CARPATHIAN J. MATH.
Volume **41** (2025), No. 4,
Pages 1015-1025

Online version at `https://www.carpathian.cunbm.utcluj.ro/`
ISSN 1584-2851 (Print edition); ISSN 1843-4401 (Electronic)
DOI: https://doi.org/10.37193/CJM.2025.04.11

A New Existence Theorem of the Solution of a Functional Fredholm Integral Equation in *b*-metric Spaces

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ABSTRACT. Using some recent results on nonlinear integral equations with new admissibility types in b-metric spaces, this paper investigates the existence of solutions to a functional Fredholm integral equation in a b-metric space. We establish the conditions under which a solution of the considered integral equation exists and then formulate an existence theorem for such solutions. This theorem complements the results obtained over time in the study of solutions to this type of functional Fredholm integral equation. The paper concludes with two illustrative examples that demonstrate the applicability of this existence theorem.

1. INTRODUCTION

In the 1970s, a group of Romanian physics professors developed, during practical investigations (which were not formally published), a mathematical model governed by the following functional Fredholm integral equation:

$$x(t) = \int_a^b K(t, r, x(r), x(a), x(b))dr + f(t), \quad t \in [a, b],$$

where $a, b \in \mathbb{R},\ a < b,\ K \in C([a, b] \times [a, b] \times \mathbb{R}^3),\ f \in C[a, b]$ are known functions, and $x \in C[a, b]$ is the unknown function.

The mathematical analysis of this equation - particularly regarding the existence and uniqueness of its solution - was later carried out and published by their colleagues in the field of mathematics, including the first author of this paper. Although the original physical investigations were not formally documented in scientific literature, they served as the primary motivation for the subsequent mathematical study, which focused on studying the properties of the solution of the integral equation above. Using classical theorems, the following aspects were investigated: existence and uniqueness of the solution, continuous dependence on data, differentiability, Ulam-Hyers stability, as well as comparison theorems, integral inequalities, and methods for approximating the solution. The results obtained were published in several articles that can be found in the mathematical literature. The first study addressing the existence and uniqueness of the solution to this Fredholm integral equation with modified arguments was published in 1978, in [2]. This development represents an early example of interdisciplinary collaboration, where practical needs from applied physics provided the motivation for theoretical advancements in mathematical analysis. The 1978 article demonstrated that, under the assumptions of Banach's fixed point theorem, this type of integral equation admits a unique solution. Since the exact solution could not always be determined, the same paper presented a method for approximating the solution, using the successive approximations method combined with the trapezoidal quadrature rule. This Fredholm integral equation formed the basis of the mathematical model studied by the group of physicists.

Received: 11.11.2024. In revised form: 23.03.2025. Accepted: 10.07.2025
2020 *Mathematics Subject Classification.* 47H10, 45G10.
Key words and phrases. *Fredholm integral equation, solution, admissibility type, b-metric space.*
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In [22], the authors introduced the notion of an $\alpha_{s,\varepsilon}$-contraction, with $\varepsilon > 1$, which is defined on a b-metric space with coefficient $s \geq 1$. This notion was used to prove several fixed point theorems for an $\alpha_{s,\varepsilon}$-contractions, with $\varepsilon > 1$, in a b-complete b-metric space with coefficient $s > 1$. These fixed point results, along with the α-admissible functions introduced in [27] were applied in [22], to establish the existence of a solution to the nonlinear Fredholm integral equation:

$$x(t) = \int_a^b K(t, r, x(r))dr + f(t),$$

where $a, b \in \mathbb{R},$ with $a < b,$ $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ are given functions and $x \in C[a, b]$ is the unknown function.

To complement the study of the solution to the integral equation investigated in the 1970s, the notions and results from [7], [22] and [27] were employed, leading to the establishment of new conditions for the existence of its solution. The obtained result is presented in paper [12].

Over time, several modifications to the arguments of the original integral equation proposed in the 1970s were introduced, depending on specific conditions.

We now consider this functional nonlinear Fredholm integral equation with one of the proposed modifications of the argument, by introducing a known continuous function $g \in C([a, b], [a, b])$:

$$x(t) = \int_a^b K(t, r, x(r), x(g(r)), x(a), x(b))dr + f(t), \quad t \in [a, b], \tag{1.1}$$

where $a, b \in \mathbb{R},$ $a < b$, with $a < b$; $K : [a, b] \times [a, b] \times \mathbb{R}^4 \to \mathbb{R}$, $g : [a, b] \to [a, b]$ and $f : [a, b] \to \mathbb{R}$ are the given functions and $x \in C[a, b]$ is the unknown function.

Some properties of the solution of this integral equation were studied using known classical theorems and the obtained results can be found in several papers, of which we mention [8]–[11]. To obtain these results, some presented results in [6], [23] and [24] were used.

In this paper, we aim to contribute to the study of the functional Fredholm integral equation (1.1) by establishing new conditions for the existence of its solution. These conditions are derived using the notions of admissibility types defined on a b-metric space, as introduced in [27], and by applying a fixed point theorem from [22]. For certain notions related to b-metric spaces, we have consulted papers [7] and [20]. For a general overview of early developments in fixed point theory on b-metric spaces, see also [5].

Our paper is organized into five sections.

Section 1 provides a brief presentation of the integral equation under study (1.1), along with some mathematical details specific to it. It also recalls for the reader the notions and results that will be used to establish the main result of the paper.

In Section 2, Preliminaries, we recall some definitions and results related to recent notions introduced in ([22], [27]), which will be used to establish an existence theorem for the solution of the functional Fredholm integral equation (1.1).

The main result is presented in Section 3, as an existence theorem for the solution of the functional Fredholm integral equation under study. In Section 4, two examples are provided to illustrate the application of this result.

The final section presents some concluding remarks regarding the result established in this paper.

2. Preliminaries

First of all, for the reader's convenience, the following notions will be recalled: b-metric on a nonempty set X, b-metric space, b-continuous function, the altering distance function, the α-admissible function, the α-admissible function type s.

Let X be a nonempty set and $s \geq 1$ be a given real number.

Definition 2.1. *A function $d : X \times X \to [0, +\infty)$ is called b-metric on X, if for all $x, y, z \in X$, it satisfies the following conditions:*

(i) $d(x, y) = 0$ *if and only if* $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a b-metric space with the coefficient $s \geq 1$.

Remark 2.1. *If $s = 1$, then the b-metric space is the usual metric space.*

The notion of b-continuity, meaning the preservation of convergence in b-metric spaces, is used in fixed point results for generalized contraction mappings (see [7], [1] and applications in [22]).

Definition 2.2. *Let (X, d) be a b-metric space. A function $f : X \to X$ is said to be* b-continuous *if for every sequence $(x_n) \subset X$ that converges to $x \in X$ in the b-metric (i.e., $d(x_n, x) \to 0$), it holds that*

$$d(f(x_n), f(x)) \to 0 \quad \text{as } n \to \infty.$$

The notions of b-convergence, b-completeness, and b-Cauchy sequence in a b-metric space can be found in [7] and [27].

In [17], Khan et al. introduced the notion of the altering distance function as a control function that modifies the distance between two points in a metric space (see also [21]).

Definition 2.3. *A function $\varphi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if it satisfies the following two properties:*

(i) φ *is continuous and non-decreasing;*
(ii) $\varphi(t) = 0$ *if and only if* $t = 0$.

Next, we present the notions of the α-admissible function and the α-admissible function type s (see [27]).

Let X be a nonempty set, and let $\alpha : X \times X \to [0, +\infty)$ be a given function.

Definition 2.4 ([27]). *A function $f : X \to X$ is said to be an α-admissible function if it satisfies the following condition:*

$$x, y \in X,\ \alpha(x, y) \geq 1 \implies \alpha(f(x), f(y)) \geq 1$$

and the set of these functions was denoted

$$\mathcal{A}(X, \alpha) = \{f : X \to X / f \ is \ an \ \alpha - admissible \ function\}.$$

In addition, let s be a given real number such that $s \geq 1$.

Definition 2.5 ([27]). *A function $f : X \to X$ is said to be an α-admissible function type s, if it satisfies the following condition:*

$$x, y \in X,\ \alpha(x, y) \geq s \implies \alpha(f(x), f(y)) \geq s$$

and the set of these functions was denoted

$$\mathcal{A}_s(X, \alpha) = \{f : X \to X / f \ is \ an \ \alpha - admissible \ function \ type \ s\}.$$

Also, the notions of the weak α-admissible function and the weak α-admissible function type s, presented below, were introduced in [27].

Definition 2.6 ([27]). *A function $f : X \to X$ is said to be a weak α-admissible function if it satisfies the following condition:*

$$x \in X,\ \alpha(x, f(x)) \geq 1 \implies \alpha(f(x), f(f(x)) \geq 1$$

and the set of these functions was denoted

$$\mathcal{WA}(X, \alpha) = \{f : X \to X/f \ is\ a\ weak\ \alpha - admissible\ function\}.$$

Definition 2.7 ([27]). *A function $f : X \to X$ is said to be a weak α-admissible function type s, if it satisfies the following condition:*

$$x \in X,\ \alpha(x, f(x)) \geq s \implies \alpha(f(x), f(f(x))) \geq s$$

and the set of these functions was denoted

$$\mathcal{WA}_s(X, \alpha) = \{f : X \to X/f \ is\ a\ weak\ \alpha - admissible\ function\ type\ s\}.$$

Remark 2.2. *It is observed that $\mathcal{A}(X, \alpha) \subseteq \mathcal{WA}(X, \alpha)$ and $\mathcal{A}_s(X, \alpha) \subseteq \mathcal{WA}_s(X, \alpha)$.*

In [27] the notion of the $(\alpha, \psi, \varphi)_s$-contraction function was also introduced, defined as follows:

Definition 2.8 ([27]). *Let (X, d) be a b-metric space with coefficient $s \geq 1$, let $\alpha : X \times X \to [0, +\infty)$ be a given function and let $\psi, \varphi : [0, +\infty) \to [0, +\infty)$ be two altering distance functions. Then a function $f : X \to X$ is an $(\alpha, \psi, \varphi)_s$-contraction function if the following condition holds:*

$$x, y \in X, \alpha(x, y) \geq s \implies \psi(s^3 d(f(x), f(y))) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)),$$

where

$$M_s(x, y) := \max\left\{d(x, y), d(x, f(x)), d(y, f(y)), \frac{d(x, f(y)) + d(y, f(x))}{2s}\right\},$$

and is denoted by $\Omega_s(X, \alpha, \psi, \varphi)$, the collection of all $(\alpha, \psi, \varphi)_s$-contraction functions on a b-metric space (X, d) with coefficient $s \geq 1$.

In [22], the notion of the $\alpha_{s,\varepsilon}$-contraction function, where $\varepsilon > 1$, defined on a *b*-metric space with coefficient $s \geq 1$, was introduced.

Definition 2.9. *Let (X, d) be a b-metric space with coefficient $s \geq 1$ and let $\alpha : X \times X \to [0, +\infty)$ be a given function. Then a function $f : X \to X$ is said to be an $\alpha_{s,\varepsilon}$-contraction function, where $\varepsilon > 1$, if it satisfies the following condition:*

$$x, y \in X,\ \alpha(x, y) \geq s \implies s^{\varepsilon} d(f(x), f(y)) \leq M_s(x, y). \tag{2.2}$$

In [22], this notion was used to obtain a new fixed point theorem, that serves as a useful tool to determine the existence conditions for solution of a fixed point equation. To establish the main result of our paper, we will use this fixed point theorem, which we present below.

Theorem 2.1 ([22]). *Let (X, d) be a b-complete b-metric space with coefficient $s > 1$, $\alpha : X \times X \to [0, +\infty)$ a given function and $f : X \to X$ an $\alpha_{s,\varepsilon} - contraction\ function,\ where\ \varepsilon > 1$. Suppose that the following conditions hold:*

(s_1) $f \in \mathcal{WA}_s(X, \alpha)$;
(s_2) $there\ exists\ x_0 \in X\ such\ that\ \alpha(x_0, f(x_0)) \geq s$;
(s_3) f is b-continuous.

Then, f has at least one fixed point in X.

Remark 2.3. *If $s > 1$, then this theorem (Theorem 2.2 in* [22]*) is slight obtained from the results presented in* [27]*.*

This existence result (Theorem 2.1) can be stated in a particular case where the generalized contractive condition involves the distance $d(x, y)$ directly, as presented below.

Theorem 2.2. *Let (X, d) be a complete b-metric space and let $f : X \to X$ be a mapping such that*

$$d(f(x), f(y)) \leq d(x, y), \quad \text{for all } x, y \in X.$$

If f satisfies an α-admissibility condition and f is b-continuous, then f has at least one fixed point in X.

In [19], Miculescu and Mihail presented a lemma (Lemma 2.2) that serves as a useful auxiliary result supporting the convergence part of Theorem 2.1, provided that the contraction condition imposed by the mapping f ensures a suitable decay of successive distances. For convenience, we restate this result as Lemma 2.1 in the present work.

Lemma 2.1 **([19]).** *Every sequence $(x_n)_{n\in\mathbb{N}}$ in a b-metric space (X, d) that satisfies the condition that there exists $\gamma \in [0, 1)$ such that*

$$d(x_{n+1}, x_n) \leq \gamma\, d(x_n, x_{n-1}),$$

for every $n \in \mathbb{N}$, is a Cauchy sequence.

Lemma 2.1 can be effectively applied within the framework of Theorem 2.1, provided that the appropriate contractive condition is satisfied.

More precisely, Lemma 2.1 states that if a sequence (x_n) in a b-metric space satisfies the inequality

$$d(x_{n+1}, x_{n+2}) \leq k \cdot d(x_n, x_{n+1}),$$

for all $n \in \mathbb{N}$ and for some constant $k \in (0, 1)$, then the sequence (x_n) is Cauchy.

In Theorem 2.1, the sequence is constructed using the iterative scheme $x_{n-1} = f(x_n)$, where f is a self-mapping on a b-metric space. If f satisfies a generalized contractive condition that leads to the recursive inequality above, then Lemma 2.1 can be directly applied to ensure the convergence of the sequence to a fixed point of f.

We emphasize that Lemma 2.1 provides a valuable tool for analyzing the convergence of iterative sequences.

To establish the main result of this paper, Theorem 2.1 will be used in the next section.

3. Main result

By using an α-admissible function type s and applying Theorem 2.1, we establish in this section an existence result for the solution of the functional nonlinear Fredholm integral equation (1.1):

$$x(t) = \int_a^b K(t, r, x(r), x(g(r)), x(a), x(b))dr + f(t),$$

where $a, b \in \mathbb{R}$, with $a < b$, $K \in C([a, b] \times [a, b] \times \mathbb{R}^4)$, $g \in C([a, b], [a, b])$ *and* $f \in C[a, b]$ are given functions and $x \in C[a, b]$ is the unknown function.

Let $X = C[a, b]$, and let $d : X \times X \to [0, +\infty)$ be a b-metric defined by the relation:

$$d(x, y) := \sup_{t\in[a,b]} |x(t) - y(t)|^p, \tag{3.3}$$

for all $x, y \in X$ and $p > 1$.

Then (X, d) is a b-complete b-metric space with the coefficient $s = 2^{p-1}$.

To this integral equation, we associate the operator $A : X \to X$ defined by the relation:

$$A(x)(t) := \int_a^b K(t, r, x(r), x(g(r)), x(a), x(b))dr + f(t), \tag{3.4}$$

for all $x \in X$ and $t \in [a, b]$.

In what follows, we establish the conditions under which the operator A has at least one fixed point. In other words, we construct the proof of the theorem stated at the end of this section, which represents the main result of this paper.

To this end, we define the function $\alpha : X \times X \to [0, +\infty)$ by the relation:

$$\alpha(x, y) := \begin{cases} 2^{p-1}, & x(t) \le y(t) \\ \tau, & \text{otherwise} \end{cases} \tag{3.5}$$

where $\tau \in (0, 2^{p-1})$.

Suppose that the function K is non-decreasing in its last four arguments. Then we obtain:

$$A \in \mathcal{A}_s(X, \alpha) \subseteq \mathcal{WA}_s(X, \alpha).$$

Next, we suppose that there exists $x_0 \in X$ such that

$$x_0(t) \le \int_a^b K(t, r, x(r), x(g(r)), x(a), x(b))dr + f(t)$$

for all $t \in [a, b]$.

Consequently, it follows: $\alpha(x_0, A(x_0)) \ge 2^{p-1}$.

Let $p \ge 1$ and $q < \infty$ be two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$.

In what follows, suppose that

$$\Big|K(t, r, x(r), x(g(r)), x(a), x(b)) - K(t, r, y(r), y(g(r)), y(a), y(b))\Big| \le \tag{3.6}$$

$$\le \gamma(t, r)\Big[|x(r) - y(r)|^p + |x(g(r)) - y(g(r))|^p + |x(a) - y(a)|^p + |x(b) - y(b)|^p\Big]$$

where $t, r \in [a, b]$, $x, y \in X$ with $x(r) \le y(r)$ for all $r \in [a, b]$, $p > 1$ and $\gamma \in C([a, b] \times [a, b], [0, \infty))$ is a function that satisfies

$$\sup_{t\in[a,b]} \left(\int_a^b \Big(\gamma(t, r)\Big)^p dr \right) < \frac{1}{2^{\varepsilon(p^2-p)+2p}(b - a)^{p-1}},\ for\ \varepsilon > 1. \tag{3.7}$$

Now, using (3.4), (3.6) and (3.3), we have:

$$\left(2^{\varepsilon(p-1)}\Big|A(x)(t) - A(y)(t)\Big|\right)^p \le 2^{\varepsilon(p^2-p)}\cdot$$

$$\left(\int_a^b \Big|K(t, r, x(r), x(g(r)), x(a), x(b)) - K(t, r, y(r), y(g(r)), y(a), y(b))\Big|dr\right)^p$$

$$\le 2^{\varepsilon(p^2-p)}\Bigg[\left(\int_a^b 1^q dr\right)^{\frac{1}{q}}\cdot$$

$$\left(\int_a^b \Big|K(t, r, x(r), x(g(r)), x(a), x(b)) - K(t, r, y(r), y(g(r)), y(a), y(b))\Big|^p dr\right)^{\frac{1}{p}}\Bigg]^p$$

$$\le 2^{\varepsilon(p^2-p)}(b - a)^{\frac{p}{q}}\left(\int_a^b \big(\gamma(t, r)\big)^p\cdot\right.$$

$$\left.\Big(|x(r) - y(r)|^p + |x(g(r)) - y(g(r))|^p + |x(a) - y(a)|^p + |x(b) - y(b)|^p\Big)^p dr\right)$$

$$\le 2^{\varepsilon(p^2-p)}(b-a)^{\frac{p}{q}}\int_a^b \Big(\gamma(t,r)\Big)^p \cdot \Big(4d(x,y)\Big)^p dr$$

$$\le 2^{\varepsilon(p^2-p)+2p}(b-a)^{p-1}\int_a^b \Big(\gamma(t,r)\Big)^p \cdot \Big(M_s(x,y)\Big)^p dr$$

$$\le 2^{\varepsilon(p^2-p)+2p}(b-a)^{p-1}\Big(M_s(x,y)\Big)^p \cdot \sup_{t\in[a,b]}\left(\int_a^b \Big(\gamma(t,r)\Big)^p dr\right),$$

and from (3.7), it results:

$$\Big(2^{\varepsilon(p-1)}|A(x)(t)-A(y)(t)|\Big)^p \le \Big(M_s(x,y)\Big)^p.$$

Therefore, we have:

$$2^{\varepsilon(p-1)}\Big|A(x)(t)-A(y)(t)\Big| \le M_s(x,y),$$

i.e.

$$s^\varepsilon d\big(A(x),A(y)\big) \le M_s(x,y),$$

and thus, using (2.2), it results that the operator A is an $\alpha_{s,\varepsilon}$-contraction, where $\varepsilon > 1$ and the function $\alpha : C[a,b]\times C[a,b] \to [0,+\infty)$ is defined by the relation (3.5).

We now observe that the conditions of Theorem 2.1 are satisfied.

Therefore, by applying Theorem 2.1, an existence result for the solution of the functional nonlinear Fredholm integral equation (1.1) can be formulated.

This existence result represents a new property of the solution to this type of functional Fredholm integral equations. We present this theorem below.

Theorem 3.3 (Main Theorem)**.** *Consider the functional nonlinear Fredholm integral equation (1.1). Suppose that the following conditions hold:*

(i) $K \in C([a,b]\times[a,b]\times\mathbb{R}^4)$, $g \in C([a,b],[a,b])$ *and* $f \in C[a,b]$*;*

(ii) K *is non-decreasing in the last four arguments;*

(iii) *for each* $r,t \in [a,b]$ *and* $x,y \in X$ *with* $x(s) \le y(s)$ *for all* $s \in [a,b]$*, we have:*

$$|K(t,r,x(r),x(g(r)),x(a),x(b)) - K(t,r,y(r),y(g(r)),y(a),y(b))|$$

$$\le \gamma(t,r)\Big[|x(r)-y(r)|^p + |x(g(r))-y(g(r))|^p + |x(a)-y(a)|^p +$$

$$+|x(b)-y(b)|^p\Big],\ p>1$$

where $\gamma \in C([a,b]\times[a,b],[0,\infty))$ *satisfies:*

$$\sup_{t\in[a,b]}\left(\int_a^b (\gamma(t,r))^p dr\right) < \frac{1}{2^{\varepsilon(p^2-p)+2p}(b-a)^{p-1}},\ for\ \varepsilon > 1;$$

(iv) *there exists* $x_0 \in C[a,b]$ *such that*

$$x_0(t) \le \int_a^b K(t,r,x(r),x(g(r)),x(a),x(b))dr + f(t)$$

for all $t \in [a,b]$*.*

Then, the functional nonlinear Fredholm integral equation (1.1) has at least one solution in $C[a,b]$*.*

This existence theorem complements the previously studied properties of the solutions to this type of functional Fredholm integral equations.

4. Applications

In the following, we present two applications of the main result (Theorem 3.3). Specifically, we consider two integral equations of this type, for which we verify the hypotheses of Theorem 3.3 and establish the existence of their solutions.

Example 4.1. *Consider the following functional Fredholm integral equation:*

$$x(t) = \int_0^1 \left(\frac{x(r) + x(r/2)}{t + r + 8} + \frac{tx(0) + rx(1)}{8} \right) dr + 1, \quad t \in [0,1], \tag{4.8}$$

where $K : [0,1] \times [0,1] \times \mathbb{R}^4 \to \mathbb{R}$, $K(t,r,x(r),x(r/2),x(0),x(1)) = \frac{x(r)+x(r/2)}{t+r+8} + \frac{tx(0)+rx(1)}{8}$, $g : [0,1] \to [0,1]$, $g(r) = \frac{r}{2}$ *and* $f : [0,1] \to \mathbb{R}$, $f(t) = 1$ *are continuous functions and* $x \in C[0,1]$ *is the unknown function.*

In this case, we use the space $C[0,1]$ *endowed with the b-metric* $d : C[0,1] \times C[0,1] \to [0,\infty)$ *defined by the relation:*

$$d(x,y) := \sup_{t \in [0,1]} |x(t) - y(t)|^p, \ for \ all \ x, y \in C[0,1].$$

The space $\big(C[0,1], d\big)$ *is a b-complete b-metric space with the coefficient* $s = 2^{p-1}$.

To this integral equation, we associate the operator $A : C[0,1] \to C[0,1]$ *defined by the relation:*

$$A(x)(t) := \int_0^1 \left(\frac{x(r) + x(r/2)}{t + r + 8} + \frac{tx(0) + rx(1)}{8} \right) dr + 1,$$

for all $x \in C[0,1]$ *and* $t, r \in [0,1]$.

In what follows, the conditions of Theorem 3.3 are verified.

It is observed that the function K *is nondecreasing in its last four arguments.*

At this moment, for $t, r \in [0,1]$, $x, y \in C[0,1]$ *with* $x(r) \le y(r)$ *for all* $r \in [0,1]$, $p > 1$, *we estimate the difference:*

$$\begin{aligned}
&\Big| K(t,r,x(r),x(r/2),x(0),x(1)) - K(t,r,y(r),y(r/2),y(0),y(1)) \Big| \\
&= \left| \frac{x(r) + x(r/2)}{t + r + 8} + \frac{tx(0) + rx(1)}{8} - \frac{y(r) + y(r/2)}{t + r + 8} - \frac{ty(0) + ry(1)}{8} \right| \\
&\le \left| \frac{x(r) - y(r)}{t + r + 8} \right| + \left| \frac{x(r/2) - y(r/2)}{t + r + 8} \right| + \left| \frac{t}{8} \right| \cdot \big| x(0) - y(0) \big| + \left| \frac{r}{8} \right| \cdot \big| x(1) - y(1) \big| \\
&\le \left| \frac{1}{8} \right| \cdot \big| x(r) - y(r) \big| + \left| \frac{1}{8} \right| \cdot \big| x(r/2) - y(r/2) \big| + \left| \frac{1}{8} \right| \cdot \big| x(0) - y(0) \big| + \left| \frac{1}{8} \right| \cdot \big| x(1) - y(1) \big| \\
&\le \frac{1}{8} \cdot \Big(\big| x(r) - y(r) \big|^p + \big| x(r/2) - y(r/2) \big|^p + \big| x(0) - y(0) \big|^p + \big| x(1) - y(1) \big|^p \Big)
\end{aligned}$$

and it follows that the condition (3.6) is satisfied and $\gamma(t,r) = \frac{1}{8}$.

In this case, we have

$$\left(\frac{1}{8} \right)^p < \frac{1}{2^{\varepsilon(p^2 - p) + 2p}}.$$

It follows that

$$2^3 > 2^{\varepsilon(p-1)+2} \implies 3 > \varepsilon(p-1) + 2$$

and it results that for $\varepsilon > 1$ *and* $p > 1$, *the condition (3.7) is satisfied. Therefore, hypothesis* (iii) *is also satisfied.*

Using the properties of the functions K, g *and* f, *we observe that the hypothesis* (iv) *is satisfied for* $x_0 = f \in C[0,1]$.

We now observe that the conditions of Theorem 3.3 are satisfied and it follows that the integral equation (4.8) has a solution.

Example 4.2. *In this example, we consider another functional Fredholm integral equation, namely:*

$$x(t) = \int_0^1 \frac{tr}{64}\Big(sin\big(x(r)\big) + sin\big(x\big((2r+1)/3\big)\big) + x(0) + x(1)\Big)dr + cos(t), \tag{4.9}$$

where $t \in [0,1]$, $K : [0,1] \times [0,1] \times \mathbb{R}^4 \to \mathbb{R}$, $K\big(t, r, x(r), x\big((2r+1)/3\big), x(0), x(1)\big) = \frac{tr}{64}\Big(sin\big(x(r)\big) + sin\big(x\big((2r+1)/3\big)\big) + x(0) + x(1)\Big)$, $g : [0,1] \to [0,1]$, $g(r) = (2r+1)/3$ *and* $f : [0,1] \to \mathbb{R}$, $f(t) = cos(t)$ *are continuous functions and* $x \in C[0,1]$ *is the unknown function.*

The space $C[0,1]$ *endowed with the b-metric* $d : C[0,1] \times C[0,1] \to [0,\infty)$ *defined by the relation (3.3), form together a b-complete b-metric space with the coefficient* $s = 2^{p-1}$.

To the integral equation, we associate We attach to the integral equation (4.9), the operator $A : C[0,1] \to C[0,1]$ *defined by the relation:*

$$A(x(t) = \int_0^1 \frac{tr}{64}\Big(sin\big(x(r)\big) + sin\big(x\big((2r+1)/3\big)\big) + x(0) + x(1)\Big)dr + cos(t),$$

for all $x \in C[0,1]$ *and* $t, r \in [0,1]$.

Next, the conditions of Theorem 3.3 are verified.

It is clear that the function K *is nondecreasing in its last four arguments.*

Now, for $t, r \in [0,1]$, $x, y \in C[0,1]$ *with* $x(r) \leq y(r)$ *for all* $r \in [0,1]$, $p > 1$, *we estimate the difference:*

$$\begin{aligned}
&\Big|K\big(t, r, x(r), x\big((2r+1)/3\big), x(0), x(1)\big) - K\big(t, r, y(r), y\big((2r+1)/3\big), y(0), y(1)\big)\Big| \\
&= \left|\frac{tr}{64}\right| \cdot \Big|sin\big(x(r)\big) + sin\big(x\big((2r+1)/3\big)\big) + x(0) + x(1) - \\
&\qquad -sin\big(y(r)\big) - sin\big(y\big((2r+1)/3\big)\big) - y(0) - y(1)\Big| \\
&\leq \left|\frac{tr}{64}\right| \cdot \Big(\Big|sin\big(x(r)\big) - sin\big(y(r)\big)\Big| + \Big|sin\big(x\big((2r+1)/3\big)\big) - sin\big(y\big((2r+1)/3\big)\big)\Big| + \\
&\qquad + \big|x(0) - y(0)\big| + \big|x(1) - y(1)\big|\Big) \\
&\leq \left|\frac{tr}{64}\right| \cdot \Big(2\Big|sin\Big(\frac{x(r) - y(r)}{2}\Big)\Big| \cdot \Big|cos\Big(\frac{x(r) + y(r)}{2}\Big)\Big| \\
&+2\Big|sin\Big(\frac{x\big((2r+1)/3\big) - y\big((2r+1)/3\big)}{2}\Big)\Big| \cdot \Big|cos\Big(\frac{x\big((2r+1)/3\big) + y\big((2r+1)/3\big)}{2}\Big)\Big| \\
&\qquad + \big|x(0) - y(0)\big| + \big|x(1) - y(1)\big|\Big) \\
&\leq \left|\frac{tr}{64}\right| \cdot \Big(\big|x(r) - y(r)\big| + \big|x\big((2r+1)/3\big) - y\big((2r+1)/3\big)\big| \\
&\qquad + \big|x(0) - y(0)\big| + \big|x(1) - y(1)\big|\Big)
\end{aligned}$$

and it follows that the condition (3.6) is satisfied and $\gamma(t, r) = \frac{tr}{64}$.

The conditions (3.7) becomes:

$$\sup_{t \in [0,1]} \int_0^1 \left(\frac{tr}{64}\right)^p dr < \frac{1}{2^{\varepsilon(p^2-p)+2p}}$$

and it follows that $\varepsilon(p^2 - p) < 4p + log_2(p+1)$. *Now, it results that there exists* $\varepsilon > 1$ *and* $p > 1$ *such that the condition (3.7) is satisfied. Therefore, the hypothesis* (iii) *is satisfied too.*

Using the properties of the functions K, g *and* f, *it is observed that hypothesis* (iv) *is satisfied for* $x_0 = f \in C[0,1]$.

Now, we observe that the conditions of Theorem 3.3 are satisfied and consequently, it results that the integral equation (4.9) has a solution.

5. Conclusions

The Fredholm integral equation is one of the most well-known types of integral equations. In general, a nonlinear Fredholm integral equation has the following form:

$$x(t) = \int_{\Omega} K(t, r, x(r))\, ds + f(t),\ t \in \Omega,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $K : \overline{\Omega} \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $f : \overline{\Omega} \to \mathbb{R}$ are given continuous functions and $x : \overline{\Omega} \to \mathbb{R}$ is the unknown function.

This equation has been studied in various particular cases for the bounded domain Ω, among which we mention: $\Omega = (a,b) \subset \mathbb{R}$ and $\Omega = (a,b) \times (a,b) \subset \mathbb{R}^2$, including cases involving modified arguments.

Using the notions of admissibility types defined on a *b*-metric space, (see [27]), we apply the fixed point result given by Theorem 2.1, ([22]), to establish new conditions for the existence of a solution to the functional nonlinear Fredholm integral equation (1.1). Accordingly, the necessary conditions for applying Theorem 2.1 are established, and the resulting existence criterion is formulated as Existence Theorem 3.2, which provides an additional property of the studied integral equation. The paper concludes with two examples illustrating the application of the main result.

As a final remark, within the framework of the existence theorem for the solution of the functional nonlinear Fredholm integral equation (1.1), once the existence of a solution is established, several additional properties may be investigated—particularly those related to the theory of Picard operators, such as well-posedness, Ulam–Hyers stability, and Ostrowski stability. For this type of integral equation, these three concepts will be the subject of a separate study, thereby leaving an open problem for future research.

Acknowledgments

The authors would like to thank the University of Petroşani for their support in the publication of this research article.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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