CARPATHIAN J. MATH.
Volume **40** (2024), No. 3,
Pages 805 - 817

Online version at https://www.carpathian.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2024.03.17

Weak and Strong Convergence of Split Douglas-Rachford Algorithms for Monotone Inclusions

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ABSTRACT. We are concerned in this paper with the convergence analysis of the primal-dual splitting (PDS) and the split Douglas-Rachford (SDR) algorithms for monotone inclusions by using an operator-oriented approach. We shall show that both PDS and SDR algorithms can be driven by a (firmly) nonexpansive mapping in a product Hilbert space. We are then able to apply the Krasnoselskii-Mann and Halpern fixed point algorithms to PDS and SDR to get weakly and strongly convergent algorithms for finding solutions of the primal and dual monotone inclusions. Moreover, an additional projection technique is used to derive strong convergence of a modified SDR algorithm.

1. INTRODUCTION

Let $\mathcal{H}$ and $\mathcal{K}$ be real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Consider a primal convex optimization problem of the form

$$\underset{x\in\mathcal{H}}{\text{minimize}}\ F(x) := f(x) + g(Lx), \tag{1.1}$$

where $f \in \Gamma_0(\mathcal{H})$ and $g \in \Gamma_0(\mathcal{K})$, and $L : \mathcal{H} \to \mathcal{K}$ is a bounded linear operator. Here $\Gamma_0(\mathcal{H})$ (resp., $\Gamma_0(\mathcal{K})$) denotes the family of all proper, lower semicontinuous, convex functions on $\mathcal{H}$ (resp., $\mathcal{K}$) taking values in $\overline{\mathbb{R}} := (-\infty, \infty]$.

The dual problem of (1.1) (assuming $F \in \Gamma_0(\mathcal{H})$) is

$$\underset{v\in\mathcal{K}}{\text{maximize}}\ F^*(v) := -f^*(L^*v) - g^*(-v). \tag{1.2}$$

Here f^* and g^* are the conjugate of f and g, respectively, and L^* is the adjoint of L defined via the relation: $\langle Lx, v\rangle = \langle x, L^*v\rangle$ for all $x \in \mathcal{H}$ and $v \in \mathcal{K}$. Recall that f^* is defined by $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in \mathcal{H}\}$ for $x^* \in \mathcal{H}$, and g^* is similarly defined.

Note that the dual problem (1.2) can equivalently be rewritten as a minimization problem as follows:

$$\underset{v\in\mathcal{K}}{\text{minimize}}\ (-F^*(v)) := f^*(L^*v) + g^*(-v). \tag{1.3}$$

Problem (1.1), together with its dual (1.2), can be extended to monotone inclusions. Indeed, let $A : \mathcal{H} \to 2^{\mathcal{H}}$ and $B : \mathcal{K} \to 2^{\mathcal{K}}$ be maximal monotone operators. Consider the inclusion problem:

$$\text{Find } (\hat{x}, \hat{v}) \in \mathcal{H} \times \mathcal{K} \quad \text{such that} \quad 0 \in A\hat{x} + L^*\hat{v},\ 0 \in B^{-1}\hat{v} - L\hat{x}. \tag{1.4}$$

We shall use $\mathcal{Z}$ to denote the set of solutions of (1.4), and always assume $\mathcal{Z} \neq \emptyset$ throughout the rest of this paper.

Note that when $A = \partial f$ and $B = \partial g$, the subdifferentials of $f \in \Gamma_0(\mathcal{H})$ and $g \in \Gamma_0(\mathcal{K})$, respectively, the inclusion (1.4) is reduced to the problem of finding solutions $\hat{x} \in \mathcal{H}$ of

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Received: 02.04.2024. In revised form: 15.04.2024. Accepted: 17.04.2024
2010 *Mathematics Subject Classification.* 47J26, 47H09, 47H05, 90C46.
Key words and phrases. *Monotone inclusion, nonexpansive mapping, fixed-point, primal-dual splitting, split Douglas-Rachford, nonstandard metric, projection.*
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the primal problem (1.1) and $\hat{v} \in \mathcal{K}$ of the dual problem (1.2), namely, the first-order optimality conditions hold:

$$(1.5) \qquad 0 \in \partial f(\hat{x}) + L^* \circ B \circ L(\hat{x}),\ 0 \in (\partial g)^{-1}(\hat{v}) - L \circ (\partial f)^{-1} \circ L^*(-\hat{v}).$$

This leads to the general primal and dual inclusions as follows:

$$(1.6) \qquad \text{Primal inclusion: find } \hat{x} \in \mathcal{H} \quad \text{with the property} \quad 0 \in A\hat{x} + L^* B L \hat{x}$$

and

$$(1.7) \qquad \text{Dual inclusion: find } \hat{v} \in \mathcal{K} \quad \text{with the property} \quad 0 \in B^{-1}\hat{v} - LA^{-1}L^*(-\hat{v}).$$

Observe that the inclusion (1.4) is equivalent to the inclusions (1.6) and (1.7) in the sense that if $(\hat{x}, \hat{v}) \in \mathcal{Z}$, then $\hat{x}$ and $\hat{v}$ solve (1.6) and (1.7), respectively, and conversely, if $\hat{x} \in \mathcal{H}$ is a solution to the primal inclusion (1.6), then there exists $\hat{v} \in \mathcal{K}$ (indeed, $\hat{v} \in BL\hat{x}$ such that $0 \in A\hat{x} + L^*\hat{v}$), which is a solution to the dual inclusion (1.7), such that $(\hat{x}, \hat{v}) \in \mathcal{Z}$.

In the case where $L = I$ (assuming $\mathcal{H} = \mathcal{K}$), then the primal inclusion (1.6) turns out to be the problem of finding a zero of the sum of A and B, that is,

$$(1.8) \qquad \text{find } \hat{x} \in \mathcal{H} \quad \text{with the property} \quad 0 \in A\hat{x} + B\hat{x}.$$

The set of solutions of (1.8) is denoted by $\operatorname{zer}(A + B)$ and assume that it is nonempty.

This problem can be solved by the Douglas-Rachford (DR) splitting method of Lions and Mercier [12, Theorem 1], which generates a sequence (z_n) by the following iteration:

$$(1.9) \qquad z_{n+1} = J_{\tau B}(2J_{\tau A} z_n - z_n) + z_n - J_{\tau A} z_n, \quad n = 0, 1, \cdots,$$

where $\tau > 0$, the initial point $z_0 \in \mathcal{H}$ is chosen arbitrarily, and $J_{\tau A} = (I + \tau A)^{-1}$ is the resolvent of τA. Lions and Mercier [12] proved that (z_n) converges weakly to a point $\hat{z}$ and $J_{\tau A}\hat{z} \in \operatorname{zer}(A + B)$.

Turning to the general inclusion (1.6) in the case where L is not the identity I, one needs the maximal monotonicity of L^*BL in order to directly apply DR (1.9), which is computationally costly due to sub-iterations for computing the resolvent $J_{\tau L^*BL}$, as pointed out in [2]. Consequently, splitting methods for separating the operator L from the resolvent of B are needed. In [23], Vũ introduced such a splitting method, known as primal-dual splitting (PDP) method, as follows:

Algorithm:

$$(1.10) \qquad \begin{cases} x_{n+1} = (1 - \lambda_n)x_n + \lambda_n J_{\tau A}(x_n - \tau L^* v_n) \\ v_{n+1} = (1 - \lambda_n)v_n + \lambda_n J_{\sigma B^{-1}}(v_n + \sigma L(2J_{\tau A}(x_n - \tau L^* v_n) - x_n)), \end{cases}$$

where the initial guess $(x_0, v_0) \in \mathcal{H} \times \mathcal{K}$ and the parameters $\tau, \sigma > 0$ satisfy the condition $\tau\sigma\|L\|^2 < 1$.

A more general splitting method, referred to as split Douglas-Rachford (SDR), is studied in [2], which updates iterations as follows (we here use Ω to replace Υ in [2, Algorithm 1.2]).

Algorithm:

$$(1.11) \qquad \begin{cases} v_n = \Sigma(I - J_{\Sigma^{-1}B})(Lx_n + \Sigma^{-1}u_n) \\ z_n = x_n - \Omega L^* v_n \\ x_{n+1} = J_{\Omega A} z_n \\ u_{n+1} = \Sigma L(x_{n+1} - x_n) + v_n \end{cases}$$

where $\Omega : \mathcal{H} \to \mathcal{H}$ and $\Sigma : \mathcal{K} \to \mathcal{K}$ are strongly monotone, self-adjoint, linear operators such that $\Omega^{-1} - L^*\Sigma L$ is monotone (equivalently, $\|\Sigma^{\frac{1}{2}} L \Omega^{\frac{1}{2}}\| \leq 1$). Algorithm (1.11) is a nonstandard metric version of DR for (1.6) [2].

Primal-dual and Douglas-Rachford methods were extensively studied (see [4, 5, 3, 17, 8, 12, 13] and references therein).

We are aimed in this paper at addressing the convergence analysis of the primal-dual splitting (PDS) and split Douglas-Rachford (SDR) algorithms for monotone inclusions by using an operator-oriented approach. More precisely, we show that both PDS and SDR algorithms can be driven by a (firmly) nonexpansive mapping in a product Hilbert space. This enables us to apply the Krasnoselskii-Mann (KM) and Halpern fixed point algorithms to PDS and SDR to get weakly and strongly convergent algorithms for finding solutions of the primal and dual problems (1.6) and (1.7), i.e., elements of $\mathcal{Z}$. Our obtained results improve and generalize the corresponding results of [23] and [2].

The paper is organized as follows. In the next section we introduce some concepts of nonexpansive and monotone mappings together with basic tools and the KM and Halpern fixed point algorithms. In Section 3 we prove the main convergence results on PDP (1.10) and SDR algorithm (1.11).

2. PRELIMINARIES

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. We use I to denote the identity on $\mathcal{H}$. Given mappings $G : \mathcal{H} \to \mathcal{H}$ and $A : \mathcal{H} \to 2^{\mathcal{H}}$. We say that

- G is nonexpansive [6] if $\|Gx - Gy\| \leq \|x - y\|$ for all $x, y \in \mathcal{H}$.
- G is firmly nonexpansive [6] if $\langle x - y, Gx - Gy \rangle \geq \|Gx - Gy\|^2$ for all $x, y \in \mathcal{H}$; equivalently, $2G - I$ is nonexpansive.
- A is monotone [19] if $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$ for all $x_i \in \text{dom}(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$. Here $\text{dom}(A) = \{x \in \mathcal{H} : Ax \neq \emptyset\}$ is the (effective) domain of A. Furthermore, A is maximal monotone if A is monotone and its graph, $\text{gra}(A) := \{(x, y) \in \mathcal{H} \times \mathcal{H} : x \in \text{dom}(A),\ y \in Ax\}$, is not properly contained in the graph of any other monotone operator.

The set of fixed point of G is denoted as $\text{Fix}(G)$, that is, $\text{Fix}(G) = \{x \in \mathcal{H} : Gx = x\}$.

A typical example of firmly nonexpansive mappings is projections. Recall that the projection from $\mathcal{H}$ onto a nonempty closed convex subset C of $\mathcal{H}$ is defined as

$$P_C(x) := \arg\min_{y \in C} \|x - y\|^2, \quad x \in \mathcal{H}.$$

Recall that the resolvent of a monotone mapping $A : \mathcal{H} \to 2^{\mathcal{H}}$ is defined as $J_A = (I + A)^{-1}$. It is known that if A is maximal monotone, then J_A is firmly nonexpansive on the entire space H. More details on monotone operators can be found from the monographs [20, 16, 1].

To prove the convergence of our algorithms in Section 3, we need two fundamental fixed point iteration algorithms, the Krasnoselskii-Mann (KM) [10, 14] and Halpern [7, 9] algorithms.

The KM algorithm for the fixed point problem $x = Gx$ generates a sequence (u_n) via the iteration process:

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n G u_n, \quad n = 0, 1, \cdots, \tag{2.12}$$

where the initial guess $u_0 \in \mathcal{H}$ and $(\alpha_n) \subset [0, 1]$ (in certain circumstances, $(\alpha_n) \subset [0, 2]$).

The Halpern algorithm for the fixed point problem $x = Gx$ generates a sequence (u_n) through the iteration procedure:

$$u_{n+1} = \beta_n u + (1 - \beta_n) G u_n, \quad n = 0, 1, \cdots, \tag{2.13}$$

where the initial guess $u_0 \in \mathcal{H}$ and $(\beta_n) \subset [0,1]$. The fixed point $u \in H$ is referred to as anchor.

The following convergence for KM and Halpern algorithms (2.12) and (2.15) are known.

Theorem 2.1. [18] *Let $G : \mathcal{H} \to \mathcal{H}$ be nonexpansive with Fix$(G) \neq \emptyset$ and suppose $(\alpha_n) \subset [0,1]$ satisfies the divergence condition*

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty. \tag{2.14}$$

Then the sequence (u_n) generated by the KM algorithm (2.12) *converges weakly to a point of Fix(G). Moreover, if G is firmly nonexpansive, then (α_n) can be relaxed to the range $(\alpha_n) \subset [0,2]$ satisfying the divergence condition*

$$\sum_{n=1}^{\infty} \alpha_n(2-\alpha_n) = \infty.$$

Theorem 2.2. [24, 22, 11] *Let $G : \mathcal{H} \to \mathcal{H}$ be nonexpansive with Fix$(G) \neq \emptyset$ and suppose $(\beta_n) \subset (0,1)$ satisfies the conditions:*

(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$,
(ii) *either* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ *or* $\lim_{n\to\infty}(\beta_{n+1}/\beta_n) = 1$.

Then the sequence (u_n) generated by the Halpern algorithm (2.15) *converges strongly to $P_{Fix(G)}u$. Moreover, if G is firmly nonexpansive, then the condition (ii) is superfluous. In addition, if one takes $\beta_n = \frac{1}{n+1}$ for $n \geq 0$ and $u = u_0$, that is,*

$$u_{n+1} = \frac{1}{n+1}u + \frac{n}{n+1}Gu_n, \quad n = 0, 1, \cdots, \tag{2.15}$$

then one has

$$\|u_n - Gu_n\| \leq \frac{2\|u_0 - u^*\|}{n+1}, \quad n \geq 0,\ u^* \in Fix(G).$$

In the rest of this paper, we shall use the standard notation:

- $x_n \rightharpoonup x$ means that (x_n) converges to x weakly,
- $x_n \to x$ means that (x_n) converges to x strongly.

3. Convergence Analysis of Algorithms

Let $\mathcal{H}$ and $\mathcal{K}$ be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. We begin with an improved simple case of Vũ's algorithm [23].

3.1. **Convergence of Vũ's Algorithm** (1.10)**.** Vũ [23, Algorithm (3.3), p. 671] introduced a splitting algorithm for solving (1.6) and (1.7). The case of $m = 1$ with no perturbation errors is the algorithm (1.10). Because Vũ's proof of the general case (see the proof of [23, Theorem 3.1]) is complicated (i.e., not easily understood), we will include a detailed simpler proof of the following theorem which also relaxes the choice of the step-size parameters (λ_n) from the interval $[\varepsilon, 1]$ for some $0 < \varepsilon < 1$ [23, Theorem 3.1] to the interval $[0, 2]$.

Theorem 3.3. *Suppose the primal and dual problems* (1.6) *and* (1.7) *are solvable, i.e., $\mathcal{Z} \neq \emptyset$. Suppose $\tau\sigma\|L\|^2 < 1$ and $(\lambda_n) \subset [0,2]$ satisfies the divergence condition*

$$\sum_{n=1}^{\infty} \lambda_n(2-\lambda_n) = \infty. \tag{3.16}$$

Then the sequences (x_n) *and* (v_n) *defined by the PDP algorithm* (1.10) *converge weakly to a solution* $\hat{x}$ *of the primal problem* (1.6) *and a solution* $\hat{v}$ *of the dual problem* (1.7), *respectively, i.e.,* $(\hat{x}, \hat{v}) \in \mathcal{Z}$.

Proof. We endow the product space $\mathcal{H} \times \mathcal{K}$ with the standard inner product and norm:

$$\langle (x, v), (x', v') \rangle = \langle x, v \rangle + \langle x', v' \rangle, \quad \|(x, v)\| = \sqrt{\|x\|^2 + \|v\|^2}$$

for $(x, v), (x', v') \in \mathcal{H} \times \mathcal{K}$.

Define operators $M, S : \mathcal{H} \times \mathcal{K} \to 2^{\mathcal{H} \times \mathcal{K}}$ by

$$M(x, v) = (Ax, B^{-1}v), \quad (x, v) \in \mathcal{H} \times \mathcal{K}$$

and

$$S(x, v) = (L^*v, -Lx), \quad (x, v) \in \mathcal{H} \times \mathcal{K}.$$

Then we have that M, S are maximal monotone; moreover, S is skew (i.e., $S^* = -S$).

Note that we have

$$(M + S)(x, v) = (Ax + L^*v, B^{-1}v - Lx), \quad (x, v) \in \mathcal{H} \times \mathcal{K} \tag{3.17}$$

is maximal monotone. Moreover,

$$(\hat{x}, \hat{v}) \in \text{zer}(M + S) \iff (\hat{x}, \hat{v}) \in \mathcal{Z}. \tag{3.18}$$

Namely, finding a point in the solution set $\mathcal{Z}$ is equivalently converted to finding a zero of the maximal monotone mapping $(M + S)$. To this end we define a linear operator $V : \mathcal{H} \times \mathcal{K} \to \mathcal{H} \times \mathcal{K}$ by

$$V(x, v) = (\frac{1}{\tau}x - L^*v, \frac{1}{\sigma}v - Lx), \quad (x, v) \in \mathcal{H} \times \mathcal{K}.$$

Then it is not hard to find that V is self-adjoint (i.e., $V^* = V$) and ρ-strongly positive (see also a generalization in Lemma 3.1), with

$$\rho = \min\{\tau^{-1} - \|L\|^2/\eta, \sigma^{-1} - \eta\},$$

where η is such that $\tau\|L\|^2 < \eta < \sigma^{-1}$ (which is possible due to the assumption $\tau\sigma\|L\|^2 < 1$), in particular, we may take $\eta = (\tau\|L\|^2 + \sigma^{-1})/2$.

As a matter of fact, we have, for each $(x, v) \in \mathcal{H} \times \mathcal{K}$,

$$\begin{aligned}
\langle (x, v), V(x, v) \rangle &= \langle x, \frac{1}{\tau}x - L^*v \rangle + \langle v, \frac{1}{\sigma} - Lx \rangle \\
&= \frac{1}{\tau}\|x\|^2 + \frac{1}{\sigma}\|v\|^2 - 2\langle Lx, v \rangle \\
&\geq \frac{1}{\tau}\|x\|^2 + \frac{1}{\sigma}\|v\|^2 - (\eta^{-1}\|L\|^2\|x\|^2 + \eta\|v\|^2) \\
&= (\frac{1}{\tau} - \frac{\|L\|^2}{\eta})\|x\|^2 + (\frac{1}{\sigma} - \eta)\|v\|^2 \\
&\geq \min\left\{\frac{1}{\tau} - \frac{\|L\|^2}{\eta}, \frac{1}{\sigma} - \eta\right\}(\|x\|^2 + \|v\|^2) \\
&= \rho\|(x, v)\|^2.
\end{aligned}$$

This also implies that $\|V^{-1}\| \leq \rho^{-1}$.

Set $p_n = J_{\tau A}(x_n - \tau L^* v_n)$, $y_n = 2p_n - x_n$, and $q_n = J_{\sigma B^{-1}}(v_n + \sigma L y_n)$. This simplifies the algorithm (1.10) as

$$\begin{cases} x_{n+1} = (1 - \lambda_n)x_n + \lambda_n p_n = x_n + \lambda_n(p_n - x_n), \\ v_{n+1} = (1 - \lambda_n)v_n + \lambda_n q_n = v_n + \lambda_n(q_n - v_n). \end{cases} \tag{3.19}$$

Following the definitions of p_n and q_n, we also have

$$\begin{aligned}&\frac{1}{\tau}(x_n - p_n) - L^* v_n \in A(p_n),\\&\frac{1}{\sigma}(v_n - q_n) - L(x_n - p_n) \in B^{-1}(q_n) - L(p_n).\end{aligned}$$

Set $w_n = (x_n, v_n)$ and $w'_n = (p_n, q_n)$. Then it is readily seen that

$$V(w_n - w'_n) = (\frac{1}{\tau}(x_n - p_n) - L^*(v_n - q_n), \frac{1}{\sigma}(v_n - q_n) - L(x_n - p_n)).$$

Thus, we get $V(w_n - w'_n) \in (M+S)w'_n$ and $V(w_n) \in (M+S+V)w'_n$. Consequently,

$$w'_n = (M+S+V)^{-1}V(w_n) = (I + V^{-1}(M+S))^{-1}w_n = J_{V^{-1}(M+S)}w_n.$$

In the product space $\mathcal{H} \times \mathcal{K}$, the algorithm (3.19) is rewritten as

$$\begin{aligned}w_{n+1} &= w_n + \lambda_n(w'_n - w_n)\\&= w_n + \lambda_n\left(J_{V^{-1}(M+S)}w_n - w_n\right).\end{aligned} \tag{3.20}$$

Renorm $\mathcal{H} \times \mathcal{K}$ by

$$\langle w, w'\rangle_V = \langle w, V(w')\rangle, \quad \|w\|_V = \sqrt{\langle w, V(w)\rangle}$$

for $w = (x, v)$, $w' = (x', v') \in \mathcal{H}\times\mathcal{K}$. We now show that $V^{-1}(M+S)$ is maximal monotone with respect to this new inner product. Indeed, since $M+S$ is monotone, we have for $w = (x, v)$, $w' = (x', v') \in \mathcal{H} \times \mathcal{K}$

$$\langle w - w', V^{-1}(M+S)w - V^{-1}(M+S)w'\rangle_V = \langle w - w', (M+S)w - (M+S)w'\rangle \geq 0.$$

Because $(M+S)$ is maximal monotone and V is strongly positive, $V^{-1}(M+S)$ is maximal monotone. It then follows that the resolvent $J_{V^{-1}(M+S)}$ is firmly nonexpansive. We may now apply Theorem 2.1 to (3.20) to assert that (w_n) converges weakly to a point $\hat{w} = (\hat{x}, \hat{v}) \in \mathrm{Fix}(J_{V^{-1}(M+S)}) = \mathrm{zer}(M+S) = \mathcal{Z}$ under the divergence condition (3.16). Hence, $x_n \rightharpoonup \hat{x}$ and $v_n \rightharpoonup \hat{v}$, and $(\hat{x}, \hat{v}) \in \mathcal{Z}$. □

Taking $\lambda_n = 1$ for all n, the algorithm (1.10) is reduced to the following algorithm:

$$\begin{cases} x_{n+1} = J_{\tau A}(x_n - \tau L^* v_n)\\ v_{n+1} = J_{\sigma B^{-1}}(v_n + \sigma L(2J_{\tau A}(x_n - \tau L^* v_n) - x_n)).\end{cases} \tag{3.21}$$

The result below follows immediately.

Corollary 3.1. *Suppose the primal and dual problems* (1.6) *and* (1.7) *are solvable and* $\tau\sigma\|L\|^2 < 1$. *Then the sequences* (x_n) *and* (v_n) *defined by the algorithm* (3.21) *converge weakly to a solution* $\hat{x}$ *of the primal problem* (1.6) *and a solution* $\hat{v}$ *of the dual problem* (1.7), *respectively.*

3.2. **Convergence of Split Douglas-Rachford Algorithm.** In this subsection we discuss the convergence analysis of the algorithm (1.11). This algorithm is known as split Douglas-Rachford (SDR) algorithm [2] which uses nonstandard metrics Ω and Σ. Here $\Omega : \mathcal{H} \to \mathcal{H}$ and $\Sigma : \mathcal{K} \to \mathcal{K}$ are strongly monotone self-adjoint linear operators.

Lemma 3.1. *Suppose* $\|\Sigma^{\frac{1}{2}} L \Omega^{\frac{1}{2}}\| < 1$ *and define* $V : \mathcal{H} \times \mathcal{K} \to \mathcal{H} \times \mathcal{K}$ *by*

$$V(x, v) = (\Omega^{-1}x - L^* v, \Sigma^{-1}v - Lx), \quad (x, v) \in \mathcal{H} \times \mathcal{K}. \tag{3.22}$$

Then V *is* ρ*-strongly monotone, self-adjoint, and linear, with*

$$\rho = \frac{1 - \|\Sigma^{\frac{1}{2}} L \Omega^{\frac{1}{2}}\|^2}{\|\Omega^{\frac{1}{2}}\|^2 + \|\Sigma^{\frac{1}{2}}\|^2}.$$

Proof. For $(x, v) \in \mathcal{H} \times \mathcal{K}$, we derive that

$$\begin{aligned}\langle (x,v), V(x,v)\rangle &= \langle x, \Omega^{-1}x - L^*v\rangle + \langle v, \Sigma^{-1}v - Lx\rangle \\ &= \langle x, (\Omega^{-1} - L^*\Sigma L)x\rangle + \langle \Sigma Lx - v, Lx\rangle + \langle v, \Sigma^{-1}v - Lx\rangle \\ &= \langle x, (\Omega^{-1} - L^*\Sigma L)x\rangle + \|\Sigma^{-1}v - Lx\|_\Sigma^2. \end{aligned} \tag{3.23}$$

Similarly, we also derive that

$$\langle (x,v), V(x,v)\rangle = \langle v, (\Sigma^{-1} - L\Omega L^*)v\rangle + \|\Omega^{-1}x - L^*v\|_\Omega^2. \tag{3.24}$$

Set $\theta := \|\Sigma^{\frac{1}{2}} L \Omega^{\frac{1}{2}}\| < 1$. Then $\|\Sigma^{\frac{1}{2}} L \Omega^{\frac{1}{2}} y\| \le \theta \|y\|$ for each $y \in \mathcal{H}$. Now it follows that for each $x \in \mathcal{H}$

$$\begin{aligned}\langle x, (\Omega^{-1} - L^*\Sigma L)x\rangle &= \langle x, \Omega^{-1}x\rangle - \langle \Sigma Lx, Lx\rangle \\ &= \|\Omega^{-\frac{1}{2}}x\|^2 - \|\Sigma^{\frac{1}{2}} L \Omega^{\frac{1}{2}} \Omega^{-\frac{1}{2}} x\|^2 \\ &= \|y\|^2 - \|\Sigma^{\frac{1}{2}} L \Omega^{\frac{1}{2}} y\|^2 \quad (\text{with } y = \Omega^{-\frac{1}{2}}x) \\ &\ge (1-\theta^2)\|y\|^2 \\ &\ge \frac{1-\theta^2}{\|\Omega^{\frac{1}{2}}\|^2}\|x\|^2 = \frac{1 - \|\Sigma^{\frac{1}{2}} L \Omega^{\frac{1}{2}}\|^2}{\|\Omega^{\frac{1}{2}}\|^2}\|x\|^2 \end{aligned} \tag{3.25}$$

since $\|y\| = \|\Omega^{-\frac{1}{2}}x\| \ge \|\Omega^{\frac{1}{2}}\|^{-1}\|x\|$.

Similarly, we get

$$\langle v, (\Sigma^{-1} - L\Omega L^*)v\rangle \ge \frac{1 - \|\Omega^{\frac{1}{2}} L^* \Sigma^{\frac{1}{2}}\|^2}{\|\Sigma^{\frac{1}{2}}\|^2}\|v\|^2. \tag{3.26}$$

Multiply $t \in (0,1)$ to Eq. (3.25) and $(1-t)$ to Eq. (3.26), respectively, and add them up to get (noticing $\|\Sigma^{\frac{1}{2}} L \Omega^{\frac{1}{2}}\| = \|\Omega^{\frac{1}{2}} L^* \Sigma^{\frac{1}{2}}\|$)

$$\begin{aligned}\langle (x,v), V(x,v)\rangle &\ge t\frac{1 - \|\Sigma^{\frac{1}{2}} L \Omega^{\frac{1}{2}}\|^2}{\|\Omega^{\frac{1}{2}}\|^2}\|x\|^2 + (1-t)\frac{1 - \|\Omega^{\frac{1}{2}} L^* \Sigma^{\frac{1}{2}}\|^2}{\|\Sigma^{\frac{1}{2}}\|^2}\|v\|^2 \\ &= (1 - \|\Sigma^{\frac{1}{2}} L \Omega^{\frac{1}{2}}\|^2)\left(\frac{t}{\|\Omega^{\frac{1}{2}}\|^2}\|x\|^2 + \frac{1-t}{\|\Sigma^{\frac{1}{2}}\|^2}\|v\|^2\right) \\ &\ge (1 - \|\Sigma^{\frac{1}{2}} L \Omega^{\frac{1}{2}}\|^2)\min\left\{\frac{t}{\|\Omega^{\frac{1}{2}}\|^2}, \frac{1-t}{\|\Sigma^{\frac{1}{2}}\|^2}\right\}(\|x\|^2 + \|v\|^2). \end{aligned} \tag{3.27}$$

Notice that the $\min$ in (3.27) is maximized at $t' := \frac{\|\Omega^{\frac{1}{2}}\|^2}{\|\Omega^{\frac{1}{2}}\|^2 + \|\Sigma^{\frac{1}{2}}\|^2}$, with minimum value of $\frac{1}{\|\Omega^{\frac{1}{2}}\|^2 + \|\Sigma^{\frac{1}{2}}\|^2}$. It then follows from (3.27) that

$$\langle (x,v), V(x,v)\rangle \ge \frac{1 - \|\Sigma^{\frac{1}{2}} L \Omega^{\frac{1}{2}}\|^2}{\|\Omega^{\frac{1}{2}}\|^2 + \|\Sigma^{\frac{1}{2}}\|^2}(\|x\|^2 + \|v\|^2). \tag{3.28}$$

This completes the proof. □

3.2.1. *Weak Convergence.* We convert Algorithm (1.11) to another formulation in terms of (x_n, v_n) which reads as follows:

$$\begin{cases} x_{n+1} = J_{\Omega A}(x_n - \Omega L^* v_n) \\ v_{n+1} = J_{\Sigma B^{-1}}(v_n + \Sigma L(2J_{\Omega A}(x_n - \Omega L^* v_n) - x_n)). \end{cases} \tag{3.29}$$

Note that v_{n+1} can also be written as

$$v_{n+1} = J_{\Sigma B^{-1}}(v_n + \Sigma L(2x_{n+1} - x_n)). \tag{3.30}$$

As a matter of fact, to get v_{n+1} in (3.29), we first utilize [1, Proposition 23.34(iii)] to obtain $I - \Sigma J_{\Sigma^{-1}B}\Sigma^{-1} = J_{\Sigma B^{-1}}$. We then derive from (1.11) that

$$\begin{aligned} v_{n+1} &= \Sigma(I - J_{\Sigma^{-1}B})(Lx_{n+1} + \Sigma^{-1}u_{n+1}) \\ &= \Sigma(Lx_{n+1} + \Sigma^{-1}u_{n+1}) - \Sigma J_{\Sigma^{-1}B}(Lx_{n+1} + \Sigma^{-1}u_{n+1}) \\ &= \Sigma Lx_{n+1} + u_{n+1} - \Sigma J_{\Sigma^{-1}B}\Sigma^{-1}(\Sigma Lx_{n+1} + u_{n+1}) \\ &= (I - \Sigma J_{\Sigma^{-1}B}\Sigma^{-1})(\Sigma Lx_{n+1} + u_{n+1}) \\ &= J_{\Sigma B^{-1}}(\Sigma L(2x_{n+1} - x_n) + v_n) \\ &= J_{\Sigma B^{-1}}(\Sigma L(2J_{\Omega A}(x_n - \Omega L^* v_n) - x_n) + v_n). \end{aligned}$$

This is (3.29).

Now consider the operator T on the product space $\mathcal{H} \times \mathcal{K}$ defined by

$$T(x, v) = (J_{\Omega A}(x - \Omega L^* v), J_{\Sigma B^{-1}}(\Sigma L(2J_{\Omega A}(x - \Omega L^* v) - x) + v)) =: (x^+, v^+) \tag{3.31}$$

for $(x, v) \in \mathcal{H} \times \mathcal{K}$, where $x^+ = J_{\Omega A}(x - \Omega L^* v)$ and $v^+ = J_{\Sigma B^{-1}}(\Sigma L(2x^+ - x) + v)$.

Lemma 3.2. *We have* $Fix(T) = \mathcal{Z}$.

Proof. Let $(x, v) \in \mathcal{H} \times \mathcal{K}$ be given. We have

$$\begin{aligned} T(x, v) = (x, v) &\Leftrightarrow x = x^+ = J_{\Omega A}(x - \Omega L^* v) \text{ and } v = v^+ = J_{\Sigma B^{-1}}(\Sigma L(2x^+ - x) + v) \\ &\Leftrightarrow x - \Omega L^* v \in (I + \Omega A)x \text{ and } \Sigma L(2x^+ - x) + v \in (I + \Sigma B^{-1})v \\ &\Leftrightarrow -L^* v \in Ax \text{ and } L(2x^+ - x) \in B^{-1}v \\ &\Leftrightarrow 0 \in Ax + L^* v \text{ and } 0 \in B^{-1}v - Lx \ (\text{note}: \ 0 \in Ax + L^* v \ \Rightarrow \ x^+ = x) \\ &\Leftrightarrow (x, v) \in \mathcal{Z}. \end{aligned}$$

□

Lemma 3.3. *We have* $Tw = J_{V^{-1}(M+S)}w$ *for* $w = (x, v) \in \mathcal{H} \times \mathcal{K}$. *Thus, T is firmly nonexpansive under the condition* $\|\Sigma^{\frac{1}{2}} L \Omega^{\frac{1}{2}}\| < 1$, *with respect to the norm* $\| \cdot \|_V$.

Proof. Let $w = (x, v) \in \mathcal{H} \times \mathcal{K}$ be given and set $J_{V^{-1}(M+S)}w = w' = (x', v')$. Let us prove that $w' = Tw = w^+ = (x^+, v^+)$. We proceed as follows.

$$J_{V^{-1}(M+S)}w = w' \quad \Rightarrow \quad w \in (I + V^{-1}(M + S))w' \quad \Rightarrow \quad V(w) \in (V + M + S)w'.$$

By the definitions of V and $M + S$ in (3.22) and (3.17), respectively, the last relation is split into the relations:

$$\begin{aligned} \Omega^{-1}x - L^* v &\in \Omega^{-1}x' - L^* v' + Ax' + L^* v' = \Omega^{-1}x' + Ax', \\ \Sigma^{-1}v - Lx &\in \Sigma^{-1}v' - Lx' + B^{-1}v' - Lx' = \Sigma^{-1}v' - 2Lx' + B^{-1}v'. \end{aligned}$$

It turns out that

$$\begin{aligned} x - \Omega L^* v \in (I + \Omega A)x' \quad &\Rightarrow \quad x' = J_{\Omega A}(x - \Omega L^* v) = x^+, \\ \Sigma L(2x' - x) + v \in v' + \Sigma B^{-1}v' \quad &\Rightarrow \quad v' = J_{\Sigma B^{-1}}(\Sigma L(2x^+ - x) + v) = v^+. \end{aligned}$$

Finally, if $\|\Sigma^{\frac{1}{2}} L \Omega^{\frac{1}{2}}\| < 1$, then V is strongly positive, self-adjoint, and linear (see Lemma 3.1), thus $V^{-1}(M + S)$ is maximal monotone w.r.t. the inner product $\langle \cdot, \cdot \rangle_V$. Consequently, as the resolvent of a maximal monotone operator, T is firmly nonexpansive. The proof is complete. □

Now we extend the algorithm (1.11) (or (3.29)) using the KM technique as follows:

$$(3.32)\qquad \begin{cases} x_{n+1} = (1-\lambda_n)x_n + \lambda_n J_{\Omega A}(x_n - \Omega L^* v_n) \\ v_{n+1} = (1-\lambda_n)v_n + \lambda_n J_{\Sigma B^{-1}}(\Sigma L(2J_{\Omega A}(x_n - \Omega L^* v_n) - x_n) + v_n)) \end{cases}$$

where the initial guess $(x_0, v_0) \in \mathcal{H} \times \mathcal{K}$ and $(\lambda_n) \subset [0, 2]$.

We have the following convergence result.

Theorem 3.4. *Suppose* $\|\Sigma^{\frac{1}{2}} L \Omega^{\frac{1}{2}}\| < 1$ *and the divergence condition* (3.16) *holds. Then the sequence* $\{(x_n, v_n)\}$ *generated by the algorithm* (3.32) *converges weakly to a point in* $\mathcal{Z}$.

Proof. Setting $w_n = (x_n, v_n)$ and by virtue of (3.31), we can rewrite the algorithm (3.32) as

$$(3.33)\qquad w_{n+1} = (1-\lambda_n)w_n + \lambda_n T w_n,$$

where $T = J_{V^{-1}(M+S)}$ is firmly nonexpansive by Lemma 3.3.

Applying Theorem 2.1, we assert that (w_n) converges weakly to a point in $\text{Fix}(T) = \text{zer}(M+S) = \mathcal{Z}$. $\square$

Corollary 3.2. *Suppose* $\|\Sigma^{\frac{1}{2}} L \Omega^{\frac{1}{2}}\| < 1$. *Then the sequence* $\{(x_n, v_n)\}$ *generated by the algorithm* (1.11) *converges weakly to a point in* $\mathcal{Z}$.

Proof. This is a special case of Theorem 3.4 where $\lambda_n = 1$ for all n. $\square$

3.2.2. *Strong Convergence.* The Krasnoselskii-Mann version of SDR algorithm (3.32), which includes SDR (1.11), is not strongly convergent in the infinite-dimensional setting, in general, unless additional conditions are assumed to satisfy. In this subsection we apply Halpern's method for modifying SDR (3.32) in order to get strongly convergent algorithms. With the starting point $(x_0, v_0) \in \mathcal{H} \times \mathcal{K}$, we define (x_n, v_n) iteratively by

$$(3.34)\qquad \begin{cases} x_{n+1} = \beta_n \tilde{x} + (1-\beta_n) J_{\Omega A}(x_n - \Omega L^* v_n) \\ v_{n+1} = \beta_n \tilde{v} + (1-\beta_n) J_{\Sigma B^{-1}}(\Sigma L(2J_{\Omega A}(x_n - \Omega L^* v_n) - x_n) + v_n)) \end{cases}$$

for $n = 0, 1, \cdots$, where $(\tilde{x}, \tilde{v}) \in \mathcal{H} \times \mathcal{K}$ is the anchor and $(\beta_n) \subset [0, 1]$.

Theorem 3.5. *Assume* $\mathcal{Z} \neq \emptyset$ *and* $\|\Sigma^{\frac{1}{2}} L \Omega^{\frac{1}{2}}\| < 1$. *Assume also* (β_n) *satisfies the condition (i) of Theorem 2.2, i.e.,* $\beta_n \to 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$. *Then the sequence* $\{(x_n, v_n)\}$ *generated by the algorithm* (3.34) *converges strongly to a point* $(\hat{x}, \hat{v}) \in \mathcal{Z}$ *such that* $(\hat{x}, \hat{v}) = P_{\mathcal{Z}}(\tilde{x}, \tilde{v})$. *Moreover, if we take* $\beta_n = \frac{1}{n+1}$ *and* $(\tilde{x}, \tilde{v}) = (x_0, v_0)$, *that is,* (3.34) *reduces to*

$$(3.35)\qquad \begin{cases} x_{n+1} = \frac{1}{n+1} x_0 + \frac{n}{n+1} J_{\Omega A}(x_n - \Omega L^* v_n) \\ v_{n+1} = \frac{1}{n+1} v_0 + \frac{n}{n+1} J_{\Sigma B^{-1}}(\Sigma L(2J_{\Omega A}(x_n - \Omega L^* v_n) - x_n) + v_n)) \end{cases}$$

then we have the $O(1/n)$ *rate of convergence to zero for the asymptotic regularity of* $\|w_n - Tw_n\|$ *with* $w_n = (x_n, v_n)$:

$$\|w_n - Tx_w\|_V \le \frac{2\|w_0 - w^*\|_V}{n+1}, \quad n \ge 0,\ w^* \in \mathcal{Z}.$$

Proof. In terms of the firmly nonexpansive mapping T as defined in (3.31), the algorithm (3.34) can be rewritten in the product space $\mathcal{H} \times \mathcal{K}$ compactly as:

$$(3.36)\qquad w_{n+1} = \frac{1}{n+1} w_0 + \frac{n}{n+1} T w_n,$$

where $w_n = (x_n, v_n)$ for all $n \ge 0$. The results now follow from Theorem 2.2. $\square$

Projection methods are extensively employed in optimization and fixed points. Additional forcing projections can improve the weak convergence of the proximal point algorithm [19] to strong convergence [21]. We now apply this idea to the SDR algorithm (1.11). Our algorithm is defined as follows. For the sake of convenience, below we drop the subscript V from both the inner product $\langle\cdot,\cdot\rangle_V$ and norm $\|\cdot\|_V$. Choose the initial guess $w_0=(x_0,v_0)\in\mathcal{H}\times\mathcal{K}$ arbitrarily. After $w_n=(x_n,v_n)$ is generated, we set

$$\begin{aligned} y_n &= Tw_n = w_n^+ = (x_n^+, v_n^+), \\ x_n^+ &= J_{\Omega A}(x_n - \Omega L^* v_n), \\ v_n^+ &= J_{\Sigma B^{-1}}(v_n + \Sigma L(2J_{\Omega A}(x_n - \Omega L^* v_n) - x_n)), \end{aligned} \tag{3.37}$$

$$E_n = \{w\in\mathcal{H}\times\mathcal{K} : \langle w - y_n, w_n - y_n\rangle \le 0\}, \tag{3.38}$$

$$G_n = \{w\in\mathcal{H}\times\mathcal{K} : \langle w - w_n, w_0 - w_n\rangle \le 0\}. \tag{3.39}$$

The $(n+1)$th iterate w_{n+1} is then defined as the projection of the initial guess w_0 onto the intersection of the above constructed two half-spaces E_n and G_n, that is,

$$w_{n+1} = P_{E_n\cap G_n}(w_0). \tag{3.40}$$

We first demonstrate that the algorithm (3.40) is well defined, namely, $E_n\cap G_n\neq\emptyset$ for all $n\ge 0$. As a matter of fact, we have the following result.

Lemma 3.4. *We have $E_n\cap G_n\supset\mathcal{Z}$ for $n=0,1,2,\cdots$.*

Proof. First observe that we always have $E_n\supset\mathcal{Z}$ for each $n\ge 0$. This is because $w_n-y_n=w_n-J_{V^{-1}(M+S)}w_n\in V^{-1}(M+S)(w_n)$. Thus, if $w\in\mathcal{Z}$, i.e., $0\in V^{-1}(M+S)(w)$, then the monotonicity of $V^{-1}(M+S)$ immediately implies that $\langle w-w_n, w_n-y_n\rangle\le 0$. Namely, $w\in E_n$. Next, we use induction to prove the conclusion. For $n=0$, since $G_0=\mathcal{H}\times\mathcal{K}$, we get $E_0\cap G_0\supset\mathcal{Z}$. Assume now $E_n\cap G_n\supset\mathcal{Z}$ for some $n>0$; then w_{n+1} is well defined through the projection in (3.40), which implies that

$$\langle w_0 - w_{n+1}, w - w_{n+1}\rangle \le 0 \quad \forall w\in E_n\cap G_n\supset\mathcal{Z}.$$

This particularly shows that $G_{n+1}\supset\mathcal{Z}$; hence, $E_{n+1}\cap G_{n+1}\supset\mathcal{Z}$ and the Lemma is proved. □

We also need the following lemma.

Lemma 3.5. [15, Lemma 1.5] *Let K be a nonempty closed convex subset of a Hilbert space H. Let $\{x_n\}$ be a sequence in H and $u\in H$. Let $q=P_K u$. If $\{x_n\}$ satisfies the conditions*

(i) $\omega_w(x_n)\subset K$,

(ii) $\|x_n-u\|\le\|u-q\|$ for all n.

Then $x_n\to q$.

Theorem 3.6. *Assume $\mathcal{Z}\neq\emptyset$ and $\|\Sigma^{\frac12}L\Omega^{\frac12}\|<1$. Then the sequence $\{w_n\}=\{(x_n,v_n)\}$ generated by the algorithm* (3.40) *converges strongly to a point $w^*=(x^*,v^*)\in\mathcal{Z}$ such that $w^*=P_{\mathcal{Z}}(w_0)$.*

Proof. First we observe from (3.40) and Lemma 3.4 that for each $w'\in\mathcal{Z}$, $\|w_{n+1}-w'\|=\|P_{E_n\cap G_n}(w_0)-w'\|\le\|w_0-w'\|$. Hence, $\{w_n\}$ is bounded.

By definition of G_n we get $w_n=P_{G_n}(w_0)$. This, together with the fact that $w_{n+1}\in G_n$, implies that

$$\begin{aligned} \|w_{n+1}-w_n\|^2 &= \|P_{G_n}(w_{n+1}) - P_{G_n}(w_0)\|^2 \\ &\le \|w_{n+1}-w_0\|^2 - \|(I-P_{G_n})w_{n+1} - (I-P_{G_n})w_0\|^2 \\ &= \|w_{n+1}-w_0\|^2 - \|w_n-w_0\|^2. \end{aligned}$$

It turns out that the sequence $\{\|w_n - w_0\|^2\}$ is increasing, and due to boundedness, $\{\|w_n - w_0\|^2\}$ converges. It then also turns out that $\|w_{n+1} - w_n\|^2 \to 0$.

Since $w_{n+1} \in E_n$, it follows that

$$\begin{aligned} 0 &\le 2\langle w_{n+1} - y_n, y_n - w_n\rangle \\ &= \|w_{n+1} - w_n\|^2 - \|w_{n+1} - y_n\|^2 - \|y_n - w_n\|^2. \end{aligned}$$

Hence, $\|y_n - w_n\| \le \|w_{n+1} - w_n\| \to 0$. Since $y_n = Tw_n$, we get $\|w_n - Tw_n\| \to 0$. By the demiclosedness principle of nonexpansive mappings in a Hilbert space, we must have $\omega_w(w_n) \subset \mathrm{Fix}(T) = \mathcal{Z}$; that is, each weak cluster point of (w_n) lies in $\mathcal{Z}$.

Now set $w^* = P_{\mathcal{Z}}(w_0)$. By the algorithm (3.40), it is evident that, for all $n \ge 1$,

$$\|w_n - w^*\| = \|P_{E_{n-1}\cap G_{n-1}} w_0 - w^*\| \le \|w_0 - w^*\|.$$

Since we have already proved that $\omega_w(w_n) \subset \mathcal{Z}$, Lemma 3.5 is applicable and we finally obtain $w_n \to w^*$. The proof is complete. □

We conclude the paper with the following strong convergence result on SDR (3.29) with additional conditions imposed on the operators A and B, or on the solution set $\mathcal{Z}$, respectively.

Theorem 3.7. *Suppose $\|\Sigma^{\frac{1}{2}} L \Omega^{\frac{1}{2}}\| < 1$ and one of the conditions below is satisfied:*

(i) *A and B are odd; i.e., $A(-x) = -A(x)$ for all $x \in \mathcal{H}$ and $B(-v) = -B(v)$ for all $v \in \mathcal{K}$,*

(ii) *$\mathcal{Z}$ has nonempty interior.*

Then the sequence $\{(x_n, v_n)\}$ generated by SDR (3.29) *converges in norm to a point in $\mathcal{Z}$.*

Proof. First observe that the sequence $\{(x_n, v_n)\}$ generated by SDR (3.29) can be rewritten as ($w_n = (x_n, v_n)$)

$$w_n = Tw_{n-1} = \cdots = T^n w_0, \quad n = 0, 1, 2, \cdots.$$

Since $T = J_{V^{-1}M}$ is firmly nonexpansive (w.r.t. the norm $\|\cdot\|_V$; below we drop the subscript V for the sake of convenience). Thus T is asymptotically regular, that is,

$$\lim_{n\to\infty} \|T^{n+1}w - T^n w\| = 0, \quad \forall w \in \mathcal{H} \times \mathcal{K}. \tag{3.41}$$

(i) It is easily seen that the oddness of A and$B$ implies that the resolvents $J_{\Omega A}$ and $J_{\Sigma B^{-1}} = I - \Sigma J_{\Sigma^{-1}B}\Sigma^{-1}$ are odd as well. It then turns out that the mapping T defined by (3.31) is also odd, which implies that $T(0) = 0$. Hence, $(0, 0) \in \mathcal{Z}$. And for each $w \in \mathcal{H} \times \mathcal{K}$, the real nonnegative sequence $\{\|T^n w\|\}$ is nonincreasing, hence convergent. Set $r = \lim_{n\to\infty} \|T^n w_0\|$. Again since T is odd, we get for each fixed $i \ge 0$,

$$\|T^{n+i}w_0 + T^n w_0\| = \|T^{n+i}w_0 - T^n(-w_0)\| \le \|T^{n-1+i}w_0 + T^{n-1}w_0\|.$$

This shows that $\{\|T^{n+i}w_0 + T^n w_0\|\}_{n=0}^{\infty}$ is nonincreasing. We further have, for $n > m \ge 0$,

$$\begin{aligned} 2r \le 2\|T^n w_0\| &= \|T^n w_0 + T^{n+i}w_0 + T^n w_0 - T^{n+i}w_0\| \\ &\le \|T^n w_0 + T^{n+i}w_0\| + \|T^n w_0 - T^{n+i}w_0\| \\ &\le \|T^m w_0 + T^{m+i}w_0\| + \|T^n w_0 - T^{n+i}w_0\| \quad (\text{letting } n \to \infty) \\ &\to \|T^m w_0 + T^{m+i}w_0\| \le \|T^m w_0\| + \|T^{m+i}w_0\| \le 2\|T^m w_0\| \to 2r. \end{aligned}$$

Consequently,

$$\lim_{n,m\to\infty} \|T^n w_0 + T^m w_0\| = 2r.$$

Moreover, as $n, m \to \infty$,

$$\|T^n w_0 - T^m w_0\|^2 = 2(\|T^n w_0\|^2 + \|T^m w_0\|^2) - \|T^n w_0 + T^m w_0\|^2 \to 2(r^2 + r^2) - (2r)^2 = 0$$

and $\{T^n w_0\}$ is Cauchy, hence convergent in norm.

(ii) By assumption, we have some $p \in \mathrm{int}(\mathcal{Z}) = \mathrm{int}(\mathrm{Fix}(T))$. In other words, there exists $\delta > 0$ such that $p + \delta w \in \mathrm{Fix}(T)$ for all $w \in \mathcal{H} \times \mathcal{K}$ suhc that $\|w\| \le 1$. As $w_{n+1} = T w_n$, we derive (for each $\|w\| \le 1$)

$$\begin{aligned}&\|w_{n+1} - p\|^2 - 2\delta\langle w_{n+1} - p, w\rangle + \delta^2\|w\|^2 = \|w_{n+1} - (p + \delta w)\|^2 = \|Tw_n - (p + \delta w)\|^2 \\ &\le \|w_n - (p + \delta w)\|^2 = \|w_n - p\|^2 - 2\delta\langle w_n - p, w\rangle + \delta^2\|w\|^2.\end{aligned}$$

This results in that

$$2\delta\langle w_n - w_{n+1}, w\rangle \le \|w_n - p\|^2 - \|w_{n+1} - p\|^2.$$

Since this is valid for each w such that $\|w\| \le 1$, it follows immediately that

$$2\delta\|w_n - w_{n+1}\| \le \|w_n - p\|^2 - \|w_{n+1} - p\|^2.$$

This implies that the series $\sum_{n=1}^{\infty} \|w_n - w_{n+1}\|$ is convergent, which ensures that $\{w_n\}$ is Cauchy, hence convergent. This finishes the proof. □

Remark 3.1. *An example of an odd maximal monotone operator is $A = \partial\varphi$, where $\varphi \in \Gamma_0(\mathcal{H})$ is even, i.e., $\varphi(-x) = \varphi(x)$ for all $x \in \mathcal{H}$.*

Acknowledgments. The authors thank the referees for their comments and suggestions which improved the presentation of this manuscript.

HKX was supported in part by National Natural Science Foundation of China grant U1811461, and by Australian Research Council Discovery Project DP200100124.

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