup u^*, x_n \rightharpoonup u^*$, and u^* is a solution of problem (SFP). To this end, we note from boundedness of $\{u_n\}$ that there exists a subsequence $\{u_{n_j}\} \subseteq \{u_n\}$ such that $u_{n_j} \rightharpoonup u^* \in H_1$.
Since $\lim\limits_{n\to\infty} ||u_{n_j} - Tu_{n_j}|| = 0$, we obtain from demiclosedness of $I - T$ at zero that $u^* \in F(T)$
From the fact that A is a bounded linear operator and $u_n \rightharpoonup u^*$, we have that $Au_n \rightharpoonup Au^* \in H_2$. Thus, we can deduce from (3.66) and demiclosedness of $I - S$ at zero that $Au^* \in F(S)$. Therefore, $u^* \in \Gamma$. Since every Hilbert space satisfies the Opial property and $\{u_n\}$ has a subsequence $\{u_{n_j}\}$ which converges weakly to a point $u^* \in \Gamma$, it follows from a standard argument that $\{u_n\}$ converges weakly to $u_n \rightharpoonup u^* \in \Gamma$. □

Theorem 3.2. *Suppose that the assumptions of* **theorem** (3.1) *are met. Assume, in addition, that the mappings S and T are also semicompact. Then, for any initial point u_0 the iterative sequence $\{(u_n\}_{n\geq 1}$ derived from* (3.50) *converges strongly to a solution of problem* (1.1).

Proof. Since S, T are semicompact, it follows from boundedness of $\{(u_n\}_{n\geq 1}$ and (3.65) that there exists subsequence $\{u_{n_j}\} \subseteq \{u_n\}$ such that $u_{n_j} \to p^* \in F(T)$. Since $u_n \rightharpoonup u^*$ while the limits $\lim\limits_{n\to\infty} ||u_n - \alpha p||$ exists for all $\alpha p \in \Gamma$, then $u^* = p^*$ and it follows from (3.65) that $u_n \to p^*$. This completes the proof of **theorem** (3.2). □

4. Numerical examples

In this section, numerical illustration of the convergence of the iterative scheme (3.50) is represented. The setting for the numerical example is that of a real Hilbert space. Using the example, we show , by use of a table of numerical values, the convergence results discussed in this paper. All codes are written in MATLAB.

Example 4.5. Let $H_2 = \mathbb{R}^2$. Let the mapping $S : H_2 \to H_2$ be defined by

$$S(X) = QX, \tag{4.67}$$

where,

$$\begin{pmatrix} 0 & -1+\sqrt{2} \\ -1-\sqrt{2} & 0 \end{pmatrix} \text{ and } X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

For arbitrary $\bar{x}, \bar{y} \in \mathbb{R}^2$ with $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$ we have that

$$\begin{aligned}
\|S\bar{x} - S\bar{y}\|^2 &= \left\| \begin{pmatrix} 0 & -1+\sqrt{2} \\ -1-\sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 & -1+\sqrt{2} \\ -1-\sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\|^2 \\
&= ((-1+\sqrt{2})(x_2 - y_2))^2 + ((-1-\sqrt{2})(x_1 - y_1))^2 \\
&= (3-2\sqrt{2})(x_2 - y_2)^2 + (3+2\sqrt{2})(x_1 - y_1)^2 \\
&\leq (3+2\sqrt{2})(x_2 - y_2)^2 + (3+2\sqrt{2})(x_1 - y_1)^2 \\
&= (3+2\sqrt{2})[(x_2 - y_2)^2 + (x_1 - y_1)^2] \\
&= (3+2\sqrt{2})\|\bar{x} - \bar{y}\|^2.
\end{aligned}$$

Thus,

$$\|S\bar{x} - S\bar{y}\| \leq \sqrt{3+2\sqrt{2}}\|\bar{x} - \bar{y}\|.$$

Similarly,

$$\begin{aligned} \|S^2\bar{x} - S^2\bar{y}\| &= \| - \bar{x} - (-\bar{y})\| \\ &= \|\bar{x} - \bar{y}\|. \end{aligned}$$

Furthermore,

$$\begin{aligned} \|S\bar{x} - S\bar{y}\|^2 &= \left\| \begin{pmatrix} 0 & -1+\sqrt{2} \\ -1-\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix} - \begin{pmatrix} 0 & -1+\sqrt{2} \\ -1-\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} -y_1 \\ -y_2 \end{pmatrix} \right\|^2 \\ &= ((-1+\sqrt{2})(y_2 - x_2))^2 + ((-1-\sqrt{2})(y_1 - x_1))^2 \\ &= (3-2\sqrt{2})(y_2 - x_2)^2 + (3+2\sqrt{2})(y_1 - x_1)^2 \\ &\leq (3+2\sqrt{2})(y_2 - x_2)^2 + (3+2\sqrt{2})(y_1 - x_1)^2 \\ &= (3+2\sqrt{2})[(x_2 - y_2)^2 + (x_1 - y_1)^2] \\ &= (3+2\sqrt{2})\|\bar{x} - \bar{y}\|^2. \end{aligned}$$

Thus,

$$\|S^3\bar{x} - S^3\bar{y}\| \leq \sqrt{3+2\sqrt{2}}\|\bar{x} - \bar{y}\|.$$

While,

$$\begin{aligned} \|S^4\bar{x} - S^4\bar{y}\| &= \|\bar{x} - \bar{y}\| \\ &= \|\bar{x} - \bar{y}\|. \end{aligned}$$

This implies that

$$\|S^n\bar{x} - S^n\bar{y}\| \leq \sqrt{3+2\sqrt{2}}\|\bar{x} - \bar{y}\| \ \forall n \in \mathbb{N} \ and \ \forall \bar{x}, \bar{y} \in \mathbb{R}^2.$$

Thus, S is uniformly Lipschitzian with Lipschitz constant $L = \sqrt{3+2\sqrt{2}}$. Next, we show that S is asymptotically $\alpha-$hemicontractive for any $\alpha \geq 1$. Clearly, $F(S) = \{(0,0)\}$ and for every $\bar{x} = (x, y) \in \mathbb{R}^2$ with $\kappa_n = \{1 + \frac{1}{n^2}\}$ and $p \in F(S)$, we have that

$$\begin{aligned} \langle \bar{x} - S\bar{x}, \bar{x} - \alpha p \rangle &= \langle (x,y) - ((-1+\sqrt{2})y, (-1-\sqrt{2})x), (x,y) - \alpha(0,0) \rangle \\ &= \langle (x - (-1+\sqrt{2})y, y - (-1-\sqrt{2})x), (x,y) \rangle \\ &= x(x - (-1+\sqrt{2})y) + y(y - (-1-\sqrt{2})x) \\ &= x^2 - (-1+\sqrt{2})xy + y^2 - (-1-\sqrt{2})xy \\ &= x^2 + 2xy + y^2 \\ &= (x+y)^2 \\ &\geq 0 \\ &\geq -\frac{\kappa_n - 1}{2}\|\bar{x} - \alpha p\|^2. \end{aligned}$$

Similarly,

$$\begin{aligned}\langle \bar{x} - S^2\bar{x}, \bar{x} - \alpha p\rangle &= \langle (x,y) - (-x,-y), (x,y) - \alpha(0,0)\rangle \\ &= \langle (2x,2y), (x,y)\rangle \\ &= 2x^2 + 2y^2 \\ &= 2(x^2+y^2) \\ &\geq 0 \\ &\geq -\frac{\kappa_n - 1}{2}\|\bar{x} - \alpha p\|^2.\end{aligned}$$

It also follows that,

$$\begin{aligned}\langle \bar{x} - S^3\bar{x}, \bar{x} - \alpha p\rangle &= \langle (x,y) - (-(-1+\sqrt{2})y, -(-1-\sqrt{2})x), (x,y) - \alpha(0,0)\rangle \\ &= \langle (x + (-1+\sqrt{2})y, y + (-1-\sqrt{2})x), (x,y)\rangle \\ &= x(x + (-1+\sqrt{2})y) + y(y + (-1-\sqrt{2})x) \\ &= x^2 + (-1+\sqrt{2})xy + y^2 + (-1-\sqrt{2})xy \\ &= x^2 - 2xy + y^2 \\ &= (x-y)^2 \\ &\geq 0 \\ &\geq -\frac{\kappa_n - 1}{2}\|\bar{x} - \alpha p\|^2.\end{aligned}$$

While,

$$\begin{aligned}\langle \bar{x} - S^4\bar{x}, \bar{x} - \alpha p\rangle &= \langle (x,y) - (x,y), (x,y) - \alpha(0,0)\rangle \\ &= \langle (0,0), (x,y)\rangle \\ &= 0 \\ &\geq -\frac{\kappa_n - 1}{2}\|\bar{x} - \alpha p\|^2.\end{aligned}$$

Obviously, values of $S^n\bar{x}$ is cyclic. Thus, S is asymptotically α-hemicontractive with any $\alpha \geq 1$ and $\kappa_n = \{1 + \frac{1}{n^2}\}$ which satisfies the requirement $\sum\limits_{n=1}^{\infty}(\kappa_n^2 - 1) < \infty$.

Next, let us consider

Example 4.6. Let $H_1 = \mathbb{R}^2$. Let the mapping $T : H_1 \to H_1$ be defined by

(4.68) $$T(x,y) = (2-x, 2-y),\ \forall (x,y) \in \mathbb{R}^2.$$

For arbitrary $\bar{x}, \bar{y} \in \mathbb{R}^2$ with $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$ we have that

$$\begin{aligned}\|T\bar{x} - T\bar{y}\| &= \|(2-x_1, 2-x_2) - (2-y_1, 2-y_2)\| \\ &= \|(y_1 - x_1, y_2 - x_2)\| \\ &= \|\bar{x} - \bar{y}\|.\end{aligned}$$

Observe that,

$$\begin{aligned}\|T^2\bar{x} - T^2\bar{y}\| &= \|(x_1, x_2) - (y_1, y_2)\| \\ &= \|(x_1 - y_1, x_2 - y_2)\| \\ &= \|\bar{x} - \bar{y}\|.\end{aligned}$$

This implies that

$$\|T^n\bar{x} - T^n\bar{y}\| \leq \|\bar{x} - \bar{y}\|\ \forall n \in \mathbb{N}\ and\ \forall \bar{x}, \bar{y} \in \mathbb{R}^2.$$

Thus, T is uniformly Lipschitzian with Lipschitz constant $L = 1$. Next, we show that T is asymptotically $\alpha-$hemicontractive for $\alpha = 1$. Clearly, $F(T) = \{(1,1)\}$ and for every $\bar{x} = (x,y) \in \mathbb{R}^2$ with $\kappa_n = \{1 + \frac{1}{n^2}\}$ and $p \in F(T)$, we have that

$$\begin{aligned}
\langle \bar{x} - T\bar{x}, \bar{x} - \alpha p\rangle &= \langle (x,y) - (2-x, 2-y), (x,y) - \alpha(1,1)\rangle \\
&= \langle (2x-2, 2y-2), (x-1, y-1)\rangle \\
&= \langle 2(x-1, y-1), (x-1, y-1)\rangle \\
&= 2\langle (x-1, y-1), (x-1, y-1)\rangle \\
&\geq 0 \\
&\geq -\frac{\kappa_n - 1}{2}\|\bar{x} - \alpha p\|^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle \bar{x} - T^2\bar{x}, \bar{x} - \alpha p\rangle &= \langle (x,y) - (x,y), (x,y) - \alpha(1,1)\rangle \\
&= \langle 0, (x-1, y-1)\rangle \\
&= 0 \\
&\geq -\frac{\kappa_n - 1}{2}\|\bar{x} - \alpha p\|^2.
\end{aligned}$$

Obviously, values of $T^n\bar{x}$ is cyclic. Thus, T is asymptotically α-hemicontractive with $\alpha = 1$ and $\kappa_n = \{1 + \frac{1}{n^2}\}$ which satisfies the requirement $\sum_{n=1}^{\infty}(\kappa_n^2 - 1) < \infty$.
We have $L = max\{\sqrt{3 + 2\sqrt{2}}, 1\} = \sqrt{3 + 2\sqrt{2}}$ and

$$\begin{aligned}
\frac{1}{\sqrt{\frac{(k+1)^2}{4} + L^2} + \frac{(k+1)}{2}} &= \frac{1}{\sqrt{\frac{(2+1)^2}{4} + (\sqrt{3 + 2\sqrt{2}})^2} + \frac{(2+1)}{2}} \\
&= \frac{1}{1.5 + \sqrt{5.25 + 2\sqrt{2}}}.
\end{aligned}$$

Thus, we consider $a_n = \beta_n = e_n = \frac{1}{1.5+\sqrt{9}} - \frac{1}{n^2(1.5+\sqrt{9})}$ and $b_n = 1 - e_n$.
Next, let us consider a bounded linear operator, $A : H_1 \to H_2$ defined by $A(x,y) = (x - y, -x+y)$ the adjoint of which is given by $A^* : H_1 \to H_2$ defined by $A(x,y) = (x+y, x+y)$. Using initial point, $u_0 = (20, 20)$, $\epsilon = 0.00001$ and S as defined in (4.67) together with the corresponding $D_n(Au_n) = e_nAu_n + (1 - e_n)S^n[(1 - b_n)Au_n + b_nS^n(Au_n)]$ and step size,

$$\gamma_n = \begin{cases} 0.00001 + \frac{(1-e_n)||(D_n - I)Au_n||^2}{1 + ||A^*(D_n - I)Au_n||^2} \; if \; u_n \notin \Gamma \\ 0.00001, \; otherwise. \end{cases}$$

By using stopping criterion of $\|u_n - p\| \leq 1e - 7$ $u_0 = (20, 20)$ we have from the table that (3.50) converges to a point in $\Gamma = \{(1,1)\}$ the solution of the SCFFP, irrespective of the starting point. We illustrate with .

s/n	u_n	$\|u_n - p\|$	s/n	u_n	$\|u_n - p\|$
1	(25.42438 ,25.42438)	34.54129	150	(1.000021, 1.000021)	0.00003024975
2	(25.42438, 25.42438)	34.54129	151	(1.000018, 1.000018)	0.00002502184
3	(22.14518, 22.14518)	29.90380	152	(1.000018, 1.000018)	0.00002502184
4	(22.14518, 22.14518)	29.90380	153	(1.000015, 1.000015)	0.00002069743
5	(18.77662, 18.77662)	25.13994	154	(1.000015, 1.000015)	0.00002069743
6	(18.77662, 18.77662)	25.13994	155	(1.000012, 1.000012)	0.00001712039
7	(15.82598, 15.82598)	20.96710	156	(1.000012, 1.000012)	0.00001712039
8	(15.82598, 15.82598)	20.96710	157	(1.000010, 1.000010)	0.00001416154
9	(13.32476, 13.32476)	17.42984	158	(1.000010, 1.000010)	0.00001416154
10	(13.32476,13.32476)	17.42984	159	(1.000008, 1.000008)	0.00001171405
⋮	⋮	⋮	⋮	⋮	⋮
95	(1.003589 ,1.003589)	0.005075390	201	(1.000000, 1.000000)	0.0000002178744
96	(1.003589, 1.003589)	0.005075390	202	(1.000000, 1.000000)	0.0000002178744
97	(1.002969, 1.002969)	0.004198342	203	(1.000000, 1.000000)	0.0000001802189
98	(1.002969, 1.002969)	0.004198342	204	(1.000000, 1.000000)	0.0000001802189
99	(1.002456, 1.002456)	0.003472846	205	(1.000000, 1.000000)	0.0000001490714
100	(1.002456, 1.002456)	0.003472846	206	(1.000000, 1.000000)	0.0000001490714
101	(1.002031, 1.002031)	0.002872714	207	(1.000000, 1.000000)	0.0000001233071
102	(1.002031, 1.002031)	0.002872714	208	(1.000000, 1.000000)	0.0000001233071
103	(1.001680, 1.001680)	0.002376286	209	(1.000000, 1.000000)	0.0000001019957
104	(1.001680, 1.001680)	0.002376286	210	(1.000000, 1.000000)	0.0000001019957
⋮	⋮	⋮	⋮	⋮	⋮

In the example given above the stopping criteria is $\|u_n - p\| \leq 10^{-7}$. This implies that the error of approximating the fixed point of the given maps is negligible as seen in the table. From the table , it is clear that for $n \geq 210$, terms of the sequence get close enough (as close as a difference of $10^{=7}$)to (1,1) which is a fixed point of T with its image under the bounded linear operator $A(1,1) = (0,0)$ which, in turn, is a fixed point of the map S. Thus, the sequence generated by algorithm (3.50) converges, indeed, to a solution of (1.1). A lower stopping criterion will make no much difference while a higher stopping criteria will truncate the computation too early which might lead to higher computational error.

5. Conclusions

We have introduced an interesting class of asymptotically $\alpha-$hemicontractive mappings and exhibited some of its important relationship with existing related families of mappings. We further established certain interesting properties of the fixed-point set of the new class of mappings and proposed and investigated a new iterative algorithm for solving split common fixed point problem associated with uniformly L-Lipschitzian asymptotically $\alpha-$hemicontractive mappings. In particular, weak and strong convergence theorems for solving split common fixed point problem associated with uniformly L-Lipschitzian asymptotically $\alpha-$hemicontractive mappings were proved without prior knowledge of the norm of the transfer operator in Hilbert spaces. Our results extend and improve the results of Censor and Segal [13], Moudafi [23, 24], Chima and Osilike [16], Fan *et al* [19] and host of other related results in literature. It will be interesting to extend our results to spaces more general than Hilbert spaces. Furthermore, in our results, it is certainly of interest to relax the uniformly Lipschitzian property requirement on our operators.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly to this research work. All authors read and approved the final manuscript.

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