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Periodic cycles for an extension of generalized $3x + 1$ functions

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ABSTRACT. The Collatz conjecture is an open problem involving the $3x+1$ function. The function belongs to a class of generalized $3x + 1$ functions of relatively prime type. This paper focuses on exploring periodic cycles for an extension of a generalized $3x + 1$ function of relatively prime type. By extending its domain to $\mathbb{R}$, the result shows that every integer periodic point is isolated in the usual topology on $\mathbb{R}$. Moreover, every positive integer periodic cycle for the extension is attracting if the generalized $3x + 1$ function is specified by parameters under some conditions.

1. INTRODUCTION

The Collatz conjecture was proposed by L. Collatz in 1937. It is also known as the $3x+1$ conjecture, the Ulam conjecture, the Twaites hypothesis, or the Syracuse problem [8]. To explain this conjecture, suppose that x is a positive integer as an initial number. Divide x by 2 if x is even. Otherwise, multiply x by 3 then add 1. These two arithmetic operations could be defined as the *Collatz function* $C : \mathbb{N} \longrightarrow \mathbb{N}$, given by

$$C(x) = \begin{cases} \frac{x}{2} & \text{if} \quad x \equiv 0 \pmod 2, \\ 3x + 1 & \text{if} \quad x \equiv 1 \pmod 2. \end{cases}$$

The $3x + 1$ *function* $T : \mathbb{N} \cup \{0\} \longrightarrow \mathbb{N} \cup \{0\}$ is an alternative function for exploring the conjecture, defined by

$$T(x) = \begin{cases} \frac{x}{2} & \text{if} \quad x \equiv 0 \pmod 2, \\ \frac{3x+1}{2} & \text{if} \quad x \equiv 1 \pmod 2. \end{cases}$$

Let the sequence $\mathrm{Orb}_T(x) := \{x, T(x), T^2(x), \ldots\}$ be an *orbit* of x iterated by T. In 1976, R. Terras [14] showed that there is a point in the orbit $\mathrm{Orb}_T(x)$ that is less than the initial number x for almost all integers x. We note that the technical term "almost all" is defined in the sense of natural density. C. J. Everett independently proposed this result in 1977 [6] and several authors improved it later (see more in [1, 7]). Especially, in 2022, T. Tao [13] showed that for almost every integer x (in the sense of logarithmic density), there is a point in the orbit $\mathrm{Orb}_C(x)$ that is less than $f(x)$, where $f : \mathbb{N} \longrightarrow \mathbb{R}$ is a function satisfying $f(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Based on the definition provided by K. R. Matthews and A. M. Watts [8, 11, 10], the following definition could be used to generalize the $3x + 1$ function:

Definition 1.1 ([10], Definition 1)**.** A function $T_{(d,\vec{m},\vec{r})} : \mathbb{Z} \longrightarrow \mathbb{Z}$ is called a *generalized* $3x + 1$ *function* if there are parameters $d \in \mathbb{N} \setminus \{1\}$, d-tuple of non-zero integers $\vec{m} =$

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$(m_0, \ldots, m_{d-1})$, and d-tuple of integers $\vec{r} = (r_0, \ldots, r_{d-1})$ satisfying $r_j \equiv jm_j \pmod{d}$, such that

$$T_{(d,\vec{m},\vec{r})}(x) = \frac{m_j x - r_j}{d} \quad \text{if} \quad x \equiv j \pmod{d}.$$

In addition, it is said to be *relatively prime* if $\gcd(m_j, d) = 1$ for all $j \in \{0, \ldots, d-1\}$.

Note that the Collatz function and the $3x + 1$ function are both special cases of generalized $3x + 1$ functions. Specifically, the $3x + 1$ function is relatively prime, while the Collatz function is not.

Embedding a problem within a larger class of problems is another strategy for solving problems. It provides more tools for exploring the problem. In 1996, M. Chamberland [2] proposed the smooth extension of the $3x + 1$ function $\widetilde{T}_1 : \mathbb{R} \longrightarrow \mathbb{R}$, defined by

$$\widetilde{T}_1(x) = \left(\frac{x}{2}\right)\cos^2\left(\frac{x\pi}{2}\right) + \left(\frac{3x+1}{2}\right)\sin^2\left(\frac{x\pi}{2}\right). \tag{1.1}$$

The results show that every positive integer periodic cycle for $\widetilde{T}_1$ is attracting. Moreover, this function has a negative Schwarzian on $\mathbb{R}^+$, which provides strong constraints for fixed and periodic points by applying Singer's theorem. In 2003, J. P. Dumont and C. A. Reiter [4] explored the real dynamics of the alternative function $\widetilde{T}_2 : \mathbb{R} \longrightarrow \mathbb{R}$, defined by

$$\widetilde{T}_2(x) = \frac{3^{\sin^2\left(\frac{\pi x}{2}\right)} x + \sin^2\left(\frac{\pi x}{2}\right)}{2}.$$

The results from this extension are similar to the results from M. Chamberland. Moreover, there is evidence that this extension is a natural extension of the $3x + 1$ function, which follows from the sense of the real dynamics of this extension related to the τ-stopping time of R. Terras [14].

In the most recent literature published in 2010, the latest paper involving an extension of generalized $3x + 1$ functions was proposed by M. Pierantoni and V. Curéié in 1996 [8]. The extension found in this reference was defined from $\mathbb{R}$ to $\mathbb{C}$ and explored by using the discrete Fourier transform and its inverse. Although extensions of the $3x+1$ function were proposed, there is no extension defined from $\mathbb{R}$ to itself for generalized $3x + 1$ functions. Moreover, papers studying in this manner have rarely been proposed since 2010.

This paper aims to demonstrate the process of constructing an extension of generalized $3x + 1$ functions. The extension is a function defined from $\mathbb{R}$ to itself. Integer periodic cycles for this extension in the case of relatively prime type will be explored. We will see that every integer periodic point is isolated in the usual topology on $\mathbb{R}$. Moreover, every positive integer periodic cycle is attracting if the extension is specified by parameters $d \in \mathbb{N} \setminus \{1\}$, d-tuple of positive integers $\vec{m}$, and d-tuple of non-positive integers $\vec{r}$. It shows that Chamberland's extension is also a natural extension of the $3x + 1$ function in the same sense.

The structure of the paper is as follows: In Section 2, the extension of generalized $3x+1$ functions is established. In Section 3, results involving periodic cycles for a generalized $3x + 1$ function of relatively prime type are proposed. Numerical results are presented in Section 4 to verify the theoretical results from the previous section. For the last section, some conclusions are provided.

2. Extension of generalized $3x + 1$ functions

In this section, an extension of generalized $3x + 1$ functions defined from $\mathbb{R}$ to itself will be provided. The extension is established by using squared trigonometric functions. To accomplish this, we begin with the following definition:

Definition 2.2. A function $I_{(d,r)} : \mathbb{R} \longrightarrow \mathbb{R}$ is called an *indicator function* if there are parameters $d \in \mathbb{N} \setminus \{1\}$ and $r \in \{0, \ldots, d-1\}$, such that

$$I_{(d,r)}(x) = \frac{4^{d-1}}{d^2} \prod_{\substack{j=0 \\ j \neq r}}^{d-1} \sin^2\left(\frac{(x-j)\pi}{d}\right).$$

Lemma 2.1. *Let $I_{(d,r)} : \mathbb{R} \longrightarrow \mathbb{R}$ be an indicator function. Then,*

(i) $I_{(d,r)}(x) = 1$ *for all $x \in \mathbb{Z}$ with $x \equiv r \pmod d$;*
(ii) $I_{(d,r)}(x) = 0$ *for all $x \in \mathbb{Z}$ with $x \not\equiv r \pmod d$.*

Proof. Let $x \in \mathbb{Z}$ with $x \equiv r \pmod d$ and consider the Euclidean division of x by d, that is $x = dq + r$ for some $q \in \mathbb{Z}$ and $r \in \{0, \ldots, d-1\}$. (i) can be proved by applying $\sin^2(q\pi + \theta) = \sin^2\theta$ for all $\theta \in \mathbb{R}$ and the formula for the product of sines of multiple arcs in [12], that is $\prod_{j=1}^{d-1} \sin\left(\frac{j\pi}{d}\right) = \frac{d}{2^{d-1}}$ to show that $I_{(d,r)}(x) = 1$ for all $x \in \mathbb{Z}$ with $x \equiv r \pmod d$. To prove (ii), let $x \in \mathbb{Z}$ with $x \not\equiv r \pmod d$. Clearly, there are $q \in \mathbb{Z}$ and $r \neq r^* \in \{0, \ldots, d-1\}$ such that $x \equiv r^* \pmod d$ and $x = dq + r^*$. By applying $\sin^2(q\pi + \theta) = \sin^2\theta$ for all $\theta \in \mathbb{R}$, it follows that $I_{(d,r)}(x) = 0$ for all $x \in \mathbb{Z}$ with $x \not\equiv r \pmod d$. □

Since the squared trigonometric functions are infinitely differentiable on $\mathbb{R}$, it follows that indicator functions are smooth. Thus, it goes without saying that derivatives of the indicator functions must exist. Moreover, the first-order derivative of these functions vanishes for all integer points, which is claimed in the following lemma:

Lemma 2.2. *Let $I_{(d,r)} : \mathbb{R} \longrightarrow \mathbb{R}$ be an indicator function. Then, the first-order derivative of $I_{(d,r)}$ vanishes for all integer points.*

Proof. By Definition 2.2, the first-order derivative of $I_{(d,r)}$ with respect to x is given by

$$I'_{(d,r)}(x) = \frac{2 \cdot 4^{d-1}\pi}{d^3} \sum_{\substack{\ell=0 \\ \ell \neq r}}^{d-1} \sin\left(\frac{(x-\ell)\pi}{d}\right) \cos\left(\frac{(x-\ell)\pi}{d}\right) \times \prod_{\substack{j=0 \\ j \neq r \wedge j \neq \ell}}^{d-1} \sin^2\left(\frac{(x-j)\pi}{d}\right). \tag{2.2}$$

In the case of $x \equiv r \pmod d$, we observe that $\sin\left(\frac{(x-\ell)\pi}{d}\right) \neq 0$ for all $\ell \in \{0, \ldots, d-1\} \setminus \{r\}$. By Lemma 2.1 (i), (2.2) can be written as

$$I'_{(d,r)}(x) = \frac{2\pi}{d} \sum_{\substack{\ell=0 \\ \ell \neq r}}^{d-1} \cot\left(\frac{(x-\ell)\pi}{d}\right) I_{(d,r)}(x) = -\frac{2\pi}{d} \sum_{\substack{\ell=0 \\ \ell \neq r}}^{d-1} \cot\left(\frac{(\ell-r)\pi}{d}\right).$$

After shifting the index ℓ and applying the remark from §A7. Sums of $\cot^{2p}\left(\frac{k\pi}{n}\right)$ in [3], which is $\sum_{\ell=1}^{d-1} \cot\left(\frac{\ell\pi}{d}\right) = 0$. It follows that $I'_{(d,r)}(x) = 0$ for all $x \equiv r \pmod d$. Let us consider the case of $x \not\equiv r \pmod d$, (2.2) can be written as

$$I'_{(d,r)}(x) = \frac{2 \cdot 4^{d-1}\pi}{d^3} \sum_{\substack{\ell=0 \\ \ell \neq r}}^{d-1} \cos\left(\frac{(x-\ell)\pi}{d}\right) \prod_{\substack{j=0 \\ j \neq r \wedge j \neq \ell}}^{d-1} \sin\left(\frac{(x-j)\pi}{d}\right) \prod_{\substack{j=0 \\ j \neq r}}^{d-1} \sin\left(\frac{(x-j)\pi}{d}\right).$$

It follows that $I'_{(d,r)}(x) = 0$ for all $x \not\equiv r \pmod d$. Therefore, $I'_{(d,r)}(x) = 0$ for all $x \in \mathbb{Z}$. □

At this point, we have explored the properties of indicator functions. According to (1.1), Chamberland's extension can be written in the form of indicator functions as follows:

$$\widetilde{T}_1(x) = \left(\frac{x}{2}\right) I_{(2,0)}(x) + \left(\frac{3x+1}{2}\right) I_{(2,1)}(x).$$

By the similar procedure, we can extend the domain of any generalized $3x+1$ function $T_{(d,\vec{m},\vec{r})}$ from $\mathbb{Z}$ to $\mathbb{R}$ as the function $\widetilde{T}_{(d,\vec{m},\vec{r})} : \mathbb{R} \longrightarrow \mathbb{R}$, defined by

$$\widetilde{T}_{(d,\vec{m},\vec{r})}(x) = \sum_{j=0}^{d-1} \left(\frac{m_j x - r_j}{d}\right) I_{(d,j)}(x). \tag{2.3}$$

Theorem 2.1. *Let $T_{(d,\vec{m},\vec{r})} : \mathbb{Z} \longrightarrow \mathbb{Z}$ be a generalized $3x+1$ function. Then, the function $\widetilde{T}_{(d,\vec{m},\vec{r})} : \mathbb{R} \longrightarrow \mathbb{R}$, defined by (2.3) is a smooth extension of $T_{(d,\vec{m},\vec{r})}$ from $\mathbb{Z}$ to $\mathbb{R}$.*

Proof. Since the linear function $f_j : \mathbb{R} \longrightarrow \mathbb{R}$, defined by $f_j(x) = \frac{m_j x - r_j}{d}$ is well-defined on $\mathbb{R}$ for all $j \in \{0, \ldots, d-1\}$, the product between f_j and $I_{(d,j)}$ is also well-defined on $\mathbb{R}$. It follows that $f_0 I_{(d,0)} + \cdots + f_{d-1} I_{(d,d-1)}$ is well-defined on $\mathbb{R}$. Next, we note that Lemma 2.1 (i) and (ii) imply that $\widetilde{T}_{(d,\vec{m},\vec{r})}\Big|_{\mathbb{Z}} = T_{(d,\vec{m},\vec{r})}$. For the last part, $\widetilde{T}_{(d,\vec{m},\vec{r})}$ is smooth since f_j and $I_{(d,j)}$ are infinitely differentiable on $\mathbb{R}$ for all $j \in \{0, \ldots, d-1\}$. □

Before moving to the next section, we consider in the case of parameters $d = 2$, $\vec{m} = (1, 3)$, and $\vec{r} = (0, -1)$. These parameters correspond to the $3x+1$ function. By Theorem 2.1, $\widetilde{T}_{(d,\vec{m},\vec{r})}$ is an extension of the $3x+1$ function. Moreover, the extension $\widetilde{T}_{(d,\vec{m},\vec{r})}$ is exactly the same as Chamberland's extension $\widetilde{T}_1$.

3. MAIN RESULTS

In this section, we illustrate some results involving integer periodic cycles for the extension of generalized $3x+1$ functions in the case of relatively prime type, which is claimed by Theorem 2.1. The first result shows that every integer periodic K-point is isolated in the usual topology on $\mathbb{R}$. This evidence demonstrates that each integer periodic point of the extension might be characterized by the stability of fixed or periodic points. By showing that all positive integer periodic cycles are attracting, it follows that Chamberland's extension is also a natural extension of the $3x+1$ function. Indeed, the extensions $\widetilde{T}_1$ and $\widetilde{T}_2$ are natural extensions of the $3x+1$ function in the sense of the real dynamics of these extensions related to the τ-stopping time of R. Terras [14].

Again, let the sequence $\mathrm{Orb}_f(x) := \{x, f(x), f^2(x), \ldots\}$ be an orbit of x iterated by any function f. An orbit Ω is said to be a *periodic cycle* (or *periodic K-cycle*) for f if there is a positive integer K such that $f^K(\omega) = \omega$ for all $\omega \in \Omega$. Every point in Ω is called a *periodic point* (or *periodic K-point*). In the case of $K = 1$, the single point in Ω is called a *fixed point* of f. We remark that this notation allows us to interpret all periodic 2-cycles and periodic 3-cycles to be periodic 6-cycles.

Theorem 3.2. *Let $\widetilde{T}_{(d,\vec{m},\vec{r})}$ be the extension of the generalized $3x+1$ function of relatively prime type $T_{(d,\vec{m},\vec{r})}$ defined by (2.3) and P_K be a union of periodic K-cycles for $\widetilde{T}_{(d,\vec{m},\vec{r})}$. Then, every integer point in P_K is isolated in the usual topology on $\mathbb{R}$.*

Proof. It is easy to see that for every $\omega \in P_K$, we have $\widetilde{T}^K_{(d,\vec{m},\vec{r})}(\omega) = \omega$. Suppose that there is an integer point $\omega^* \in P_K$ such that ω^* is not isolated. Without loss of generality, there is a sequence in P_K, denoted by $\{\omega_j\}_{j\in\mathbb{N}}$ such that $\omega_j \longrightarrow \omega^*$ as $j \longrightarrow \infty$ together with $\omega_{j+1} \neq \omega_j$ and $\omega_j \neq \omega^*$ for all $j \in \mathbb{N}$. Let $f = \widetilde{T}^K_{(d,\vec{m},\vec{r})}$, we have $f(\omega^*) = \omega^*$ and $f(\omega_j) = \omega_j$ for all $j \in \mathbb{N}$. Note that for every $j \in \mathbb{N}$, we have

$$\frac{f(M_j) - f(m_j)}{M_j - m_j} = 1,$$

where $m_j = \min\{\omega_j, \omega_{j+1}\}$ and $M_j = \max\{\omega_j, \omega_{j+1}\}$. Since f is differentiable on $\mathbb{R}$, there is $\xi_j \in (m_j, M_j)$ such that $f'(\xi_j) = 1$ for all $j \in \mathbb{N}$ by Mean Value Theorem. Thus, we obtain $f'(\omega^*) = 1$. Since $T_{(d,\vec{m},\vec{r})}$ is a generalized $3x + 1$ function of relatively prime type, $\gcd(m_j, d) = 1$ for all $j \in \{0, \ldots, d-1\}$. The first-order derivative of $\widetilde{T}_{(d,\vec{m},\vec{r})}$ with respect to x is given by

$$\widetilde{T}'_{(d,\vec{m},\vec{r})}(x) = \sum_{j=0}^{d-1} \left[\left(\frac{m_j}{d}\right) I_{(d,j)}(x) + \left(\frac{m_j x - r_j}{d}\right) I'_{(d,j)}(x) \right]. \tag{3.4}$$

Let $\omega_0 := \omega^*$ be an integer point in P_K. If $K \neq 1$, there are $\omega_1, \ldots, \omega_{K-1} \in P_K$ such that $\{\omega_0, \ldots, \omega_{K-1}\}$ is an integer periodic K-cycle for $\widetilde{T}_{(d,\vec{m},\vec{r})}$. By applying Chain Rule, it follows that

$$f'(\omega^*) = \widetilde{T}'_{(d,\vec{m},\vec{r})}(\omega_0) \cdots \widetilde{T}'_{(d,\vec{m},\vec{r})}(\omega_{K-1}).$$

By Lemma 2.1 (i) and (ii) together with Lemma 2.2, we observe that

$$f'(\omega^*) = \frac{\hat{m}_0 \cdots \hat{m}_{K-1}}{d^K} = 1$$

for some $\hat{m}_0, \ldots, \hat{m}_{K-1} \in \{m_0, \ldots, m_{d-1}\}$. Thus, we have $d \mid \hat{m}_0 \cdots \hat{m}_{K-1}$. So, there is $j \in \{0, \ldots, d-1\}$ such that $d \mid m_j$. This is a contradiction, which completes the proof. □

According to Chamberland's extension, it can be seen that there is no interval of periodic points of $\widetilde{T}_1$ on $\mathbb{R}^+$. It follows from Singer's theorem with the negative Schwarzian of $\widetilde{T}_1$ on $\mathbb{R}^+$. This is similar to the result from Theorem 3.2 when focusing only on all integer periodic points with parameters $d = 2$, $\vec{m} = (1, 3)$, and $\vec{r} = (0, -1)$. While these two results are similar, the result from Theorem 3.2 does not require the Schwarzian derivative.

Theorem 3.2 also implies that for every integer point ω in the union of periodic K-cycles P_K, there is a sufficiently small $\delta > 0$ such that $((\omega - \delta, \omega + \delta) \cap P_K) \setminus \{\omega\} = \emptyset$. This also means that every point in the neighborhood cannot perform a periodic K-cycle, excluding ω. It is possible that the orbit of $x \in (\omega - \delta, \omega + \delta) \setminus \{\omega\}$ iterated by $\widetilde{T}^K_{(d,\vec{m},\vec{r})}$ might converge to ω, diverge, or neither. This involves the stability of fixed and periodic points. The following definition can be used to determine fixed and periodic points.

Definition 3.3 ([9]). A fixed point ω of a smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be *attracting* if there is a sufficiently small $\delta > 0$ such that for every $x \in (\omega - \delta, \omega + \delta)$, $f^k(x) \longrightarrow \omega$ as $k \longrightarrow \infty$. On the other hand, it is said to be *repelling* if there is a sufficiently small $\delta > 0$ such that for every $x \in (\omega - \delta, \omega + \delta) \setminus \{\omega\}$, there is $k \in \mathbb{N}$ such that $f^k(x) \notin (\omega - \delta, \omega + \delta)$. In general, a periodic K-cycle Ω for f is said to be *attracting* (*repelling*) if for every $\omega \in \Omega$, (Ω is an attracting (a repelling)) fixed point of f^K.

Proposition 3.1 ([9]). *Let $\Omega = \{\omega_0, \ldots, \omega_{K-1}\}$ be a periodic K-cycle for a smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}$. Then, Ω is attracting if $|f'(\omega_0) \cdots f'(\omega_{K-1})| < 1$. On the other hand, Ω is repelling if $|f'(\omega_0) \cdots f'(\omega_{K-1})| > 1$.*

As mentioned, M. Chamberland showed that every positive integer periodic cycle for the extension $\widetilde{T}_1$ is attracting [2]. We observe that the stability of their periodic cycles could be proved by using the parameter $\vec{r} = (0, -1)$. For generalized $3x+1$ functions, this observation can be stated as the following lemma:

Lemma 3.3. *Let $T_{(d,\vec{m},\vec{r})}$ be a generalized $3x+1$ function of relatively prime type and Ω be an integer periodic cycle for $T_{(d,\vec{m},\vec{r})}$ without zero. Then, there is $\omega \in \Omega$, such that $\omega \equiv j \pmod d$ for some $j \in \{0, \ldots, d-1\}$ satisfying $r_j \neq 0$.*

Proof. Let $\Omega = \{\omega_0, \ldots, \omega_{K-1}\}$ be an integer periodic K-cycle for $T_{(d,\vec{m},\vec{r})}$ without zero. We define the index set $J = \{j \in \{0, \ldots, d-1\} : r_j \neq 0\}$. It is easy to see that the case $J = \{0, \ldots, d-1\}$ is trivial. Let us consider the case of $J \neq \{0, \ldots, d-1\}$, we have $\{0, \ldots, d-1\} \setminus J \neq \emptyset$. Suppose that for every $\omega \in \Omega$, $\omega \not\equiv j \pmod d$ for all $j \in J$. So, there is $j_\omega \in \{0, \ldots, d-1\} \setminus J$ such that $\omega \equiv j_\omega \pmod d$. Since Ω is an integer periodic K-cycle for $T_{(d,\vec{m},\vec{r})}$ without zero, we have

$$\omega_0 = \frac{m_{j_{\omega_0}} \cdots m_{j_{\omega_{K-1}}} \omega_0}{d^K} \quad \Longrightarrow \quad d \,|\, m_{j_{\omega_0}} \cdots m_{j_{\omega_{K-1}}}.$$

Thus, there is $\omega^* \in \Omega$ such that $d \,|\, m_{j_{\omega^*}}$. This is a contradiction, which completes the proof. □

Since the generalized $3x+1$ function of relatively prime type $T_{(d,\vec{m},\vec{r})}$ is considered, the extension $\widetilde{T}_{(d,\vec{m},\vec{r})}$ defined by (2.3) is an extension of $T_{(d,\vec{m},\vec{r})}$. In the case of $T_{(d,\vec{m},\vec{r})}$ specified by parameters d-tuple of positive integers $\vec{m}$ and d-tuple of non-positive integers $\vec{r}$, the stability of positive integer periodic cycles for $\widetilde{T}_{(d,\vec{m},\vec{r})}$ is shown in the following theorem:

Theorem 3.3. *Let $T_{(d,\vec{m},\vec{r})}$ ba a generalized $3x+1$ function of relatively prime type specified by parameters d-tuple of positive integers $\vec{m}$ and d-tuple of non-positive integers $\vec{r}$. Also, let $\widetilde{T}_{(d,\vec{m},\vec{r})}$ be the extension of $T_{(d,\vec{m},\vec{r})}$ defined by (2.3). Then, every positive integer periodic cycle for $\widetilde{T}_{(d,\vec{m},\vec{r})}$ is attracting.*

Proof. Let Ω be a positive integer periodic K-cycle for $\widetilde{T}_{(d,\vec{m},\vec{r})}$. By Lemma 3.3, there is $\omega^* \in \Omega$ such that $\omega^* \equiv j^* \pmod d$ for some $j^* \in \{0, \ldots, d-1\}$ satisfying $r_{j^*} \neq 0$. Hence, $r_{j^*} > 0$. According to S. Eliahou [5], we have

$$\prod_{\omega \in \Omega} \omega = \prod_{\omega \in \Omega} \widetilde{T}_{(d,\vec{m},\vec{r})}(\omega) \quad \Longrightarrow \quad \prod_{\omega \in \Omega} \left| \frac{\widetilde{T}_{(d,\vec{m},\vec{r})}(\omega)}{\omega} \right| = 1.$$

We consider

$$1 = \prod_{\omega \in \Omega} \left| \frac{\widetilde{T}_{(d,\vec{m},\vec{r})}(\omega)}{\omega} \right| = \left(\frac{m_{j^*}}{d} - \frac{r_{j^*}}{d\omega^*} \right) \prod_{\omega \in \Omega \setminus \{\omega^*\}} \left[\sum_{j=0}^{d-1} \left(\frac{m_j}{d} - \frac{r_j}{d\omega} \right) I_{(d,j)}(\omega) \right].$$

Since $-\frac{r_j}{d\omega} \geq 0$ for all $j \in \{0, \ldots, d-1\}$ and $-\frac{r_{j^*}}{d\omega^*} > 0$, we obtain

$$1 > \frac{m_{j^*}}{d} \prod_{\omega \in \Omega \setminus \{\omega^*\}} \left[\sum_{j=0}^{d-1} \left(\frac{m_j}{d} \right) I_{(d,j)}(\omega) \right] = \prod_{\omega \in \Omega} \left[\sum_{j=0}^{d-1} \left(\frac{m_j}{d} \right) I_{(d,j)}(\omega) \right].$$

Since the first-order derivative of $\widetilde{T}_{(d,\vec{m},\vec{r})}$ with respect to x is given by (3.4), we obtain

$$\prod_{\omega \in \Omega} \left| \widetilde{T}'_{(d,\vec{m},\vec{r})}(\omega) \right| = \prod_{\omega \in \Omega} \left[\sum_{j=0}^{d-1} \left(\frac{m_j}{d} \right) I_{(d,j)}(\omega) \right] < 1$$

by Lemma 2.2. Therefore, Ω is attracting. □

Next, let $\{X_k\}_{k=0}^{\infty}$ be a sequence of functions defined from $\mathbb{N} \cup \{0\}$ to $\{0,1\}$. For every $k \in \mathbb{N} \cup \{0\}$, X_k defined by $X_k(x) = 0$ if $T^k(x)$ is even. Otherwise, $X_k(x) = 1$.

Theorem 3.4 ([14], Theorem 1.1). *Let $x \in \mathbb{N} \cup \{0\}$. Also, let*

$$\lambda_k = \frac{3^{X_0+\cdots+X_{k-1}}}{2^k} \quad \text{and} \quad \rho_k = \frac{\lambda_k}{2}\left(\frac{X_0}{\lambda_1} + \cdots + \frac{X_{k-1}}{\lambda_k}\right)$$

for all $k \in \mathbb{N}$. Then, the integer $T^K(x)$ can be written as $T^K(x) = \lambda_K(x)\, x + \rho_K(x)$.

We observe that every positive integer periodic K-cycle for Chamberland's extension relates to the coefficient function λ_K. This fact is expressed in the following proposition:

Proposition 3.2. *Let ω be a positive integer periodic K-point of $\widetilde{T}_1$. Then, $\left(\widetilde{T}_1^K\right)'(\omega) = \lambda_K(\omega)$. Moreover, $\lambda_K(\omega) < 1$.*

Proof. Let $\omega_0 = \omega$. If $K \neq 1$, there are $\omega_1, \ldots, \omega_{K-1} \in \mathbb{N}$ such that $\{\omega_0, \ldots, \omega_{K-1}\}$ is a periodic K-cycle for $\widetilde{T}_1$. By applying Chain Rule, it follows that

$$\left(\widetilde{T}_1^K\right)'(\omega) = \widetilde{T}_1'(\omega_0) \cdots \widetilde{T}_1'(\omega_{K-1}) = \prod_{j=0}^{K-1} \widetilde{T}_1'(\omega_j).$$

Since the first-order derivative of $\widetilde{T}_{(d,\vec{m},\vec{r})}$ with respect to x is given by (3.4), we apply it with parameters $d = 2$, $\vec{m} = (1,3)$, and $\vec{r} = (0,-1)$ to obtain

$$\begin{aligned}\left(\widetilde{T}_1^K\right)'(\omega) &= \prod_{j=0}^{K-1}\left[\frac{1}{2}\sin^2\left(\frac{(\omega_j - 1)\pi}{2}\right) + \frac{3}{2}\sin^2\left(\frac{\omega_j \pi}{2}\right)\right] = \frac{3^{X_0(\omega_0)+\cdots+X_0(\omega_{K-1})}}{2^K} \\ &= \frac{3^{X_0(\omega_0)+\cdots+X_{K-1}(\omega_0)}}{2^K} = \lambda_K(\omega).\end{aligned}$$

To see that $\lambda_K(\omega) < 1$, it follows from the proof of Theorem 3.3. □

As discussed in [4], the stopping time of a real number x for $\widetilde{T}_1$ is defined as the smallest positive integer k such that $\widetilde{T}_1^k(x) < x$. Otherwise, the stopping time is ∞. Likewise, the τ-stopping time of a real number x for $\widetilde{T}_1$ is defined as the smallest positive integer k such that $\lambda_k(x) < 1$. Otherwise, the τ-stopping time is ∞. In fact, the τ-stopping time is less than or equal to the stopping time [14]. Since Proposition 3.2 shows a relationship between Chamberland's extension and the τ-stopping time of R. Terras based on any positive integer periodic cycle of integers, it also shows that Chamberland's extension is a natural extension of the $3x+1$ function, which follows from the sense of the real dynamics of this extension related to the τ-stopping time.

4. Some numerical results

Let us consider Chamberland's extension $\widetilde{T}_1$, which is an extension of the $3x+1$ function. After applying Theorem 3.2 with the parameters $d = 2$, $\vec{m} = (1,3)$, and $\vec{r} = (0,-1)$, we see that every integer periodic K-point is isolated in the usual topology on $\mathbb{R}$. This result is similar to the result from M. Chamberland, which is obtained from Singer's theorem by showing that $\widetilde{T}_1$ has a negative Schwarzian on $\mathbb{R}^+$ [2]. Since $\{1,2\}$ is a periodic 2-cycle for $\widetilde{T}_1$, we can choose the initial numbers $x = 1.02$ and $x = 1.98$ for testing the stability. The stability result around the point $x = 2$ is illustrated in Figures 1, which corresponds to Theorem 3.4.

Next, we consider the generalized $3x+1$ function of relatively prime type $T_{(d,\vec{m},\vec{r})}$ specified by parameters $d = 3$, $\vec{m} = (2,4,4)$, and $\vec{r} = (0,-2,-4)$. Also, let $\widetilde{T}_{(d,\vec{m},\vec{r})}$ be the

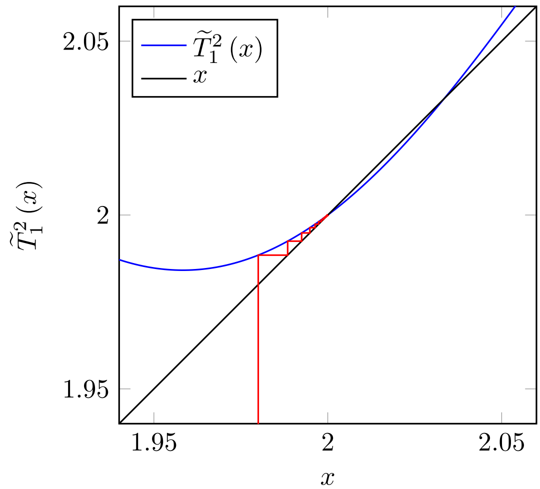

FIGURE 1. Stability testing around the point $x = 2$ with the initial number $x = 1.98$ iterated by $\widetilde{T}_1^2$

extension of $T_{(d,\vec{m},\vec{r})}$ defined by (2.3). Some integer periodic cycles for $\widetilde{T}_{(d,\vec{m},\vec{r})}$ are given by

$$\{-4\}, \quad \{-2\}, \quad \{0\}, \quad \{4,6\}, \quad \{8,12\}, \quad \text{and} \quad \{-12,-10,-8\}.$$

Since all periodic cycles listed above can be interpreted as a periodic 6-cycle, the union of the listed periodic cycles is a subset of the union of all periodic 6-cycles, denoted by P_6. Clearly, every point in P_6 must be a root of the equation $\widetilde{T}^6_{(d,\vec{m},\vec{r})}(x) - x = 0$. Figure 2 illustrates that all listed integer periodic points $\omega \in P_6$ such that $|\omega| \leq 8$, marked by red dots, are isolated points of P_6 in the usual topology on $\mathbb{R}$.

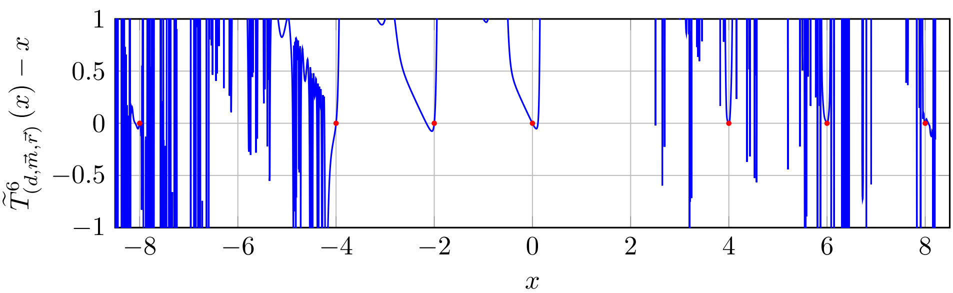

FIGURE 2. Every integer periodic point $\omega \in P_6$ such that $|\omega| \leq 8$ marked by red dot is an isolated root of the equation $\widetilde{T}^6_{(d,\vec{m},\vec{r})}(x) - x$ (in the usual topology on $\mathbb{R}$), where $d = 3$, $\vec{m} = (2,4,4)$, and $\vec{r} = (0,-2,-4)$.

For the stability of the positive integer periodic 2-cycle $\{4,6\}$, we can choose the initial numbers $x = 3.965$ and $x = 5.98$ for testing the stability of $\{4,6\}$. The stability result around the point $x = 6$ is illustrated in Figures 3, which also corresponds to Theorem 3.4.

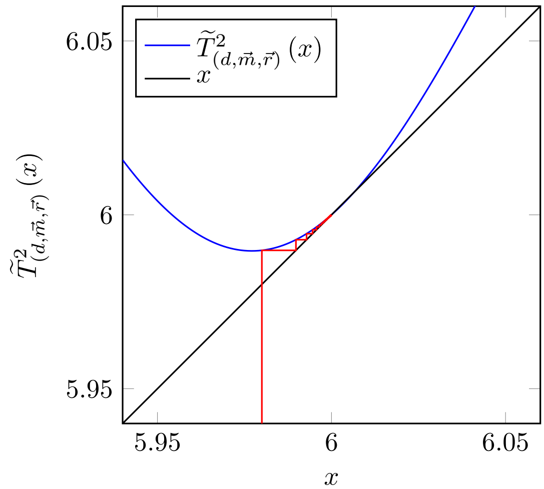

FIGURE 3. Stability testing around the point $x = 6$ with the initial number $x = 5.98$ iterated by $\widetilde{T}^2_{(d,\vec{m},\vec{r})}$, where $d = 3$, $\vec{m} = (2, 4, 4)$, and $\vec{r} = (0, -2, -4)$

5. CONCLUSION

This paper provides an extension of generalized $3x + 1$ functions. This extension is a function defined from $\mathbb{R}$ to itself. We see that every integer periodic point is isolated in the usual topology on $\mathbb{R}$. Also, every positive integer periodic cycle is attracting if the extension is specified by parameters $d \in \mathbb{N} \setminus \{1\}$, d-tuple of positive integers $\vec{m}$, and d-tuple of non-positive integers $\vec{r}$. It shows that Chamberland's extension is also a natural extension of the $3x + 1$ function in the sense of the real dynamics of this extension related to the τ-stopping time of R. Terras. Finally, some numerical results are presented to verify the theoretical results.

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