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Unit neighborhoods of zero in topological ordered rings

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ABSTRACT. A closed unit neighborhood of zero in a topological ring is an additively symmetric and multiplicatively idempotent regular closed neighborhood of zero containing the unity whose interior is multiplicatively idempotent as well. The search for nontrivial closed unit neighborhoods of zero in topological rings is an ongoing quest. A unital ordered ring is a ring endowed with a partial ordering compatible with the addition and multiplication by positive elements for which zero and the unity are comparable. A topological ordered ring is a unital ordered ring for which the order topology is a ring topology. Recently, it was posed the question whether the set of elements lying in between -1 and 1 is a closed unit neighborhood of 0 in a topological ordered ring. This question has been partially solved on topological totally ordered division rings with no holes. Here, we provide a full answer in topological ordered rings (not relying on total orderings nor on division rings).

1. INTRODUCTION

A unit ball in a topological ring is an additively symmetric and multiplicatively idempotent closed neighborhood of zero containing the unity [8, Definition 1]. There are plenty of examples of topological rings endowed with a unit ball, ranging from $\{-1, 0, 1\}$ in a discrete ring thru the whole ring (which is the unique unit ball if the ring is endowed with the trivial topology). The study of this kind of Banach-space-like structures in topological rings and modules is each passing time more popular [11, 7, 10, 9, 12, 13, 14].

A particular case of unit balls are the closed unit neighborhoods of zero [4]. Let R be a topological ring. Let U, B be subsets of R. Then U is called an open unit neighborhood of 0 provided that U is an additively symmetric ($U = -U$) and multiplicatively idempotent ($UU = U$) regular open ($\mathrm{int}(\mathrm{cl}(U)) = U$) neighborhood of 0 such that $1 \in \mathrm{cl}(U)$. On the other hand, B is called a closed unit neighborhood of 0 provided that B is regular closed ($\mathrm{cl}(\mathrm{int}(B)) = B$) and its interior is an open unit neighborhood of 0. Observe that, if U is an open unit neighborhood of 0, then $\mathrm{cl}(U)$ is a closed unit neighborhood of 0. Alternatively (and by definition), if B is a closed unit neighborhood of 0, then $\mathrm{int}(B)$ is an open unit neighborhood of 0. In [4, Theorem 3.7], it is proved that the only proper closed unit neighborhood of 0 in $\mathbb{R}$ is $[-1, 1]$ and the only proper closed unit neighborhood of 0 in $\mathbb{C}$ containing the unit complex numbers $\mathbb{T}$ is the closed unit disk $\mathbb{D}$. In [15], other proper closed unit neighborhoods of 0 in $\mathbb{C}$ are provided.

Every ring of characteristic 0 can be easily endowed, via Zorn's Lemma, with a unital ring ordering (partial ordering compatible with addition and multiplication by positive elements, for which 0 and 1 are comparable). A topological ordered ring is a unital ordered ring for which the order topology is a ring topology [5, Definition 2.11]. The main question here is to determine whether, in a topological ordered ring, the set of elements lying between -1 and 1 is a closed unit neighborhood of 0. This question was partially answered in [3, Theorem 3.8] for topological totally ordered division rings with no holes. Here, we will provide a full answer by not relying on total orderings nor on division rings.

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2. PRELIMINARIES

The following notation on posets will be used throughout the manuscript: If A is a poset and $a, b \in A$, then $\uparrow a := [a, \infty) := \{x \in A : a \leq x\}$, $\downarrow b := (-\infty, b] := \{x \in A : x \leq b\}$, $\uparrow_\times a := (a, \infty) := \{x \in A : a < x\}$ and $\downarrow^\times b := (-\infty, b) := \{x \in A : x < b\}$. The classical notation for bounded intervals will also be employed.

All rings considered in this manuscript are assumed to be associative and unitary (and nonzero). All subrings will be considered unitary subrings in the sense that they share the unity of the ring. An ordered ring is a ring R endowed with a partial order $\leq$ compatible with the addition and multiplication operations, that is, for all $r, s \in R$ such that $r \leq s$, then $r + t \leq s + t$ for all $t \in R$ (translation invariance) and $rs \in R^+$ for all $r, s \in R^+$ (homotecy invariance), where $R^+ := \uparrow 0 = \{t \in R : t \geq 0\}$. Observe that, if $0, 1$ are comparable, then $\mathrm{char}(R) = 0$, that is, R has characteristic 0 (such ring orderings are called unital). Conversely, by relying on Zorn's Lemma, any ring of characteristic 0 can be endowed with a maximal unital ring ordering. From now on, we will only consider unital ring orderings. We will also let $R^- := \downarrow 0 = \{a \in R : a \leq 0\} = -R^+$. Other basic properties of ordered rings will be recalled as long as they are required for the proofs of the main results of this work.

The order topology is typically defined for totally ordered sets with at least two points. However, it can also be transported to posets with a more subtle construction. If X is a set and $\mathcal{S}$ is a nonemtpy subset of $\mathcal{P}(X)$ (power set of X), then the set of finite intersections of $\mathcal{S}$, $\mathcal{B}(\mathcal{S}) := \left\{\bigcap_{T \in \mathcal{T}} T : \mathcal{T} \subseteq \mathcal{S} \text{ finite}\right\}$, is trivially closed under finite intersections, thus $\mathcal{B}(\mathcal{S})$ is a base for a topology on X if and only if $\bigcup_{S \in \mathcal{S}} S = X$. As a consequence, if X is a poset, then $\{\uparrow_\times x, \downarrow^\times x : x \in X\}$ is a subbase for a topology on X if and only if for every $x \in X$ either $\uparrow_\times x \neq \varnothing$ or $\downarrow^\times x \neq \varnothing$. In this situation, the order topology on X is the topology generated by the subbase $\{\uparrow_\times x, \downarrow^\times x : x \in X\}$ and we call X a topological poset. There are different ways to construct the order topology in those posets X having incomparable elements with all the rest (those elements $x \in X$ satisfying that $\uparrow_\times x = \downarrow^\times x = \varnothing$). One way is by considering the subbase $\{\uparrow_\times x, \downarrow^\times x : x \in X\} \cup \{X\}$. Nevertheless, posets with incomparable elements with all the rest will not be the case of this work. It is immediate to observe that open intervals are open in the order topology, however closed intervals are not necessarily closed in the order topology, unless, for instance, the order is total (see Lemma 3.2).

We point out that in unital ordered rings, the order topology is always well defined. Indeed, if R is a unital ordered ring, then $r + 1 \in \uparrow_\times r$ and $r - 1 \in \downarrow^\times r$ for all $r \in R$, meaning that there are not elements incomparable with all the rest in R.

Sometimes ring topologies are assumed to have continuous multiplicative inversion by default (like, for instance, in [7]) due to the fact that some technical lemmas need such continuity (see [7, Lemma 1] or [6, Lemma 2.4]). However, in classical Associative Ring Theory [1, 16, 17, 2], continuity of the inversion is not generally assumed (except for topological division rings). On topological rings for which inversion is not continuous, it is often considered a special group topology on its multiplicative group of invertibles. Let R be a topological ring whose inversion is not continuous. Let $\mathcal{U}(R)$ stand for the multiplicative group of invertibles of R. Then $\{\mathsf{U}(V) : V \text{ is a } 0\text{-neighborhood in } R\}$ is a base for a group topology on $\mathcal{U}(R)$, where $\mathsf{U}(V) := \left\{r \in \mathcal{U}(R) : r, r^{-1} \in 1 + V\right\}$, for every neighborhood V of 0 in R. Here, in this manuscript, we will not assume the continuity of the inversion by default and we will explicitly call on it when necessary.

The following definition can be found in [5, Definition 2.11] (although it was originally coined in [3]).

Definition 2.1 (Topological ordered ring)**.** *Let R be a unital ordered ring. If the order topology on R is a ring topology, then we call R a topological ordered ring.*

Notice that not every order topology on a unital ordered ring is a ring topology. Indeed, let $R := \mathbb{R}[x]$ and define $p \leq q$ whenever $p = q$ or the leading coefficient of $q-p$ is positive. According to the properties of ring topologies, for every neighborhood V of 0 in R and every $r \in R$ there exists another neighborhood W of 0 in R satisfying that $rW \subseteq V$, which cannot hold for $r := x$ and $V := (-1, 1)$.

The following question was posed in [3].

Question 1. *Let R be a topological ordered ring. Is $[-1, 1]$ a closed unit neighborhood of 0?*

Question 1 was partially answered in [3, Theorem 3.8] for topological totally ordered division rings with no holes. Here, we will provide a full answer in Theorem 3.1 by not relying on total orderings nor on division rings.

3. Main result

In [7, Proposition 1] it was proved that a unit ball in a topological ring is a closed unit neighborhood of zero if and only if it is regular closed and its interior is multiplicatively idempotent. Observe that, if R is a topological ordered ring such that $[-1, 1]$ is closed, then $[-1, 1]$ is trivially a unit ball. Therefore, in order to achieve our goal, it only suffices to check that $\operatorname{int}([-1, 1]) = (-1, 1)$, $\operatorname{cl}((-1, 1)) = [-1, 1]$ and $(-1, 1)(-1, 1) = (-1, 1)$. In other words, we have the following initial result.

Proposition 3.1. *Let R be a topological ordered ring. If $\operatorname{int}([-1, 1]) = (-1, 1)$, $\operatorname{cl}((-1, 1)) = [-1, 1]$ and $(-1, 1)(-1, 1) = (-1, 1)$, then $[-1, 1]$ is a closed unit neighborhood of 0 and $(-1, 1)$ is an open unit neighborhood of 0.*

Proof. By assumption, $[-1, 1]$ is closed, therefore it is a unit ball since it is trivially additively symmetric, multiplicatively idempotent and contains 1. Also, by assumption, $[-1, 1]$ is regular closed and its interior is multiplicatively idempotent. By [7, Proposition 1], $[-1, 1]$ is a closed unit neighborhood of 0. □

The following technical lemma is just a direct consequence of the basic properties satisfied by unital ring orderings.

Lemma 3.1. *Let R be a unital ordered ring. Then $(-1, 1)(-1, 1) \subseteq (-1, 1)$.*

Proof. it only suffices to check that $(0, 1)(0, 1) \subseteq [0, 1)$. Fix arbitrary elements $r, s \in (0, 1)$. We definition of ring ordering, $rs \geq 0$. Finally, $r < 1$ and $s > 0$ meaning that $rs \leq s < 1$. □

Next technical lemma allows a sufficient condition to assure that closed intervals are closed in the order topology. Recall that in a poset X, the set of incomparable elements to a certain $x \in X$ is denoted by θ_x.

Lemma 3.2. *Let X be a topological poset. Let $x \in X$. Then:*

(1) If θ_x is cofinal in itself, then $\uparrow x$ is closed.
(2) If θ_x is coinitial in itself, then $\downarrow x$ is closed.

Proof. Only the first item will be proved. Notice that $X \setminus \uparrow x = \downarrow^{\times} x \cup \theta_x$. We will show that $\downarrow^{\times} x \cup \theta_x$ is open by showing that it is a neighborhood of each of its points. Indeed, fix an arbitrary $y \in \downarrow^{\times} x \cup \theta_x$. If $y \in \downarrow^{\times} x$, then $y \in \downarrow^{\times} x \subseteq \downarrow^{\times} x \cup \theta_x$, so $\downarrow^{\times} x \cup \theta_x$ is a neighborhood of y by definition of order topology. Next, assume that $y \in \theta_x$. By hypothesis, θ_x is cofinal in itself, meaning that there exists $z \in \theta_x$ such that $y < z$. Let us prove that $y \in \downarrow^\times z \subseteq \downarrow^\times x \cup \theta_x$. Take any $w \in \downarrow^\times z$. If w is not comparable to x, then $w \in \theta_x$. If w is comparable to x, then we have two options. One is that $x \leq w$, which implies that $x \leq w < z$ contradicting the fact z is not comparable to x. This only leaves the other option, that is, $w < x$, meaning that $w \in \downarrow^\times x$. As a consequence, $y \in \downarrow^\times z \subseteq \downarrow^\times x \cup \theta_x$. This shows that $X \setminus \uparrow x = \downarrow^\times x \cup \theta_x$ is open because it is a neighborhood of each of its points. $\square$

A direct consequence of Lemma 3.2 is the fact that if all the sets of incomparable elements are coinitial and cofinal in themselves, then all closed intervals are closed in the order topology. On the other hand, recall that a poset Y is said to be downward directed provided that for every $y_1, y_2 \in Y$ there exists $y_3 \in Y$ with $y_3 \leq y_2$ and $y_3 \leq y_1$. In a similar way, upward directed is defined.

Lemma 3.3. *Let X be a topological poset. If $x_0 \in X$ satisfies that $\uparrow_\times x_0 \neq \varnothing$ is downward directed and $\downarrow^\times x_0 \neq \varnothing$ is upward directed, then $\{(a, b) : a < x_0 < b\}$ is a base of neighborhoods of x_0 for the order topology.*

Proof. Let $W \subseteq X$ be an x_0-neighborhood for the order topology. There can be found $a_1, \ldots, a_n, b_1, \ldots, b_m \in X$ satisfying that $x_0 \in \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_n \cap \downarrow^\times b_1 \cap \cdots \cap \downarrow^\times b_m \subseteq W$. Observe that $a_i < x_0$ and $b_j > x_0$ for all $i \in \{1, \ldots, n\}$ and all $j \in \{1, \ldots, m\}$. Since $\downarrow^\times x_0$ is upward directed, there exists $a_0 \in \downarrow^\times x_0$ such that $a_0 \geq a_i$ for all $i \in \{1, \ldots, n\}$. Similarly, since $\uparrow_\times x_0$ is downward directed, there exists $b_0 \in \uparrow_\times x_0$ such that $b_0 \leq b_j$ for all $j \in \{1, \ldots, m\}$. Finally, notice that $x_0 \in (a_0, b_0) \subseteq \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_n \cap \downarrow^\times b_1 \cap \cdots \cap \downarrow^\times b_m \subseteq W$. $\square$

From now on, we will only work with topological posets X for which $\uparrow_\times x_0$ is not empty and downward directed and $\downarrow^\times x_0$ is not empty and upward directed, for every $x_0 \in X$.

Remark 3.1. *Let R an unital ordered ring. On the one hand, for every $r \in R$, $\uparrow_\times r$ and $\downarrow^\times r$ are not empty. Indeed, note that $-1 < 0 < 1$ meaning that $-1 + r < r < 1 + r$, hence $\uparrow_\times r \neq \varnothing$ and $\downarrow^\times r \neq \varnothing$. On the other hand, it is clear that $\uparrow_\times 0$ is upward directed and $\downarrow^\times 0$ is downward directed. As a consequence, for every $r \in R^+$, $\uparrow_\times r$ is upward directed, and for every $r \in R^-$, $\downarrow^\times r$ is downward directed.*

In order to apply Lemma 3.3, we will be interested in unital ring orderings for which $\uparrow_\times r$ is downward directed and $\downarrow^\times r$ is upward directed for all $r \in R$. Recall that a poset X is said to be hole free provided that for every $x, y \in X$ with $x < y$, there exists $z \in X$ with $x < z < y$.

Lemma 3.4. *Let X be a topological poset. Let $a, b \in X$ with $a < b$. If $\uparrow_\times b \neq \varnothing$ and is downward directed, and $\uparrow b$ is hole free, then $b \notin \mathrm{int}([a, b])$. Similarly, if $\downarrow^\times a \neq \varnothing$ and is upward directed, and $\downarrow a$ is hole free, then $a \notin \mathrm{int}([a, b])$.*

Proof. We will only show that if $\uparrow_\times b \neq \varnothing$ and is downward directed, and $\uparrow b$ is hole free, then $b \notin \mathrm{int}([a, b])$. Indeed, assume on the contrary that $b \in \mathrm{int}([a, b])$. There exists $W \subseteq X$ a b-neighborhood for the order topology contained in $[a, b]$. There are three possibilities:

- There are $a_1, \ldots, a_n \in X$ satisfying that $b \in \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_n \subseteq W$. Then $\uparrow_\times b \subseteq \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_n \subseteq W$. By hypothesis, $\uparrow_\times b \neq \varnothing$, so if we take $b_0 \in \uparrow_\times b$, then $b_0 > b$ so we reach the contradiction that $b_0 \in \uparrow_\times b \subseteq \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_n \subseteq W \subseteq [a, b]$.
- There are $a_1, \ldots, a_n, b_1, \ldots, b_m \in X$ satisfying that $b \in \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_n \cap \downarrow^\times b_1 \cap \cdots \cap \downarrow^\times b_m \subseteq W$. By hypothesis, $\uparrow_\times b \neq \varnothing$ and is downward directed, thus there exists $b_0 \in \uparrow_\times b$ such that $b_0 \leq b_j$ for each $j = 1, \ldots, m$. Again, by hypothesis,

$\uparrow b$ is hole free, meaning that there exists $c \in \uparrow b$ with $b < c < b_0$. Then we reach the contradiction that $c \in \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_n \cap \downarrow^\times b_1 \cap \cdots \cap \downarrow^\times b_m \subseteq W \subseteq [a, b]$.

- There are $b_1, \ldots, b_m \in X$ satisfying that $b \in \downarrow^\times b_1 \cap \cdots \cap \downarrow^\times b_m \subseteq W$. It follows the same proof as right above.

□

Next lemma is technically similar to Lemma 3.4.

Lemma 3.5. *Let X be a topological poset. Let $a, b \in X$ with $a < b$. If $\downarrow^\times b$ is upward directed and $\downarrow b$ is hole free, then $b \in \mathrm{cl}((a, b))$. Similarly, if $\uparrow_\times a$ is downward directed and $\uparrow a$ is hole free, then $a \in \mathrm{cl}((a, b))$.*

Proof. We will only show that if $\downarrow^\times b$ is upward directed and $\downarrow b$ is hole free, then $b \in \mathrm{cl}((a, b))$. Take any b-neighborhood $W \subseteq X$ for the order topology. There are three possibilities:

- There are $a_1, \ldots, a_n \in X$ satisfying that $b \in \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_n \subseteq W$. By hypothesis, $\downarrow^\times b$ is upward directed, meaning that there exists $a_0 \in \downarrow^\times b$ such that $a \leq a_0$ and $a_i \leq a_0$ for each $i = 1, \ldots, n$. Again, by hypothesis, $\downarrow b$ is hole free, therefore there exists $c \in \downarrow b$ with $a_0 < c < b$. Finally, $c \in (a, b) \cap \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_n$.
- There are $a_1, \ldots, a_n, b_1, \ldots, b_m \in X$ satisfying that $b \in \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_n \cap \downarrow^\times b_1 \cap \cdots \cap \downarrow^\times b_m \subseteq W$. It follows the same proof as right above.
- There are $b_1, \ldots, b_m \in X$ satisfying that $b \in \downarrow^\times b_1 \cap \cdots \cap \downarrow^\times b_m \subseteq W$. Since $\downarrow b$ is hole free, we can find $c \in \downarrow b$ with $a < c < b$. Then $c \in (a, b) \cap \downarrow^\times b_1 \cap \cdots \cap \downarrow^\times b_m$.

□

The following technical lemma is the last tool that we need to prove our main theorem and it is an slight improvement of [7, Lemma 1]. Recall that $\mathcal{U}(R)$ stands for the multiplicative group of invertibles of a ring R.

Lemma 3.6. *Let R be a topological ring. Let U be an open subset of R. Then:*

(1) *If $\mathcal{U}(R)$ is open and $1 \in \mathrm{cl}(U)$, then $1 \in \mathrm{cl}(U \cap \mathcal{U}(R))$.*
(2) *If multiplicative inversion is continuous and $1 \in \mathrm{cl}(U \cap \mathcal{U}(R))$, then $U \subseteq UU$.*

Proof.

(1) Let V be any open subset containing 1. Then $V \cap \mathcal{U}(R)$ is an open neighborhood of 1, therefore by hypothesis, $(V \cap \mathcal{U}(R)) \cap U \neq \varnothing$, meaning that $V \cap (\mathcal{U}(R) \cap U) \neq \varnothing$. As a consequence, $1 \in \mathrm{cl}(U \cap \mathcal{U}(R))$.
(2) Fix an arbitrary $u \in U$. Let $(u_i)_{i \in I} \subseteq U \cap \mathcal{U}(R)$ be a net converging to 1. By the continuity of the inversion, we deduce that $\left(u_i^{-1}\right)_{i \in I}$ also converges to 1, therefore $\left(u_i^{-1} u\right)_{i \in I}$ converges to u, which means that we can find $j \in I$ with $u_j^{-1} u \in U$. Finally, $u = u_j \left(u_j^{-1} u\right) \in UU$.

□

We are finally in the right position to answer Question 1 in the affirmative.

Theorem 3.1. *Let R be a topological ordered ring with continuous inversion. If $\mathcal{U}(R)$ is open, $R^+ \setminus \{0\}$ is downward directed, and R^+ is closed and hole free, then $[-1, 1]$ is a closed unit neighborhood of 0 and $(-1, 1)$ is an open unit neighborhood of 0.*

Proof. In accordance with Proposition 3.1, it only suffices to show that $\mathrm{int}([-1, 1]) = (-1, 1)$, $\mathrm{cl}((-1, 1)) = [-1, 1]$ and $(-1, 1)(-1, 1) = (-1, 1)$. By hypothesis, R^+ is closed, meaning that the sets $R^- = -R^+$, $\downarrow 1 = 1 + R^-$ and $\uparrow -1 = -1 + R^+$ are closed as well since the order topology is a ring topology. In particular, $[-1, 1] = \uparrow -1 \cap \downarrow 1$ is closed.

Also, by hypothesis, $R^+ \setminus \{0\}$ is downward directed, meaning that $\uparrow_\times 1 = 1 + R^+ \setminus \{0\}$ and $\uparrow_\times -1 = -1 + R^+ \setminus \{0\}$ are downward directed by translation. Also, $R^+ \setminus \{0\}$ being downward directed implies that $R^- \setminus \{0\}$ is upward directed by additive symmetricity, so again by translation, $\downarrow^\times 1 = 1 + R^- \setminus \{0\}$ and $\downarrow^\times -1 = R^- \setminus \{0\}$ are upward directed. Since R^+ is hole free, then so is R^- by additive symmetricity, hence so are $\uparrow 1$, $\uparrow -1$, $\downarrow 1$ and $\downarrow -1$ as well by translation. In virtue of Lemmas 3.4 and 3.5 together with the fact that $[-1,1]$ is closed, we obtain that $\mathrm{int}([-1,1]) = (-1,1)$ and $\mathrm{cl}((-1,1)) = [-1,1]$. By bearing in mind Lemma 3.1, it only remains to show that $(-1,1) \subseteq (-1,1)(-1,1)$. But this is just a direct application of Lemma 3.6. $\square$

As a direct consequence of Theorem 3.1, we obtain [3, Theorem 3.7].

Corollary 3.1. *Let R be a topological totally ordered division ring with no holes. Then $[-1,1]$ is a closed unit neighborhood of 0 and $(-1,1)$ is an open unit neighborhood of 0.*

Proof. By definition, R is a topological ordered ring (whose order is total) as well as a topological division ring (hence multiplicative inversion is continuous by default). In totally ordered sets, the order topology is always Hausdorff. Therefore, R is Hausdorff, so $\{0\}$ is closed and $\mathcal{U}(R) = R \setminus \{0\}$ is open. Next, $R^+ \setminus \{0\}$ is trivially downward directed because the order is total. It only remains to show that R^+ is closed and hole free. It is hole free because the whole of R is free of holes. Finally, $R^+ = R \setminus \downarrow^\times 0$ due to the fact that the order is total, so R^+ is closed since its complementary is open in the order topology. $\square$

4. Conclusions

An affirmative answer to Question 1 has been provided in Theorem 3.1 without relying on total orders nor on division rings. Nevertheless, the proof of [3, Theorem 3.7] relies on [6, Theorem 2.6], which states that a subset B of a topological division ring R is a closed unit neighborhood of 0 if and only if B is an additively symmetric and multiplicatively idempotent regular closed neighborhood of 0 containing 1. The proof of [3, Theorem 3.7] could have been simplified a little more with the following improvement of [6, Theorem 2.6].

Theorem 4.2. *Let R be a topological division ring. A subset B of R is a closed unit neighborhood of 0 if and only if B is an additively symmetric and multiplicatively idempotent closed neighborhood of 0 such that $1 \in \mathrm{cl}(\mathrm{int}(B))$.*

Proof. All we need to show is that B is regular closed and call off [6, Theorem 2.6]. Indeed, let us prove that $\mathrm{cl}(\mathrm{int}(B)) = B$. Fix an arbitrary $b \in B$ and an arbitrary $U \subseteq R$ neighborhood of 0. The ring topology of R allows the existence of a 0-neighborhood $V \subseteq R$ such that $Vb \subseteq U$. By hypothesis, there exists $v \in V$ such that $1 + v \in \mathrm{int}(B)$. On the one hand, $b + vb \in b + Vb \subseteq b + U$. On the other hand, $b + vb = (1+v)b \in \mathrm{int}(B)B$. At this stage, note that $\mathrm{int}(B)B$ is open and contained in $BB = B$, therefore $\mathrm{int}(B)B \subseteq \mathrm{int}(B)$. The reason why $\mathrm{int}(B)B$ is open is because $\mathrm{int}(B)B = \bigcup_{c \in B\setminus\{0\}} \mathrm{int}(B)c$ is a union of open sets since c is invertible. Finally, $b + vb \in (b+U) \cap \mathrm{int}(B)$, meaning that $b \in \mathrm{cl}(\mathrm{int}(B))$.

$\square$

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