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A new type of cyclic iterated function systems via enriched cyclic weak contractions

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ABSTRACT. The purpose of this article is to introduce a new class of enriched cyclic contractions in Banach spaces, called enriched cyclic weak contractions. As an application, we define the corresponding cyclic iterated function system composed of this new class of enriched cyclic contractions. Some examples are also presented to validate the theoretical results.

1. INTRODUCTION

The term "fractal" was first introduced by Mandelbrot [17] in 1975, marking the beginning of a completely new field of study that bridges chaos theory and mathematical analysis. Hutchinson [8] established the theory about fractals in connection with fixed point theory. The term "iterated function system" (IFS) became well-known due to Barnsley (see [2, 3]). Since iterated function systems are providing one of the main techniques used to create fractals, the problem of extending the concept of these systems was taken into consideration by several authors (see, for example [11, 15, 20, 24, 27, 28, 29] and references therein).

Several researchers have obtained many fixed point results and their applications over the past 60 years (see [4, 5, 7]). In 2003, Kirk et al. [14] explored fixed points for maps that satisfy cyclic contraction conditions, attracting many researchers, who obtained a variety of fixed point results [5, 9, 10, 13, 16, 20, 22, 24, 25, 26].

Following the idea, Kirk et al. [14], Pasupathi et al. [29] discussed the concept of a cyclic iterated function system with cyclic contraction. Pasupathi et al. also constructed a cyclic ϕ IFS in [20], and a cyclic Meir-Keeler IFS in [21]. In 2022, Abbas et al. [1] investigated the IFS consisting of generalized enriched cyclic contraction mappings, and Ullah et al. [11] studied the cyclic weak ϕ IFS with weak ψ contraction mappings.

This current paper has two main goals. The first is to define a new class of mappings, called enriched cyclic weak contractions, which is also novel in the literature on fixed point theory, and to demonstrate the existence of their fixed points and their iterative approximation. The second goal is to study a cyclic iterated function system associated to enriched cyclic weak contractions.

2. ENRICHED CYCLIC WEAK CONTRACTIONS

Throughout the paper, by $\mathbb{N}$ and $\mathbb{R}$ we will denote the set of all natural numbers and the set of all real numbers, respectively. $(X, \|\cdot\|)$ denotes a normed space over the field $\mathbb{R}$, while $\{B_j : j = 1, 2, 3, \ldots, p\}$ denotes a finite family of nonempty closed subsets of the normed space X, where $p \in \mathbb{N}$.

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Let

$$\Omega = \{\phi : X \to \mathbb{R}\colon \phi(x) \neq 0 \; \forall \, x \in X\}, \quad \mho = \{\psi : X \to \mathbb{R}\colon \psi(x) \neq -1 \; \forall \, x \in X\}$$

and let Θ be the family of functions $\zeta : [0, \infty) \to [0, \infty)$ which are continuous and non-decreasing with $\zeta(t) > 0$, for $t \in (0, \infty)$, and satisfy $\zeta(0) = 0$.

In [23], the following concept was introduced. Let $T : X \to X$ and a fixed $\phi \in \Omega$. The average mapping of T is $T_\phi : X \to X$ defined by

$$T_\phi(x) = (1 - \phi(x))x + \phi(x)Tx, \; \forall \, x \in X. \tag{2.0.1}$$

A finite collection $\{B_j : j = 1, 2, 3, \ldots, p\}$ is called a *cyclic representation of* $\bigcup_{j=1}^{p} B_j$ *with respect to* T (see [25]) if

$$T(B_1) \subseteq B_2, \ldots, T(B_{p-1}) \subseteq B_p, \text{ and } T(B_p) \subseteq B_1.$$

The class of cyclic weak contractions was proposed and investigated by Karapınar [12] in the setting of metric spaces. Now we introduce the concept of cyclic weak contraction in the framework of normed spaces.

Definition 2.1. *Let $(X, \|\cdot\|)$ be a normed space and $\{B_j : j = 1, 2, 3, \ldots, p\}$ a finite collections of nonempty subsets of X. A mapping $T : \bigcup_{j=1}^{p} B_j \to \bigcup_{j=1}^{p} B_j$ is called a cyclic weak contraction if*

(1) $\{B_j : j = 1, 2, 3, \ldots, p\}$ is a cyclic representation of $\bigcup_{j=1}^{p} B_j$ with respect to T;
(2) there exists $\zeta \in \Theta$, such that for $1 \leq j \leq p$ we have

$$\|Tx - Ty\| \leq \|x - y\| - \zeta(\|x - y\|), \text{ for all } x \in B_j, y \in B_{j+1}, \tag{2.0.2}$$

where $B_{p+1} = B_1$.

Observe that this class of mappings includes many well-known contractive conditions in the current literature [13]. It was proved that a cyclic weak contraction mapping defined on a complete metric space has a unique fixed point (see Theorem 6 in [13]).

We introduce the following class of mappings.

Definition 2.2. *Let $(X, \|\cdot\|)$ be a normed space and $\{B_j : j = 1, 2, 3, \ldots, p\}$ a finite collections of nonempty closed subsets of X. A mapping $T : \bigcup_{j=1}^{p} B_j \to X$ is called enriched cyclic weak contractions (ECWC) if it satisfies the following conditions:*

(1) there exists $\psi \in \mho$ such that, for $\phi(x) = \frac{1}{1+\psi(x)}$, $\forall x \in X$, we have $\phi \in \Omega$, and the collection $\{B_j : j = 1, 2, 3, \ldots, p\}$ is a cyclic representation of $\bigcup_{j=1}^{p} B_j$ with respect to T_ϕ,
(2) there exists $\zeta \in \Theta$, such that for all $x \in B_j, y \in B_{j+1}$ for $1 \leq j \leq p$, with $B_{p+1} = B_1$, we have

$$\left\| \frac{x\psi(x) + Tx}{1 + \psi(x)} - \frac{y\psi(y) + Ty}{1 + \psi(y)} \right\| \leq \|x - y\| - \zeta(\|x - y\|). \tag{2.0.3}$$

To highlight the involvement of ϕ, ψ and ζ in (2.0.3), we shall also call T a (ϕ, ψ, ζ)-ECWC.

Enriched cyclic weak contractions are preferred over weak cyclic contractions for four reasons:

(1) They are non self-mappings, while weak cyclic contractions are self-mappings.
(2) Every T weak cyclic contraction is (ϕ, ψ, ζ) -ECWC. Indeed, if we take $\phi(x) = 1$ for all $x \in X$, then T_ϕ becomes T, and then $\{B_j : j = 1, 2, 3, \ldots, p\}$ is a cyclic representation of $\bigcup_{j=1}^{p} B_j$ with respect to $T_\phi \equiv T$. Moreover, the condition (2.0.2) for cyclic weak contraction T satisfies condition (2.0.3) for $\psi(x) = 0$, $\forall x \in X$.

(3) In the case of cyclic weak contraction the first condition is given for mapping T, that is, $\{B_j : j = 1, 2, 3, \dots, p\}$ is a cyclic representation of $\bigcup_{j=1}^{p} B_j$ with respect to T, while in the case of enriched cyclic weak contraction this condition is replaced by some averaged operator T_ϕ as given by (2.0.1). The example 2.1 demonstrates this argument: there exist a class of mapping such that $\{B_j : j = 1, 2, 3, \dots, p\}$ is not a cyclic representation of $\bigcup_{j=1}^{p} B_j$ with respect to T but there exists $\phi \in \Omega$ such that $\{B_j : j = 1, 2, 3, \dots, p\}$ is a cyclic representation of $\bigcup_{j=1}^{p} B_j$ with respect to T_ϕ.

(4) There is a class of mappings T that satisfy all of the conditions of enriched cyclic weak contraction but are not weak cyclic contractions. This argument is demonstrated by the following Example 2.1.

As a conclusion, the class of enriched cyclic weak contractions is larger than the class of cyclic weak contractions.

Example 2.1. *Let $X = \mathbb{R}$ with the usual metric. Suppose $B_1 = [-1, 0] = B_3$ and $B_2 = [0, 1] = B_4$. Define $T : \cup_{j=1}^{4} B_j \to X$ such that $Tx = -x\left(\dfrac{5 + 4|x|}{3}\right)$ for all $x \in \cup_{j=1}^{4} B_j$.*

It is easy to check that $\{B_j : j = 1, \dots, 4\}$ is not a cyclic representation of $\bigcup_{j=1}^{4} B_j$ with respect to T. Indeed, if $x = -1 \in B_1$, then $T(-1) = 3 \notin B_2$.

Let $\psi : X \to \mathbb{R}$ defined by $\psi(x) = 1 + |x|$, for all $x \in X$. It is clear that $\psi \in \mho$. Then $\phi : X \to \mathbb{R}$ is defined by $\phi(x) = \dfrac{1}{2 + |x|}$, for all $x \in X$, therefore $\phi \in \Omega$. Note that the average mapping is

$$\begin{aligned} T_\phi(x) &= (1 - \phi(x))x + \phi(x)Tx \\ &= \left(1 - \frac{1}{2 + |x|}\right)x + \left(\frac{1}{2 + |x|}\right)\left(-x\left(\frac{5 + 4|x|}{3}\right)\right) \\ &= \left(\frac{1 + |x|}{2 + |x|}\right)x + \frac{-x(5 + 4|x|)}{3(2 + |x|)} = \frac{3x(1 + |x|) - 5x - 4x|x|}{3(2 + |x|)} = \frac{-2x - x|x|}{3(2 + |x|)} \\ &= \frac{-x}{3}. \end{aligned}$$

Therefore, it follows that $T_\phi(x) = \dfrac{-x}{3}$, for all $x \in \bigcup_{j=1}^{4} B_j$. It is clear that $\{B_1, B_2, B_3, B_4\}$ is a cyclic representation of $\cup_{j=1}^{4} B_j$ with respect to T_ϕ.

Furthermore, if $\zeta : [0, \infty) \to [0, \infty)$ is defined by $\zeta(t) = \frac{t}{2}$, for all $t \in [0, \infty)$ then $\zeta \in \Theta$. Hence T is a (ϕ, ψ, ζ)-ECWC.

With $\phi(x) = \dfrac{1}{1 + \psi(x)}$, for all $x \in X$ with $\phi \in \Omega$ the (ϕ, ψ, ζ)-ECWC condition (2.0.3) becomes

$$\left|\phi(x)\left(\left(\frac{1}{\phi(x)} - 1\right)x + Tx\right) - \phi(y)\left(\left(\frac{1}{\phi(y)} - 1\right)y + Ty\right)\right| \le |x - y| - \zeta(|x - y|),$$

$$\left|\phi(x)\frac{(1 - \phi(x))x + \phi(x)Tx}{\phi(x)} - \phi(y)\frac{(1 - \phi(y))y + \phi(y)Ty}{\phi(y)}\right| \le |x - y| - \zeta(|x - y|)$$

which can be written in an equivalent form as

$$|T_\phi x - T_\phi y| \le |x - y| - \zeta(|x - y|). \tag{2.0.4}$$

This holds for all $x \in B_j, y \in B_{j+1}$ for $1 \le j \le 4$, where $B_5 = B_1$.

On the other hand, if T would be a cyclic weak contraction, then for $\zeta(t) = \frac{t}{2}$, $\forall t \in [0, \infty)$, the *contractive condition (2.0.2) becomes*

$$\left|-x\left(\frac{5+4|x|}{3}\right)+y\left(\frac{5+4|y|}{3}\right)\right| \le \frac{|x-y|}{2},$$

which for $x=0\in B_1$ and $y\in B_2$ leads to a contradiction.

Before proving the main result, we need the following lemma.

Lemma 2.1. [23] *Let $T: X\to X$ and T_ϕ be given in (2.0.1). Then for any $\phi\in\Omega$, we have*

$$F(T)=\{x\in X: Tx=x\}=\{x\in X: T_\phi x=x\}=F(T_\phi). \tag{2.0.5}$$

Proof. The proof is obvious. □

We start with the following result.

Theorem 2.1. *Let $(X,\|\cdot\|)$ be a normed space and $\{B_j: j=1,2,3,\dots,p\}$ a finite collection of nonempty closed subsets of X. If $T:\bigcup_{j=1}^p B_j\to X$ is a $(\phi,\psi,\zeta)-$ECWC, then T has a unique fixed point $x^*\in\bigcap_{j=1}^p B_j$.*

Proof. Let $\psi\in\mho$ such that if we take $\phi(x)=\dfrac{1}{1+\psi(x)}, \forall x\in X$ we have $\phi\in\Omega$. Moreover the collection $\{B_j: j=1,2,3,\dots,p\}$ is cyclic representation of $\bigcup_{j=1}^p B_j$ with respect to T_ϕ. From the condition (2.0.3) we obtain

$$\left\|\phi(x)\left(\left(\frac{1}{\phi(x)}-1\right)x+Tx\right)-\phi(y)\left(\left(\frac{1}{\phi(y)}-1\right)y+Ty\right)\right\| \le \|x-y\|-\zeta\left(\|x-y\|\right),$$

$$\left\|\phi(x)\frac{(1-\phi(x))x+\phi(x)Tx}{\phi(x)}-\phi(y)\frac{(1-\phi(y))y+\phi(y)Ty}{\phi(y)}\right\| \le \|x-y\|-\zeta\left(\|x-y\|\right)$$

which can be written in an equivalent form as

$$\|T_\phi x-T_\phi y\| \le \|x-y\|-\zeta\left(\|x-y\|\right), \tag{2.0.6}$$

for all $x\in B_j, y\in B_{j+1}$ with $1\le j\le p$ where $B_{p+1}=B_1$.

Let $x_0\in\bigcup_{j=1}^p B_j$ and set

$$x_{n+1}=(1-\phi(x_n))x_n+\phi(x_n)Tx_n=T_\phi x_n.$$

Notice that, for any $n\ge 0$, there exists $j_n\in\{1,2,3,\dots,p\}$ such that $x_n\in B_{j_n}$ and $x_{n+1}\in B_{j_n+1}$.

Then by (2.0.6), we have

$$\|x_{n+1}-x_{n+2}\|=\|T_\phi x_n-T_\phi x_{n+1}\|\le\|x_n-x_{n+1}\|-\zeta\left(\|x_n-x_{n+1}\|\right).$$

Define $\delta_n=\|x_n-x_{n+1}\|$. Then one can obtain

$$\delta_{n+1}\le\delta_n-\zeta\left(\delta_n\right)\le\delta_n, \tag{2.0.7}$$

which implies that $\{\delta_n\}$ is a non-increasing sequence. Hence, $\{\delta_n\}$ converges to $\delta\ge 0$. Assume that $\delta>0$. Having in view that ζ is non-decreasing, we get $0<\zeta(\delta)\le\zeta(\delta_n)$.

It follows from (2.0.7) that

$$\delta_{n+1}\le\delta_n-\zeta\left(\delta_n\right)\le\delta_n-\zeta\left(\delta\right)$$

and so

$$\delta_{n+2}\le\delta_{n+1}-\zeta\left(\delta_{n+1}\right)\le\delta_n-\zeta\left(\delta_n\right)-\zeta\left(\delta_{n+1}\right)\le\delta_n-2\zeta\left(\delta\right).$$

Inductively, we obtain $\delta_{n+m} \leq \delta_p - m\zeta(\delta)$, which is a contradiction for large $m \in \mathbb{N}$. Therefore we have $\delta = 0$.

Take $\varepsilon > 0$. Choose $n_0 \in \mathbb{N}$ in a way that $\|x_{n_0} - x_{n_0+1}\| \leq \min\left\{\frac{\varepsilon}{2}, \zeta\left(\frac{\varepsilon}{2}\right)\right\}$. We assert that T_ϕ is a self-mapping on the closed ball $B(x_{n_0}, \varepsilon) = \{x \in X : \|x - x_{n_0}\| \leq \varepsilon\}$. To prove our assertion, take $x \in B(x_{n_0}, \varepsilon)$. If $\|x - x_{n_0}\| \leq \frac{\varepsilon}{2}$, then due to (2.0.6) and triangle inequality, we have

$$\begin{aligned}\|T_\phi x - x_{n_0}\| &\leq \|T_\phi x - T_\phi x_{n_0}\| + \|T_\phi x_{n_0} - x_{n_0}\| \\ &\leq \|x - x_{n_0}\| - \zeta(\|x - x_{N_0}\|) + \|x_{n_0+1} - x_{n_0}\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Now consider the other case, that is $\|x - x_{n_0}\| > \frac{\varepsilon}{2}$. For sure we have, $\frac{\varepsilon}{2} < \|x - x_{n_0}\| \leq \varepsilon$ which implies that $\zeta\left(\frac{\varepsilon}{2}\right) \leq \zeta(\|x - x_{n_0}\|)$. Thus, due to (2.0.6) and triangle inequality, we have

$$\begin{aligned}\|T_\phi x - x_{n_0}\| &\leq \|T_\phi x - T_\phi x_{n_0}\| + \|T_\phi x_{n_0} - x_{n_0}\| \\ &\leq \|x - x_{n_0}\| - \zeta(\|x - x_{n_0}\|) + \|x_{n_0+1} - x_{n_0}\| \\ &\leq \varepsilon - \zeta\left(\frac{\varepsilon}{2}\right) + \zeta\left(\frac{\varepsilon}{2}\right) \\ &\leq \varepsilon.\end{aligned}$$

Therefore, in all cases, $T_\phi x \in B(x_{n_0}, \varepsilon)$.

In other words, T_ϕ is a self-mapping on the closed ball $B(x_{n_0}, \varepsilon)$ and thus $x_n \in B(x_{n_0}, \varepsilon)$ for each $n > n_0$.

Therefore, the sequence $\{x_n\}$ is Cauchy in the complete subspace $\bigcup_{j=1}^p B_j$. Since $B(x_{n_0}, \varepsilon)$ is closed, the sequence $\{x_n\}$ is convergent in $\bigcup_{j=1}^p B_j$, say $x^* \in \bigcup_{j=1}^p B_j$. By the fact that $\{B_j : j = 1, 2, 3, \ldots, p\}$ is cyclic representation of $\bigcup_{j=1}^p B_j$ with respect to T_ϕ, the sequence $\{x_n\}$ has infinite number of terms in each B_j for all $j \in \{1, ..., p\}$. Therefore $x^* \in \bigcap_{j=1}^p B_j$ and thus $\bigcap_{j=1}^p B_j \neq \emptyset$.

Consider the restriction of T_ϕ on $\bigcap_{j=1}^p B_j$, that is, $T_\phi|_{\bigcap_{j=1}^p B_j} : \bigcap_{j=1}^p B_j \to \bigcap_{j=1}^p B_j$ which satisfies the assumptions of Theorem 1 in [26] and thus, $T_\phi|_{\bigcap_{j=1}^p B_j}$ has a unique fixed point, say $z^* \in \bigcap_{j=1}^p B_j$ which is obtained by iteration from starting point x_0. We claim that for any initial value $x \in \bigcup_{j=1}^p B_j$, we get the same limit point $z^* \in \cap_{j=1}^p B_j$. Indeed, for $x \in \cup_{j=1}^p B_j$, by repeating the above process, the corresponding iterative sequence yields that $T_\phi|_{\bigcap_{j=1}^p B_j}$ has a unique fixed point, say $w^* \in \cap_{j=1}^p B_j$. Regarding that $z^*, w^* \in \cap_{j=1}^p B_j$, we have $z^*, w^* \in B_j$ for all j, hence $\|z^* - w^*\|$ and $\|T_\phi z^* - T_\phi w^*\|$ are well defined. Due to (2.0.6),

$$\|z^* - w^*\| = \|T_\phi z^* - T_\phi w^*\| \leq \|z^* - w^*\| + \zeta(\|z^* - w^*\|)$$

which is a contradiction.

In conclusion, z^* is the unique fixed point of T_ϕ for any initial starting point $x_0 \in \bigcap_{j=1}^p B_j$. □

Example 2.2. *Let $B_1 = [0, 1] = B_2$. Define a map $T : B_1 \cup B_2 \to B_1 \cup B_2$ by $Tx = x - x^2$, for all $x \in B_1 \cup B_2$. Clearly $\{B_1, B_2\}$ is a cyclic representation of $B_1 \cup B_2$ with respect to T, but T is not a weak contraction. To prove this, let $x \in B_1$ and $y \in B_2$. Then*

$$|Tx - Ty| = |(x - y) \cdot (x + y + 1)| \leq 3 \cdot |x - y|$$

Since $\zeta \in \Theta$ one can't determine ζ suct that $3 \cdot |x - y| \leq |x - y| - \zeta(|x - y|)$.

We claim that T is a (ϕ, ψ, ζ) −ECWC.

Let us take $\psi(x) = |x|$, *for all* $x \in B_1 \cup B_2$. *Then* $\phi(x) = \dfrac{1}{1+|x|}$, *with* $\phi \in \Omega$ *and* $T_\phi x = \dfrac{x}{x+1}$, *for all* $x \in B_1 \cup B_2$. *It is easy to check that* $\{B_1, B_2\}$ *is a cyclic representation of* $B_1 \cup B_2$ *with respect to* T_ϕ. *Further, we need to verify if* T_ϕ *satisfies 2.0.4. To see this, let* $x \in B_1$ *and* $y \in B_2$. *Hence*

$$|T_\phi x - T_\phi y| = \left|\frac{x}{1+x} - \frac{y}{1+y}\right| = \left|\frac{x-y}{(1+x)(1+y)}\right|$$

If we take $\zeta(t) = \dfrac{t^2}{t+2}$ *then* $\zeta \in \Theta$ *and*

$$|x-y| - \zeta(|x-y|) = \frac{|x-y|}{\dfrac{|x-y|}{2}+1} \geq |T_\phi x - T_\phi y|$$

since $\dfrac{1}{(1+x)(1+y)} \leq \dfrac{1}{\dfrac{|x-y|}{2}+1}$. *To prove this, we consider two cases:*

- *case 1: If* $0 \leq x \leq y \leq 1$, *then* $|x-y| = y-x$. *The above inequality becomes*

$$(1+x)(1+y) \geq \frac{y-x}{2} + 1 \;\Leftrightarrow\; (2x+1)(2y+3) \geq 3$$

  which is always true in this case.
- *case 2: If* $0 \leq y < x \leq 1$, *then* $|x-y| = x-y$. *The above inequality becomes*

$$(1+x)(1+y) \geq \frac{x-y}{2} + 1 \;\Leftrightarrow\; (2x+3)(2y+1) \geq 3$$

  which is always true in this case.

Therefore T *is a* (ϕ, ψ, ζ) *−ECWC, and then by Theorem 2.1* T *has a unique fixed point* $x^* = 0 \in B_1 \cap B_2$.

Corollary 2.1. *Let* $(X, \|\cdot\|)$ *be a normed space and* $\{B_j : j = 1, 2, 3, \ldots, p\}$ *a finite collections of nonempty subsets of* X. *If* $T : \bigcup_{j=1}^p B_j \to X$ *is a mapping with the property that there existes a positive integer* η *such that* T^η *is a* (ϕ, ψ, ζ)*−ECWC, then:*

(1) T *has a unique fixed point* $x^* \in \bigcap_{j=1}^p B_j$.
(2) the sequence $\{x_n\}$ *given by*

$$x_{n+1} = (1-\phi(x))x_n + \phi(x)T^\eta x_n$$

converges to x^*, *for any* $x_0 \in \bigcup_{j=1}^p B_j$.

Proof. We apply Theorem 2.1 for the mapping T^η and we obtain that T^η has a unique fixed point $x^* \in \bigcap_{j=1}^p B_j$, that means $T^\eta x^* = x^*$. We also have:

$$T^\eta(Tx^*) = T^{\eta+1}x^* = T(T^\eta x^*) = Tx^*$$

This shows that Tx^* is a fixed point of T^η. But T^η has a unique fixed point x^* hence $Tx^* = x^*$. The remaining part of the proof follows from Theorem 2.1. □

3. Application to Cyclic Iterated Function Systems

An iterated function system (IFS) on a topological space is given by a finite set of continuous maps defined on the entire space. If the space is X and the maps are $T_i : X \to X$, for all $1 \leq i \leq n$ with $n \in \mathbb{N}$ fixed, then we denote the IFS with $\{X; T_1, T_2, \ldots, T_n : n \in \mathbb{N}\}$. Let us use the notations $\mathcal{C}(X)$ for the collection of all nonempty compact subsets of the metric space (X, d).

Definition 3.3. [20] *Let $\mathcal{C}(X)$ be the collection of all nonempty compact subsets of the metric space (X, d). For $A, B \in \mathcal{C}(X)$, define*

$$\sigma(A, B) = \sup_{a \in A}\{\xi(a, B)\},$$

where $\xi(a, B) = \inf\{d(a, b),\ b \in B\}$.
Define the functional $\mathcal{H} : \mathcal{C}(X) \times \mathcal{C}(X) \to [0, \infty)$

$$\mathcal{H}(A, B) = \max\big\{\sigma(A, B), \sigma(B, A)\big\}.$$

The mapping $\mathcal{H}$ is called Pompeiu-Hausdorff metric on $\mathcal{C}(X)$ induced by d. The metric space $(\mathcal{C}(X), \mathcal{H})$ is complete (compact) provided that (X, d) is complete (compact).

Remark 3.1. [20] *As the image of $A \in \mathcal{C}(X)$ under the continuous mapping T is compact, there is a natural way to define the induced mapping $\mathbf{T} : \mathcal{C}(X) \to \mathcal{C}(X)$ by $\mathbf{T}(A) := T(A)$, for all $A \in \mathcal{C}(X)$, where $T(A)$ denotes the image of A under T.*

Definition 3.4. [2] *Let $\{X; T_1, T_2, \dots, T_n : n \in \mathbb{N}\}$ be an iterated function system (IFS) and $\mathbf{T} : \mathcal{C}(X) \to \mathcal{C}(X)$ be given by*

$$\mathbf{T}(A) = \bigcup_{i=1}^{n} T_i(A), \text{ for any } A \in \mathcal{C}(X).$$

If $A = \mathbf{T}(A)$, then A is called the attractor of the IFS, and it is a fixed point of $\mathbf{T}$.

In the sequel, we need the following lemmas.

Lemma 3.2. [1] *If $\{A_j\}_{j\in\bigwedge}$, $\{B_j\}_{j\in\bigwedge}$ are two finite collections of sets in $(\mathcal{C}(X), \mathcal{H})$, then*

$$\mathcal{H}\big(\cup_{j\in\bigwedge} A_j, \cup_{j\in\bigwedge} B_j\big) \le \max_{j\in\bigwedge} \mathcal{H}(A_j, B_j),$$

where $\bigwedge = \{1, 2, 3, ..., n\}$.

Lemma 3.3. [1] *Let (X, d) be a complete metric space. If A is a closed subset of X, then $\mathcal{C}(A)$ is also closed subset of the complete metric space $(\mathcal{C}(X), \mathcal{H})$.*

Before giving our main result in this section, we prove the following theorem.

Theorem 3.2. *Let $(X, \|\cdot\|)$ be a normed space and $\{B_j : j = 1, 2, 3, \dots, p\}$ a finite collection of nonempty closed subsets of X. If $T : \bigcup_{j=1}^{p} B_j \to X$ is an ECWC, then the induced map $\mathbf{T} : \bigcup_{j=1}^{p} \mathcal{C}(B_j) \to \mathcal{C}(X)$ satisfies the following conditions:*

(1) there exist $\psi \in \mho$ such that for $\phi(x) = \dfrac{1}{1+\psi(x)}$, $\forall x \in X$ we have $\phi \in \Omega$ and $\{\mathcal{C}(B_j) : j = 1, 2, 3, \dots, p\}$ is cyclic representation of $\bigcup_{j=1}^{p} \mathcal{C}(B_j)$ with respect to $\mathbf{T}_\phi$, provided that T_ϕ defined by (2.0.1) is continuous

(2) there exists $\zeta^* \in \Theta$ such that for each $j \in \{1, 2, ..., p\}$, $A \in \mathcal{C}(B_j)$ and $B \in \mathcal{C}(B_{j+1})$ with $B_{p+1} = B_1$, we have

$$\mathcal{H}\left(\frac{\psi(x)(A) + \mathbf{T}(A)}{1+\psi(x)}, \frac{\psi(y)(B) + \mathbf{T}(B)}{1+\psi(y)}\right) \le \mathcal{H}(A, B) - \zeta^*(\mathcal{H}(A, B)), \tag{3.0.1}$$

Moreover, the induced map $\mathbf{T}$ has a unique fixed point.

Proof. Let $\psi \in \mho$ and

$$\phi(x) = \frac{1}{1+\psi(x)}, \quad \forall x \in X. \tag{3.0.2}$$

such that $\phi \in \Omega$. The average mapping T_ϕ is defined by (2.0.1). Similarly to the proof of Theorem 2.1 the (ϕ, ψ, ζ)-ECWC condition (2.0.3) becomes

$$\|T_\phi x - T_\phi y\| \leq \|x-y\| - \zeta\left(\|x-y\|\right), \tag{3.0.3}$$

for all $x \in B_j, y \in B_{j+1}$ for $1 \leq j \leq p$ with $B_{p+1} = B_1$.

Note that in view of (3.0.2), the inequality (3.0.1) is equivalent to

$$\mathcal{H}\left(\mathbf{T}_\phi(A), \mathbf{T}_\phi(B)\right) \leq \mathcal{H}(A,B) - \zeta^*(\mathcal{H}(A,B)). \tag{3.0.4}$$

Let $A \in \mathcal{C}(B_j)$, for some $j \in \{1,2,...,p\}$. Since $\{B_j : j = 1,2,3,\ldots,p\}$ is cyclic representation of $\bigcup_{j=1}^{p} B_j$ with respect to T_ϕ, we have that

$$\mathbf{T}_\phi(A) \subseteq B_{j+1}.$$

By continuity of T_ϕ, $\mathbf{T}_\phi(A)$ is a compact set and hence

$$\mathbf{T}_\phi(A) \in \mathcal{C}(B_{j+1}).$$

For all $j \in \{1,2,...,p\}$, we have

$$\mathbf{T}_\phi(\mathcal{C}(B_j)) \subseteq \mathcal{C}(B_{j+1}). \tag{3.0.5}$$

We will prove that

$$\sigma\left(\frac{\psi(x)(A) + \mathbf{T}(A)}{1+\psi(x)}, \frac{\psi(y)(B) + \mathbf{T}(B)}{1+\psi(y)}\right) \leq \mathcal{H}(A,B) - \zeta^*(\mathcal{H}(A,B)). \tag{3.0.6}$$

Again in view of (3.0.2), the inequality (3.0.6) is equivalent to

$$\sigma\left(\mathbf{T}_\phi(A), \mathbf{T}_\phi(B)\right) \leq \mathcal{H}(A,B) - \zeta^*(\mathcal{H}(A,B)). \tag{3.0.7}$$

For arbitrary $x \in A$, we have

$$\begin{aligned} \xi(T_\phi x, \mathbf{T}_\phi(B)) &= \min\{\|T_\phi x - T_\phi y\| : y \in B\} \\ &\leq \|T_\phi x - T_\phi y\|, \ \ \forall y \in B \\ &\leq \|x-y\| - \zeta\left(\|x-y\|\right), \ \ \forall y \in B. \end{aligned} \tag{3.0.8}$$

Since $\zeta \in \Theta$ and $\|x-y\| \leq \sigma(A,B) \leq \mathcal{H}(A,B)$, we can find a real number $a_1 \geq 1$ such that

$$\frac{\zeta\left(\sigma(A,B)\right)}{a_1} \leq \zeta\left(\|x-y\|\right).$$

Similarly, we can find a real number $a_2 \geq 1$ such that

$$\frac{\zeta\left(\mathcal{H}(A,B)\right)}{a_1 a_2} = \zeta^*\left(\mathcal{H}(A,B)\right) \leq \frac{\zeta\left(\sigma(A,B)\right)}{a_1},$$

Clearly, $\zeta^* \in \Theta$. Hence, by the compactness of B and inequality (3.0.8), it follows that

$$\begin{aligned} \xi(T_\phi x, \mathbf{T}_\phi(B)) &\leq \xi(x,B) - \zeta\left(\xi(x,B)\right), \\ &\leq \sigma(A,B) - \frac{\zeta\left(\sigma(A,B)\right)}{a_1}, \\ &\leq \mathcal{H}(A,B) - \zeta^*(\mathcal{H}(A,B)). \end{aligned} \tag{3.0.9}$$

As $x \in A$ is arbitrary and $\mathbf{T}(A)$, is a compact set, we will get $z \in A$ such that

$$\sigma\left(\mathbf{T}_\phi(A), \mathbf{T}_\phi(B)\right) = \xi(T_\phi z, \mathbf{T}_\phi(B)),$$

for which the inequality (3.0.9) is true. Hence, we have

$$\sigma\left(\mathbf{T}_\phi(A), \mathbf{T}_\phi(B)\right) \leq \mathcal{H}(A,B) - \zeta^*(\mathcal{H}(A,B)). \tag{3.0.10}$$

Similarly,

$$\sigma\left(\mathbf{T}_\phi(B), \mathbf{T}_\phi(A)\right) \leq \mathcal{H}(A,B) - \zeta^*(\mathcal{H}(A,B)). \tag{3.0.11}$$

By combining (3.0.10) and (3.0.11), we get (3.0.7). Notice that $\mathbf{T}$ satisfies all the conditions of Theorem 2.1, therefore $\mathbf{T}$ has a unique fixed point. □

We introduce now the following concept.

Definition 3.5. *An enriched cyclic weak iterated function system (ECWIFS) consists in a Banach space X and a finite collection of nonempty closed subsets of X, $B_1, B_2, \dots B_p$ for some $p \in \mathbb{N}$, together with a finite set of ECWC continuous mappings, $T_i : \bigcup_{j=1}^p B_j \to X$, for all $i \in \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$ being $(\phi_i, \psi_i, \zeta_i)$-ECWC. We will use the following notation for ECWIFS:*

$$\{(X, B_1, B_2, \dots B_p); T_1, T_2, \dots T_n : n, p \in \mathbb{N}\}.$$

We mention here that since T_i are $(\phi_i, \psi_i, \zeta_i)$-ECWC mappings, the average mappings are defined by $(T_i)_{\phi_i}(x) = (1 - \phi_i(x))x + \phi_i(x)T_i x$, for all $x \in X$ and $n \in \mathbb{N}$.

Theorem 3.3. *Consider $\{(X, B_1, B_2, \dots B_p); T_1, T_2, \dots, T_n : n, p \in \mathbb{N}\}$ an ECWIFS with T_i beeing $(\phi_i, \psi_i, \zeta_i)$-ECWC for all $i \in \{1, 2, \dots, n\}$, and the map $U : \bigcup_{j=1}^p \mathcal{C}(B_j) \to \bigcup_{j=1}^p \mathcal{C}(B_j)$, defined by $U(B) = \bigcup_{i=1}^n (\mathbf{T}_i)_{\phi_i}(B)$ for every $B \in \bigcup_{j=1}^p \mathcal{C}(B_j)$. Then there exists $\zeta^* \in \Theta$ such that*

$$\mathcal{H}(U(A), U(B)) \leq \mathcal{H}(A, B) - \zeta^*(\mathcal{H}(A, B)),$$

where $\zeta^* = \min\{\zeta_i^*,\ i = 1, 2, .., n\}$. Moreover, the U has a unique fixed point (attractor or in general the fractal of IFS).

Proof. It is easy to see that $U(\mathcal{C}(B_j)) \subseteq \mathcal{C}(B_{j+1})$, for all $j \in \{1, 2, 3, \dots, p\}$.

Indeed, it follows from Theorem 3.2 that $\{\mathcal{C}(B_j) : j = 1, 2, 3, \dots, p\}$ is a cyclic representation of $\bigcup_{j=1}^p \mathcal{C}(B_j)$ with respect to each $(\mathbf{T_i})_{\phi_i}$ since each T_i are continuous $(\phi_i, \psi_i, \zeta_i)-$ECWC mappings, for all $i \in \{1, 2, 3, \dots, n\}$.

Also, it follows from Theorem 3.2, that

$$\mathcal{H}\left(\frac{\psi_i(x)(A) + \mathbf{T_i}(A)}{1 + \psi_i(x)}, \frac{\psi_i(y)(B) + \mathbf{T_i}(B)}{1 + \psi_i(y)}\right) \leq \mathcal{H}(A, B) - \zeta_i^*(\mathcal{H}(A, B)), \tag{3.0.12}$$

for all $i \in \{1, 2, 3, \dots, n\}$. Let $A \in \mathcal{C}(B_j)$ and $B \in \mathcal{C}(B_{j+1})$ for some $j \in \{1, 2, 3, \dots, p\}$.

Then, by Lemma 3.2 and (3.0.12), we have

$$\begin{aligned}
\mathcal{H}(U(A), U(B)) &= \mathcal{H}\big(\cup_{i=1}^n \{(\mathbf{T_i})_{\phi_i}(A)\}, \cup_{i=1}^n \{(\mathbf{T_i})_{\phi_i}(B)\}\big) \\
&\leq \max\big\{\mathcal{H}((\mathbf{T_1})_{\phi_1}(A), (\mathbf{T_1})_{\phi_1}(B)), \dots, H((\mathbf{T_n})_{\phi_n}(A), (\mathbf{T_n})_{\phi_n}(B))\big\} \\
&\leq \max\big\{\mathcal{H}(A, B) - \zeta_1^*(\mathcal{H}(A, B)), \dots, \mathcal{H}(A, B) - \zeta_n^*(\mathcal{H}(A, B))\big\} \\
&\leq \mathcal{H}(A, B) - \zeta^*(\mathcal{H}(A, B)),
\end{aligned}$$

where $\zeta^* = \min\{\zeta_i^*,\ i = 1, 2, .., n\}$.
Since X is a Banach space, $(\mathcal{C}(X), \mathcal{H})$ is a complete metric space. Using Lemma 3.3, $\mathcal{C}(B_j)$ is nonempty closed subset of $\mathcal{C}(X)$ for every $j \in \{1, 2, 3, \dots, p\}$. Clearly, U satisfies all the conditions of Theorem 2.1.

Hence, by Theorem 2.1, U has a unique fixed point. □

The next example supports our previous result.

Example 3.1. *Let $B_1 = [0, 2]$ and $B_2 = [1, 3]$. We define the map $T_1, T_2 : B_1 \cup B_2 \to \mathbb{R}$ by*

$$T_1(x) = \begin{cases} \dfrac{22 + x - 9x^2}{8} & \text{if } x \in [0, 2], \\[2mm] \dfrac{18 + x - 8x^2}{8} & \text{if } x \in [2, \frac{11}{4}] \\[2mm] \dfrac{62 + 7x - 16x^2}{8} & \text{if } x \in [\frac{11}{4}, 3], \end{cases}$$

and

$$T_2(x)=\begin{cases}\dfrac{28+x-9x^2}{8} & \text{if } x\in[0,2],\\ \dfrac{6+x-2x^2}{2} & \text{if } x\in[2,\dfrac{11}{4}]\\ \dfrac{34+5x-2x^2}{4} & \text{if } x\in[\dfrac{11}{4},3],\end{cases}$$

Assume that $\psi(x)=1+|x|$, *for all* $x\in\mathbb{R}$, *with* $\phi_1(x)=\phi_2(x)=\dfrac{1}{2+|x|}$, *and* $\zeta_1(t)=\zeta_2(t)=\frac{t}{3}$ *for all* $t\in[0,\infty)$. *Then we have*

$$(T_1)_{\phi_1}(x)=\begin{cases}\dfrac{11-x}{8} & \text{if } x\in[0,2],\\ \dfrac{9}{8} & \text{if } x\in[2,\frac{11}{4}]\\ \dfrac{31-8x}{8} & \text{if } x\in[\frac{11}{4},3],\end{cases}$$

and

$$(T_2)_{\phi_2}(x)=\begin{cases}\dfrac{14-x}{8} & \text{if } x\in[0,2],\\ \dfrac{3}{2} & \text{if } x\in[2,\frac{11}{4}]\\ \dfrac{17-4x}{4} & \text{if } x\in[\frac{11}{4},3],\end{cases}$$

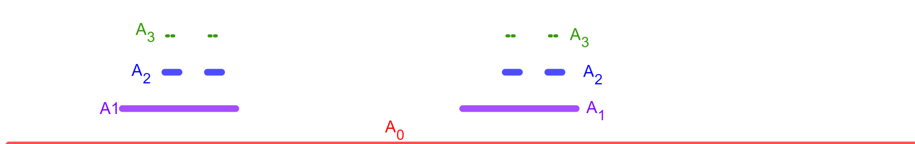
Clearly, $(T_1)_{\phi_1}$ *and* $(T_2)_{\phi_2}$ *are continuous and it is easy to check that* $(T_1)_{\phi_1}$ *and* $(T_2)_{\phi_2}$ *are* $(\phi_1,\psi_1,\zeta_1)-$*ECWC and* $(\phi_2,\psi_2,\zeta_2)-$*ECWC mappings, respectively.*

Hence, $\{(\mathbb{R},B_1,B_2);T_1,T_2\}$ *is an ECWIFS and it has the attractor* A. *The attractor* A *is similar to a Cantor set for* $[1,2]$ *with 8 sub-intervals.*

Let $A_0=[1,2]$. *We construct the sequence* $\{A_n\}$ *in the following way:*

$$\begin{aligned} A_1 &= U(A_0)=(T_1)_{\phi_1}(A_0)\cup(T_2)_{\phi_2}(A_0)=\left[\frac{9}{8},\frac{10}{8}\right]\bigcup\left[\frac{12}{8},\frac{13}{8}\right]\\ A_2 &= U(A_1)=(T_1)_{\phi_1}(A_1)\cup(T_2)_{\phi_2}(A_1)\\ &= \left[\frac{75}{8^2},\frac{76}{8^2}\right]\bigcup\left[\frac{78}{8^2},\frac{79}{8^2}\right]\bigcup\left[\frac{99}{8^2},\frac{100}{8^2}\right]\bigcup\left[\frac{102}{8^2},\frac{103}{8^2}\right]\\ A_3 &= U(A_2)=(T_1)_{\phi_1}(A_2)\cup(T_2)_{\phi_2}(A_2)\\ &= \left[\frac{601}{8^3},\frac{602}{8^3}\right]\bigcup\left[\frac{604}{8^3},\frac{605}{8^3}\right]\bigcup\left[\frac{626}{8^3},\frac{627}{8^3}\right]\bigcup\left[\frac{628}{8^3},\frac{629}{8^3}\right]\bigcup\\ \bigcup &\ \left[\frac{792}{8^3},\frac{793}{8^3}\right]\bigcup\left[\frac{796}{8^3},\frac{797}{8^3}\right]\bigcup\left[\frac{817}{8^3},\frac{818}{8^3}\right]\bigcup\left[\frac{820}{8^3},\frac{821}{8^3}\right]\\ &\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

The initial three steps of the iterative process are illustrated in the diagram below.



Also we can see that ... $\subset A_2 \subset A_1 \subset A_0$, *hence the sequence* $\{A_n\}$ *is non-increasing and* $A = \lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$.

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