CARPATHIAN J. MATH.
Volume **41** (2025), No. 4,
Pages 951-957

Online version at https://www.carpathian.cunbm.utcluj.ro/
ISSN 1584-2851 (Print edition); ISSN 1843-4401 (Electronic)
DOI: https://doi.org/

an extragradient method with conjugate gradient-type direction for solving variational inequalities with application

IBRAHIM ARZUKA[1,2], PARIN CHAIPUNYA[1], AND POOM KUMAM[1]

ABSTRACT. In this paper, we establish a fact that guarantees the strong convergence of any sequence of images of a metric projection onto a closed convex set C. We further incorporated the extragradient technique with a conjugate gradient-type direction to solve monotone variational inequality problems in Hilbert spaces. Unlike existing conjugate gradient-type methods, the proposed method does not require boundedness of the feasible set to converge to a solution of the variational inequality problem. In this regard, we establish weak convergence for the proposed method under appropriate conditions and conduct numerical experiments to showcase the computational efficacy and robustness of the method. Finally, we illustrate a potential application of the method in solving international migration equilibrium problem.

1. INTRODUCTION

In this paper, we introduce an efficient numerical method for solving variational inequality problems (VIP) of the form:

$$\text{find } x^* \in C \text{ such that } \langle W(x^*), y - x^* \rangle \geq 0, \ \forall y \in C, \tag{1.1}$$

where, C represents a closed and convex subset of a real Hilbert space H, and $W : C \to H$ is a nonlinear operator. VIP provide a powerful framework for addressing problems across social sciences, natural sciences, engineering, and other disciplines, as outlined in [1, 2, 6, 9, 10, 13, 21, 26, 28, 29]. The wide range of applications of the VIP has motivated several researchers over the years to propose iterative methods for solving such problems. Among these, the most famous is the extragradient method (EGM), proposed by Korpelovich [14],

$$\begin{cases} x_0 \in C, \\ y_j = P_C(x_j - \lambda W(x_j)), \\ x_{j+1} = P_C(x_j - \lambda W(y_j)), \end{cases} \tag{1.2}$$

where $\lambda \in \left(0, \frac{1}{L}\right)$. The sequence $\{x_j\}$ generated by EGM converges weakly to the solution of VIP for a certain class of mappings. This method requires computing both the operator W and the metric projection P_C twice per iteration, which can be computationally expensive in some cases. As a result, several authors have devoted considerable attention in addressing these challenges.

Popov [18] proposed the following modification

$$\begin{cases} x_0 \in C, \\ y_j = P_C(x_j - \lambda W(x_j)), \\ x_{j+1} = P_C(y_j - \lambda W(x_j)), \end{cases} \tag{1.3}$$

Received: 04.11.2024. In revised form: 26.05.2025. Accepted: 02.07.2025
2020 *Mathematics Subject Classification.* 65F10, 47J25, 65K15.
Key words and phrases. *Conjugate Gradient; Monotone Operators; Variational Inequality* .
Corresponding author: parin.cha@kmutt.ac.th

where $\lambda \in (0, \frac{1}{3L})$. This strategy requires the computation of the operator W only once in every iteration, but the number of projections remains the same.

Tseng [22] considered the following modification

$$\begin{cases} x_0 \in C, \\ y_j = P_C(x_j - \lambda W(x_j)), \\ x_{j+1} = y_j - \lambda(W(x_j) - W(y_j)), \end{cases} \tag{1.4}$$

where $\lambda \in (0, \frac{1}{L})$. This reduces the computation of the projection to just one but requires the computation of W twice in every iteration. However, Censor [3, 4] proposed a half-space projection technique to reduce the computational burden of the EGM. Moreover, the EGM method has been investigated and generalized in various ways, refer to [12, 15, 16, 23–25].

On the other hand, Iiduka and Uchida [11] proposed the conjugate gradient-like method to solve VIP as

$$\begin{cases} y_j = T(x_j + \lambda_j \zeta_j), \\ x_{j+1} = \tau x_j + (1-\tau) y_j, \\ \zeta_{j+1} = \nabla U(x_{j+1}) + \beta_{j+1}\zeta_j, \end{cases} \tag{1.5}$$

where $\lambda_j \in (0, 1)$, $\beta_{j+1} \in \mathbb{R}$, $\tau \in [0, 1)$, T is nonexpansive operator and U is the utility function. The sequence $\{x_j\}$ converges to the solution of the problem under the assumption that the feasible set is compact. Similarly, Iiduka [8] proposed another conjugate gradient-like method as

$$\begin{cases} x_{j+1} = T(x_j + \tau_j \zeta_j), \\ \zeta_{j+1} = \nabla f(x_{j+1}) + \beta_{j+1}\zeta_j - \gamma_{j+1} z_j, \end{cases} \tag{1.6}$$

where $r_j \in [0, 1)$, $\beta_{j+1}, \gamma_{j+1} \in \mathbb{R}_+$, and $z_j \in H$. The sequence $\{x_j\}$ converges strongly to the solution of the VIP under the assumption that the operator is strongly monotone and bounded.

Remark 1.1. *Although, the performance of algorithm* (1.5) *and* (1.6) *with a conjugate gradient type direction is encouraging, when compared with its variant. Their convergence result holds under the condition that the feasible set is compact and the operator is alpha-strongly monotone. These conditions appear to be restrictive and it will be of great interest to dispense them.*

Motivated by (1.5), (1.6) and the work in [14], we construct an extragradient method with a conjugate gradient-type direction for monotone variational inequality problems in infinite-dimensional real Hilbert spaces. A key feature of the proposed method is that it converges without requiring any of the conditions mentioned in Remark 1.1. This approach is the first of its kind to consider an extragradient method with a conjugate gradient-type direction.

The structure of the paper is outlined as follows. Section 2 compiles essential definitions and lemmas required for subsequent discussions. In Section 3, we introduce the convergence theorem associated with the sequence of images of the metric projection, the algorithm and examine its convergence. Numerical examples are provided in Section 4 to demonstrate the proposed algorithm's efficacy compared to related ones. Section 5 deals with the application in international human migration. The paper concludes with a concise summary in Section 6.

2. Preliminaries

In this section, we revisit key definitions and lemmas essential for this study. Throughout this manuscript, unless stated otherwise, the solution set of the variational inequality problem associated with the operator W over the set C is denoted by VIP(W, C).

Definition 2.1. *Let $W : H \to H$ be a mapping. Then*

(i) *W is α-strong monotone operator if there exists $\alpha > 0$ such that*

$$\langle x - y \ , \ W(x) - W(y)\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in H.$$

(ii) *W is monotone if $\forall x, y \in H$,*

$$\langle x - y \ , \ W(x) - W(y)\rangle \geq 0.$$

(iii) *W is Lipschitz continuous if there exists $L > 0$, such that*

$$\|W(x) - W(y)\| \leq L\|x - y\|, \quad \forall x, y \in H.$$

Definition 2.2. *Consider a nonempty, closed, convex subset C of a Hilbert space H. The mapping $P_C : H \to C$, which assigns each element of H to its unique nearest element in C, is referred to as a metric projection onto C. Various properties of this mapping include:*

(i) $\langle x - P_C(x) \ , \ y - P_C(x)\rangle \leq 0$, $\forall\, y \in C$ *and* $x \in H$.
(ii) $\|P_C(x) - P_C(y)\| \leq \|x - y\|$, $\forall x, y \in H$.
(iii) $\|P_C(x) - y\| \leq \|x - y\|^2 - \|x - P_C(x)\|$, $\forall x, y \in H$.

The following identity would be used in establishing the convergence of the proposed method.

Lemma 2.1. *Let $x, y \in H$ and $\alpha \in \mathbb{R}$. Then*

(i) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x \ , \ y\rangle$.
(ii) $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x \ , \ y\rangle$.
(iii) $\|(1 - \alpha)x + \alpha y\|^2 = (1 - \alpha)\|x\|^2 + \alpha\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$.

Lemma 2.2. *[20, Lemma 1] Let $\{\sigma_n\}$ and $\{b_n\}$ be nonnegative sequence of a real numbers such that*

$$\sigma_{n+1} \leq \sigma_n + b_n, \quad \forall n \geq 1.$$

If $\sum_n b_n$ converges, then the limit of the sequence $\{\sigma_n\}$ exists.

Lemma 2.3. *[17, Lemma 1]. If a sequence $\{x_n\} \subset H$ converges weakly to x, then for any $y \in H\backslash\{x\}$,*

$$\liminf_{n\to\infty} \|x_n - y\| > \liminf_{n\to\infty} \|x_n - x\|.$$

A mapping $M : H \to 2^H$ is monotone if for all $x, y \in H$, $u_1 \in M(x)$ and $u_2 \in M(y)$, we have

$$\langle u_1 - u_2, x - y\rangle \geq 0.$$

M is maximal monotone if the graph $G(M)$ of M is not properly contained in the graph of any other monotone mapping. Moreover, a monotone map M is maximal monotone if and only if for $(u_1, x) \in H \times H$, $\langle u_1 - u_2, x - y\rangle \geq 0$ for any $(u_2, y) \in G(M)$ implies $u_1 \in M(x)$. Let $W : C \to H$ be a monotone operator and $N_C(y^*)$ be a normal cone of C at $y^* \in C$ that is

$$N_C(y^*) = \{\mu \in H : \langle p - y^* \ , \ \mu\rangle \leq 0 \quad \forall p \in C\}.$$

Consider M defined by

$$M(y^*) = \begin{cases} W(y^*) + N_C(y^*), & y^* \in C, \\ \\ \emptyset, & y^* \notin C. \end{cases} \tag{2.7}$$

By (2.7), it follows that M is a maximal monotone operator and the following holds:

$$0 \in W(y^*) + N_C(y^*),$$

if and only if

$$y^* \in \mathrm{VIP}(W, C).$$

3. Convergence

The most effective approach widely used in forcing elements into a feasible region is through metric projection, especially when the region inherits closedness and convexity structure. We first state and prove the following theorem which provide a condition guaranteeing the convergence of a sequence of images of a metric projection onto the feasible region.

Lemma 3.4. *Suppose H is a real Hilbert space and C is a nonempty closed convex subset of H. Let $\{x_j\}$ be a sequence in H such that*

$$\|x_{j+1} - w\|^2 \leq \|x_j - w\|^2 + \alpha_j, \quad \forall w \in C, \tag{3.8}$$

where $\sum_j \alpha_j < \infty$. Then, $\{P_C x_j\}$ converge strongly to an element of C.

Proof. Let $w_j = P_C x_j$ and suppose that $m > j$. Then,

$$\begin{aligned}
\|w_m - w_j\|^2 &= 2\|x_m - w_m\|^2 + 2\|x_m - w_j\|^2 - 4\left\|x_m - \frac{w_m + w_j}{2}\right\|^2 \\
&\leq 2\|x_m - w_m\|^2 + 2\|x_m - w_j\|^2 - 4\|x_m - w_m\|^2 \\
&\leq 2\|x_m - w_j\|^2 - 2\|x_m - w_m\|^2 \\
&\leq 2\|x_{m-1} - w_j\|^2 - 2\|x_m - w_m\|^2 + 2\alpha_{m-1} \\
&\leq 2\|x_{m-2} - w_j\|^2 - 2\|x_m - w_m\|^2 + 2\alpha_{m-2} + 2\alpha_{m-1} \\
&\;\;\vdots \\
&\leq 2\|x_j - w_j\|^2 - 2\|x_m - w_m\|^2 + 2\sum_{k=j}^{m-1} \alpha_k \\
&\leq 2\|x_j - w_j\|^2 - 2\|x_m - w_m\|^2 + 2\sum_{k=j}^{+\infty} \alpha_k,
\end{aligned} \tag{3.9}$$

which implies

$$\begin{aligned}
2\|x_m - w_m\|^2 &\leq 2\|x_j - w_j\|^2 - \|w_m - w_j\|^2 + 2\sum_{k=j}^{+\infty} \alpha_k \\
&\leq 2\|x_j - w_j\|^2 + 2\sum_{k=j}^{+\infty} \alpha_k.
\end{aligned} \tag{3.10}$$

Thus,

$$\|x_m - w_m\|^2 \leq \|x_j - w_j\|^2 + \sum_{k=j}^{+\infty} \alpha_k. \tag{3.11}$$

By taking $\limsup$ of both side as $m \to \infty$ we have

$$\limsup \|x_m - w_m\|^2 \leq \|x_j - w_j\|^2 + \sum_{k=j}^{+\infty} \alpha_k. \tag{3.12}$$

Take $\liminf$ of both side as $j \to \infty$, we have

$$\limsup \|x_m - w_m\|^2 \leq \liminf \|x_j - w_j\|^2. \tag{3.13}$$

Thus, $\lim_{j\to\infty} \|x_j - w_j\|^2$ exist. Now, as $j, m \to \infty$ in (3.9), we have that $\{w_j\}$ is a Cauchy sequence. Hence $\{w_j\}$ converges strongly to some $\tau \in C$. □

Corollary 3.1. *Observe that if $\alpha_j = 0$ for all $j \geq 0$, then Lemma 3.4 reduces to Lemma 3.2 in [19]. Moreover, the result of Lemma 3.4 is essential for the convergence analysis of the proposed method.*

To establish the convergence, we make the following assumptions.

Assumption:

$(A_1.)$ $C \subset H$ is a nonempty, closed and convex set.
$(A_2.)$ $W : C \to H$ is a monotone operator and Lipschitz continuous.
$(A_3.)$ The solution set $\mathrm{VIP}(W, C)$ is nonempty.

Next, we present our propose algorithm as:

Algorithm 1: An extragradient method with conjugate gradient-type direction.

Initialization: Given $x_0 \in H,\ \Gamma_j \in [0, 1),\ \lambda_j \in (0, 1)$, and $\psi > 0$. Set $d_0 = -W(x_0)$,

Step 1: Compute:

$$\Theta_j = \frac{\Gamma_j}{\max\{\|d_j\|, \psi\}}$$

and

$$d_{j+1} = -W(x_j) + \Theta_j d_j. \tag{3.14}$$

Step 2: Determine y_j by

$$y_j = P_C(x_j + \lambda_j d_{j+1}).$$

Step 2: Update $x_{j+1} \in H$ as

$$x_{j+1} = P_C(x_j - \lambda_j W(y_j)).$$

Set $j = j + 1$ and go back to step 1.

Remark 3.2. *If $\Gamma_j = 0$ for all $j \geq 0$ in* (3.14), *the proposed Algorithm 1 reduces to the classical extragradient method [14].*

Next, we state our main theorem as follows:

Theorem 3.1. *Suppose H is a real Hilbert space, and $\{\Gamma_j\}$ and $\{\lambda_j\}$ are sequences such that the following conditions are satisfied:*

(i) $\sum_{j=1}^{\infty} \Gamma_j < \infty$.
(ii) $\liminf_{j\to\infty}(1 - 3L^2\lambda_j^2) > 0$.

Then, the sequence $\{x_j\}$ generated by Algorithm 1 converges weakly to $\mathrm{VIP}(W, C)$, *such that $\pi = \lim_{j\to\infty} P_C(x_j)$.*

Proof. Let us begin with the following estimate base on Algorithm 1:

$$\begin{aligned}\|x_{j+1}-y_j\|^2 &= \|P_C(x_j-\lambda_j W(y_j)) - P_C(x_j+\lambda_j d_{j+1})\|^2\\ &\le \|x_j-\lambda_j W(y_j) - x_j - \lambda_j d_{j+1}\|^2\\ &= \|-\lambda_j(W(y_j)-W(x_j)) - \lambda_j\Theta_j d_j\|^2\\ &= \lambda_j^2\|W(y_j)-W(x_j)+\Theta_j d_j\|^2\\ &= \lambda_j^2(\|W(y_j)-W(x_j)\|^2 + \Theta_j^2\|d_j\|^2 + 2\Theta_j\langle W(y_j)-W(x_j), d_j\rangle)\\ &\le 2\lambda_j^2(\|W(y_j)-W(x_j)\|^2+\Theta_j^2\|d_j\|^2)\\ &\le 2\lambda_j^2(L^2\|y_j-x_j\|^2+\Theta_j^2\|d_j\|^2)\\ &\le 2\lambda_j^2 L^2\|y_j-x_j\|^2 + 2\lambda_j^2\Theta_j^2\|d_j\|^2.\end{aligned} \tag{3.15}$$

Now, let $\tau \in C$. It follows that

$$\begin{aligned}\|x_{j+1}-\tau\|^2 &\le \|x_j-\lambda_j W(y_j)-\tau\|^2 - \|x_j-\lambda_j W(y_j) - x_{j+1}\|^2\\ &= \|x_j-\tau\|^2 + \lambda_j^2\|W(y_j)\|^2 - 2\lambda_j\langle x_j-\tau\ ,\ W(y_j)\rangle\\ &- (\|x_j-x_{j+1}\|^2 + \lambda_j^2\|W(y_j)\|^2 - 2\lambda_j\langle x_j-x_{j+1}\ ,\ W(y_j)\rangle)\\ &= \|x_j-\tau\|^2 + \lambda_j^2\|W(y_j)\|^2 - 2\lambda_j\langle x_j-\tau\ ,\ W(y_j)\rangle\\ &- \|x_j-x_{j+1}\|^2 - \lambda_j^2\|W(y_j)\|^2 + 2\lambda_j\langle x_j-x_{j+1}\ ,\ W(y_j)\rangle)\\ &= \|x_j-\tau\|^2 + 2\lambda_j\langle \tau-x_j\ ,\ W(y_j)\rangle - \|x_j-x_{j+1}\|^2\\ &+ 2\lambda_j\langle x_j-x_{j+1}\ ,\ W(y_j)\rangle)\\ &= \|x_j-\tau\|^2 - \|x_j-x_{j+1}\|^2 + 2\lambda_j\langle \tau-x_{j+1}\ ,\ W(y_j)\rangle).\end{aligned} \tag{3.16}$$

By monotonicity of W, we have

$$\begin{aligned}0 &\le \langle W(y_j)-W(\tau)\ ,\ y_j-\tau\rangle\\ &= \langle W(y_j)\ ,\ y_j-\tau\rangle - \langle W(\tau)\ ,\ y_j-\tau\rangle\\ &= \langle W(y_j)\ ,\ y_j-\tau\rangle.\end{aligned} \tag{3.17}$$

Thus,

$$\langle W(y_j)\ ,\ \tau-y_j\rangle \le 0.$$

Consequently, we have

$$\begin{aligned}\langle W(y_j)\ ,\ \tau-x_{j+1}\rangle &= \langle W(y_j)\ ,\ \tau-y_j\rangle + \langle W(y_j)\ ,\ y_j-x_{j+1}\rangle\\ &\le \langle W(y_j)\ ,\ y_j-x_{j+1}\rangle.\end{aligned} \tag{3.18}$$

Thus, (3.16) becomes

$$\begin{aligned}\|x_{j+1}-\tau\|^2 &\le \|x_j-\tau\|^2 - \|x_j-x_{j+1}\|^2 + 2\lambda_j\langle W(y_j)\ ,\ y_j-x_{j+1}\rangle\\ &\le \|x_j-\tau\|^2 - \|x_j-y_j\|^2 - \|y_j-x_{j+1}\|^2\\ &- 2\langle x_j-y_j\ ,\ y_j-x_{j+1}\rangle + 2\lambda_j\langle W(y_j)\ ,\ y_j-x_{j+1}\rangle\\ &\le \|x_j-\tau\|^2 - \|x_j-y_j\|^2 - \|y_j-x_{j+1}\|^2\\ &+ 2\langle x_j-y_j\ ,\ x_{j+1}-y_j\rangle - 2\lambda_j\langle W(y_j)\ ,\ x_{j+1}-y_j\rangle\\ &\le \|x_j-\tau\|^2 - \|x_j-y_j\|^2 - \|y_j-x_{j+1}\|^2\\ &+ 2\langle x_j-\lambda_j W(y_j)-y_j\ ,\ x_{j+1}-y_j\rangle.\end{aligned} \tag{3.19}$$

Hence,

$$\begin{aligned}\|x_{j+1}-\tau\|^2 \le \|x_j-\tau\|^2 - \|x_j-y_j\|^2 - \|y_j-x_{j+1}\|^2 \\ + 2\langle x_j-\lambda_j W(y_j)-y_j\ ,\ x_{j+1}-y_j\rangle.\end{aligned} \tag{3.20}$$

By Cauchy Schwarz, Lipschitz continuity of W and Definition 2.2(i), we have

$$\begin{aligned}\langle x_j-\lambda_j W(y_j)-y_j\ ,\ x_{j+1}-y_j\rangle &= \langle x_j-\lambda_j d_{j+1}+\lambda_j d_{j+1}-\lambda_j W(y_j)-y_j\ ,\ x_{j+1}-y_j\rangle \\ &= \langle x_j+\lambda_j d_{j+1}-y_j\ ,\ x_{j+1}-y_j\rangle \\ &+ \langle -\lambda_j d_{j+1}-\lambda_j W(y_j)\ ,\ x_{j+1}-y_j\rangle \\ &\le -\lambda_j\langle d_{j+1}+W(y_j)\ ,\ x_{j+1}-y_j\rangle \\ &\le -\lambda_j\langle W(y_j)-W(x_j)+\Theta_j d_j\ ,\ x_{j+1}-y_j\rangle \\ &\le \lambda_j\langle W(y_j)-W(x_j)\ ,\ x_{j+1}-y_j\rangle - \Theta_j\lambda_j\langle d_j\ ,\ x_{j+1}-y_j\rangle \\ &\le \lambda_j\|W(x_j)-W(y_j)\|\|x_{j+1}-y_j\| + \Theta_j\lambda_j\|d_j\|\|x_{j+1}-y_j\| \\ &\le L\lambda_j\|x_j-y_j\|\|x_{j+1}-y_j\| + \Theta_j\lambda_j\|d_j\|\|x_{j+1}-y_j\|.\end{aligned} \tag{3.21}$$

This implies that

$$\begin{aligned}2\langle x_j-\lambda_j W(y_j)-y_j\ ,\ x_{j+1}-y_j\rangle \le L^2\lambda_j^2\|x_j-y_j\|^2 + \|x_{j+1}-y_j\|^2 \\ + 2\Theta_j\lambda_j\|d_j\|\|x_{j+1}-y_j\|.\end{aligned} \tag{3.22}$$

By (3.15) and (3.22), we have the following

$$\begin{aligned}\|x_{j+1}-\tau\|^2 &\le \|x_j-\tau\|^2 - \|x_j-y_j\|^2 - \|y_j-x_{j+1}\|^2 \\ &+ L^2\lambda_j^2\|x_j-y_j\|^2 + \|x_{j+1}-y_j\|^2 + 2\Theta_j\lambda_j\|d_j\|\|x_{j+1}-y_j\| \\ &\le \|x_j-\tau\|^2 - (1-L^2\lambda_j^2)\|x_j-y_j\|^2 + 2\Theta_j\lambda_j\|d_j\|\|x_{j+1}-y_j\|, \\ &\le \|x_j-\tau\|^2 - (1-L^2\lambda_j^2)\|x_j-y_j\|^2 + \Theta_j^2\lambda_j^2\|d_j\|^2 + \|x_{j+1}-y_j\|^2, \\ &\le \|x_j-\tau\|^2 - (1-3L^2\lambda_j^2)\|x_j-y_j\|^2 + \Theta_j^2\lambda_j^2\|d_j\|^2 + 2\lambda_j^2\Theta_j^2\|d_j\|^2 \\ &\le \|x_j-\tau\|^2 - (1-3L^2\lambda_j^2)\|x_j-y_j\|^2 + 3\lambda_j^2\Theta_j^2\|d_j\|^2,\end{aligned} \tag{3.23}$$

where $\lambda_j \in \left(0, \dfrac{1}{\sqrt{3}L}\right)$. Let $\Pi_j = 1-3L^2\lambda_j^2 > 0$. By Lemma 2.2 and hypothesis (ii), we have that $\{\|x_j-\tau\|\}$ converges. Consequently, the sequence $\{x_j\}$ is bounded. It follows that there exists a subsequence $\{x_{j_i}\}$ that converges weakly to some point π. Since C is closed, we have $\pi \in C$. Moreover, by condition (ii) of Theorem 3.1, we have

$$\|x_j-y_j\|^2 \le \frac{1}{\Pi_j}((\|x_j-\tau\|^2 - \|x_{j+1}-\tau\|^2) + 4\lambda_j^2\Theta_j^2\|d_j\|^2) \to 0. \tag{3.24}$$

Now, we want to show that $\pi \in \mathrm{VIP}(W, C)$. Let

$$\Phi(d) := \begin{cases} W(d)+N_C(d), & d \in C, \\ \emptyset, & d \notin C.\end{cases} \tag{3.25}$$

Clearly, Φ is maximal monotone. Suppose $(d, r) \in G(\Phi)$, it follows that $r - W(d) \in N_C(d)$ and for $x_{j+1} \in C$, we have

$$\langle d-x_{j+1}\ ,\ r-W(d)\rangle \ge 0.$$

However, for

$$x_{j+1} = P_C(x_j-\lambda_j W(y_j)),$$

we have that

$$\langle x_j-\lambda_j W(y_j)-x_{j+1}\ ,\ x_{j+1}-d\rangle \ge 0.$$

This implies that

$$\langle d - x_{j+1}\ ,\ \frac{x_{j+1} - x_j}{\lambda_j} + W(y_j)\rangle \geq 0.$$

Now,

$$\begin{aligned}
\langle d - x_{j_i+1}\ ,\ r\rangle &\geq \langle d - x_{j_i+1}\ ,\ W(d)\rangle \\
&\geq \langle d - x_{j_i+1}\ ,\ W(d)\rangle - \langle d - x_{j_i+1}\ ,\ \frac{x_{j_i+1} - x_{j_i}}{\lambda_{j_i}} + W(y_{j_i})\rangle \\
&= \langle d - x_{j_i+1}\ ,\ -(\frac{x_{j_i+1} - x_{j_i}}{\lambda_{j_i}}) + W(d) - W(y_{j_i})\rangle \\
&= \langle d - x_{j_i+1}\ ,\ -(\frac{x_{j_i+1} - x_j}{\lambda_{j_i}}) - W(x_{j_i+1}) + W(x_{j_i+1}) + W(d) - W(y_{j_i})\rangle \\
&= -\langle d - x_{j_i+1}\ ,\ (\frac{x_{j_i+1} - x_{j_i}}{\lambda_{j_i}})\rangle - \langle d - x_{j_i+1}\ ,\ W(d) - W(x_{j_i+1})\rangle \\
&+ \langle d - x_{j+1}\ ,\ W(x_{j_i+1}) - W(y_{j_i})\rangle \\
&\geq -\langle d - x_{j_i+1}\ ,\ \frac{x_{j_i+1} - x_{j_i}}{\lambda_{j_i}}\rangle + \langle d - x_{j_i+1}\ ,\ W(x_{j_i+1}) - W(y_{j_i})\rangle.
\end{aligned} \tag{3.26}$$

Thus, $\langle d - \pi\ ,\ r\rangle \geq 0$ as $j \to \infty$. Since Φ is maximal monotone, it follows that $\pi \in \mathrm{VIP}(W, C)$.

Next, we show that the whole sequence converges weakly to π. Suppose $\{x_{j_i}\}$ and $\{x_{j_k}\}$ are subsequences of $\{x_j\}$ such that $x_{j_i} \rightharpoonup \pi$ and $x_{j_k} \rightharpoonup \pi_1$. Let us assume that $\pi \neq \pi_1$. By Lemma 2.3, we have

$$\begin{aligned}
\lim_{j\to\infty} \|x_j - \pi\| &= \liminf_{i\to\infty} \|x_{j_i} - \pi\| < \liminf_{i\to\infty} \|x_{j_i} - \pi_1\| \\
&= \lim_{j\to\infty} \|x_j - \pi_1\| = \liminf_{k\to\infty} \|x_{j_k} - \pi_1\| < \liminf_{k\to\infty} \|x_{j_k} - \pi\| \\
&= \lim_{j\to\infty} \|x_j - \pi\|,
\end{aligned}$$

which is a contradiction. Hence $\pi = \pi_1$. Thus, $\{x_j\}$ converges weakly to $\pi \in \mathrm{VIP}(W, C)$. Now, let

$$\eta_j = P_{VIP(W,C)}(x_j).$$

We want to show that

$$\pi = \lim_{j\to\infty} \eta_j.$$

Since $\pi \in \mathrm{VIP}(W, C)$, we have that

$$\langle \pi - \eta_j\ ,\ \eta_j - x_j\rangle \geq 0.$$

It follows from (3.23) and Lemma 3.4, that $\{\eta_j\}$ converges strongly to $\pi_1 \in \mathrm{VIP}(W, C)$. Thus,

$$\langle \pi - \pi_1\ ,\ \pi_1 - \pi\rangle \geq 0,$$

and therefore $\pi = \pi_1$. □

4. Numerical Example

In this section, our goal is to demonstrate the efficiency and robustness of the proposed strategy (MEG) in comparison with the projection and contraction method (PCM) [5] and Tseng's algorithm (Tseng) [22]. We used Matlab 2021a on a Dell Core i7 computer to conduct the numerical simulation. We set the control parameters as: $\Gamma_j = \frac{1}{(j+1)^5}$, $\lambda_j = \frac{0.001}{(10j+1)}$ and $\psi = 1$.

Example 4.1. *[7, Ex. 3] Consider the VIP with $W : \mathbb{R}^m \to \mathbb{R}^m$ defined by*

$$W(x) = \begin{bmatrix} x_1 + x_2 + sin(x_1) \\ -x_1 + x_2 + sin(x_2) \\ x_3 + x_4 + sin(x_3) \\ -x_3 + x_4 + sin(x_4) \\ \vdots \\ x_{2j-1} + x_{2j} + sin(x_{2j-1}) \\ -x_{2j-1} + x_{2j} + sin(x_2 j) \\ \vdots \\ x_{2m-1} + x_{2m} + sin(x_{2m-1}) \\ -x_{2m-1} + x_{2m} + sin(x_2 m) \end{bmatrix}, \tag{4.27}$$

where $x = (x_1, x_2, ...x_{2m-1}, x_{2m})$. It is evident from the paper that W is monotone and Lipschitz continuous. We set $m = 10$ and compared the performances of these methods; the numerical results are summarized in Table 1. It is evident from Table 1 that the proposed method MEG is effective,

TABLE 1. Numerical results for Example 4.1 with different stopping criteria.

	MEG		*PCM*		*Tseng*	
TOL	*Iter.*	*CPU*	*Iter.*	*CPU*	*Iter.*	*CPU*
10^{-6}	2	0.0030	15	0.0040	18	0.0035
10^{-7}	3	0.0037	22	0.0039	21	0.038
10^{-8}	5	0.0037	42	0.0041	24	0.0427

requiring fewer iterations and less computational time. Furthermore, the graphical representation of the results in Figure 1 clearly demonstrates the computational efficiency and robustness of the proposed approach.

Example 4.2. *Let us consider a simple bilevel optimization problem*

$$\min_{x \in X^*} Q(x),$$

where X^* is a set of solutions for another minimization problem with objective function, say K. This kind of problem and its well-posedness have been analyzed and considered by many researchers (see, for example, [31]). In this case, we consider the inner objective function given by

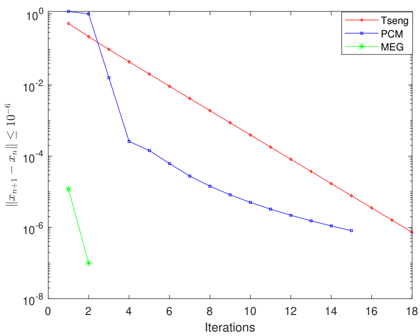
$$K(x) = \frac{1}{2}\|M(x) - c\|^2 + \delta_X(x), \tag{4.28}$$

where δ_X denotes an indicator function over the non-negative orthant. The outer objective function is given by

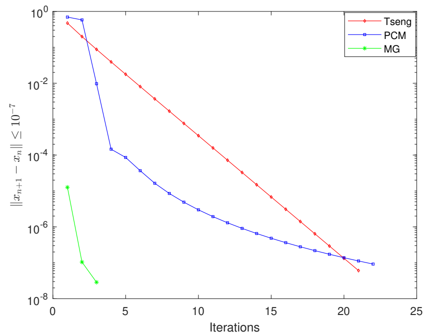
$$Q(x) = \frac{1}{2}x^T Dx. \tag{4.29}$$

Here, the variational operator $W := D$ satisfies the required assumptions with the feasible set $C := X^*$ provided that M is a positive definite and D is a semi-positive definite.

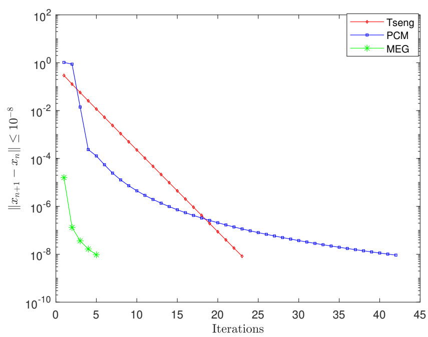
In the numerical experiment, we obtained data for M and c by discretizing the Baart and Phillip test problem as in [30]. We compared the performance of the proposed method with the other methods, the convergences plots are presented in Figures 2



(A) Tolerance $\|x_{j+1} - x_j\| \leq 10^{-6}$

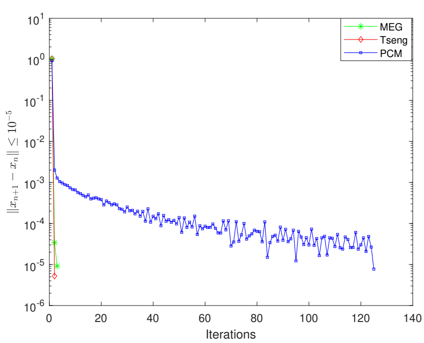


(B) Tolerance $\|x_{j+1} - x_j\| \leq 10^{-7}$

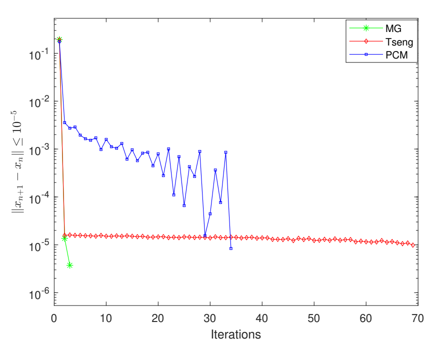


(C) Tolerance $\|x_{j+1} - x_j\| \leq 10^{-8}$

FIGURE 1. Convergence plots of Example 4.1.



(A) Philip test problem, $n = 180$



(B) Baart test problem, $n = 180$

FIGURE 2. Convergence plots for Example 4.2.

In this example, we set the tolerance to 10^{-5}. The results show that the proposed method and Tseng method are highly competitive in solving the Phillip test problem while concerning the Baart problem the proposed method exhibits faster convergence with the lowest error and fewer iterations, whereas the performance of PCM and the Tseng method alternates. In general, the proposed method is suitable for this type of problem.

5. Application

Example 5.3. *Let us consider international human migration as another example where we assume a closed economic system comprises n countries in focus, each identified by its origin i and destination j. Within this framework, there exist n classes of international migrants, with each class represented by k. It's crucial to note that a migrant class may encompass various types, such as highly skilled, skilled, or unskilled workers. Alternatively, it may represent individuals like refugees, asylum seekers, or irregular migrants. The conservation flow of the immigrants is given by*

$$q_i^k = \sum_l f_{il}^k, \tag{5.30}$$

and

$$\hat{q_i}^k = \sum_l f_{li}^k. \tag{5.31}$$

where $\hat{q_i}^k$ is the initial fixed population of class k in country i, q_i^k is the population of class k in country i, and f_{il}^k is the flow of immigrant of class k from the country i, to the country l. We define the feasible set $N = \{(q, f) \mid f \geq 0,\ (5.30)\ and\ (5.31)\ hold\}$.

The set of constraints encompasses a wide range of international migration regulations. To be specific, we examine regulations enforced by a solitary country denoted as $\bar{j}$. Let the set C^* include classes k and countries i, where $i = \bar{j}$, subject to an upper limit on the international migration flows into country $\bar{j}$, represented as U_j. The constraints can be expressed as follows

$$\sum_{i \in C^*} \sum_{k \in C^*} f_{ij}^k \leq U_j. \tag{5.32}$$

Now, we discuss some categories of regulations that (5.32) *represented.*

As an illustration, the set C^* may be defined to limit the migration flow from a particular country $\bar{i}$ and a specific migrant class $\bar{k}$, indicating that

$$f_{\bar{i}\bar{j}}^{\bar{k}} \leq U_{\bar{j}}. \tag{5.33}$$

A multi-class and flow pattern $(q^*, f^*) \in N$ is in equilibrium if for each class k=1,...,j and pair of location (i, j) $i = 1, ...j, j \neq i$

$$\mu_i^k(q^*) + c_{ij}(q^*) \begin{cases} = \mu_i^k - \lambda_j^{k*}, & if \quad f_{ij}^k > 0, \\ \\ \geq \mu_i^k - \lambda_j^{k*}, & if \quad f_{ij}^k = 0, \end{cases} \tag{5.34}$$

and

$$\lambda_i^{k*} \begin{cases} \geq 0, & if \quad \sum_{l \neq i} f_{li}^{k*} = q_i^k, \\ \\ = 0, & if \quad \sum_{l \neq i} f_{li}^{k*} < q_i^k. \end{cases} \tag{5.35}$$

The cost functions which correspond to the utility and cost of flow are given by

$$\mu_i^k(q) = -L_i^k (q_i^k)^2 + \sum_{lj} L_{ij}^{kl} q_j^l + b_i^k, \tag{5.36}$$

$$c_{ij}^k(f) = \tau_i^k (f_{ij}^{kl})^2 + \sum_{lj} \gamma_{ij}^{kl} f_{ij}^{kl} + h_{ij}^{kl}, \tag{5.37}$$

subject to the constraints (5.32), (5.31) *and* (5.30), *where* μ_i^k *denote the utility perceived by an immigrant of class* k *in country* i *and* c_{ij}^k *denote the cost of flow of immigrant for class* k, *which encompassing the economic, psychological, and social costs incurred when migrating from country* i *to country* j, $L_i^k, \tau_i^k, \gamma_{ij}^{kl}$ *and* L_{ij} *are real numbers. Now, the corresponding variational inequality formulation of the migration equilibrium is given by the following theorem:*

Theorem 5.2. *[27, Theorem 5.2] The population and migration flow pattern* $(q^*, f^*) \in N$ *satisfies the equilibrium conditions* (5.34) *and* (5.35) *if and only if it solves the variational inequality problem:*

$$\langle -u(q^*), q - q^* \rangle + \langle c(f^*), f - f^* \rangle \geq 0, \quad \forall (q, f) \in N, \tag{5.38}$$

where u *and* c *are the cost functions integrating all the* μ_i^k *and* c_{ij}^k, *respectively.*

For the proof, please refer to the book Networks economics [27].
As an illustrative example, consider the international human-migration model consisting of 10 countries of origin and 15 destination countries. The initial populations $q_i = 5000i$ $(i = 1 : 10)$ *and the data for the other parameter were generated randomly and uniformly in the following manners:* $L_i^k \in [1, 10] \times 10^{-6}$, $\tau_i^k \in [.1, .5] \times 10^{-6}$, $\gamma_{ij}^{kl} \in [.1, .5] \times 10^{-6}$, $L_{ij} \in [1, 10]$, $b_i^k \in [1, 10]$ *and* $h_{ij}^{kl} \in [1, 10]$.

The convergence plots based on this example are given in Figure 3.

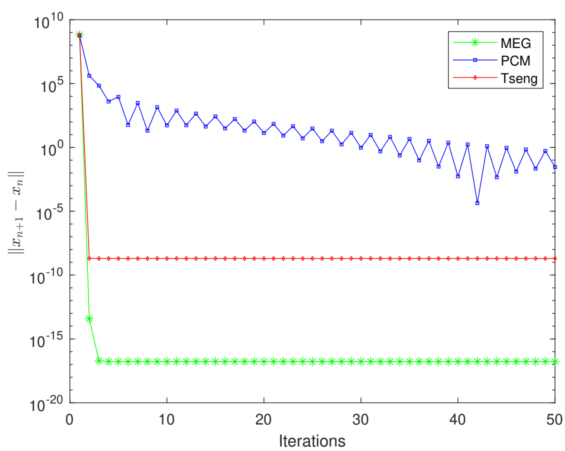


FIGURE 3. Convergence plots for the international migration problem.

In this example, we set the maximum number of iterations to 50 as the stopping criterion to determine the best method with a minimum number of errors. As depicted in Figure 3, the proposed method converges with the least error, followed by PCM, and then the Tseng method.

6. CONCLUSION

This manuscript proposed an iterative method to solve variational inequality problems. The method incorporates the extragradient technique with a conjugate gradient-type direction, accelerating convergence towards the solution. It is shown that sequences generated by the proposed method converge weakly to a solution of the problem. This convergence is facilitated by a substantial lemma that guarantees the strong convergence of a certain sequence of images of a metric projection onto C to an element of C. Examples and application are given to showcase the theoretical findings. Example 4.1 is for a special monotone operator adapted from the work of [7] while Example 4.2 is taken from [30] for solving a special bilevel programming problem. Example 5.3 illustrates the application of our proposed method in addressing problems involving international human migration within a network economy system. Additionally, the proposed method demonstrates promising performance compared to other methods for variational inequality problems

Acknowledgments

The authors acknowledged the support provided by Center of Excellence in Theoretical and Computational Science (TaCS-CoE), KMUTT and the Petchra Pra Jom Klao PhD Scholarship of King Mongkut's University of Technology Thonburi (KMUTT) with Contract No: 55/2021. Moreover, This research project is supported by King Mongkut's University of Technology Thonburi (KMUTT), Thailand Science Research and Innovation (TSRI) , and National Science, Research and Innovation Fund (NSRF) Fiscal year 2024 Grant number FRB670073/0164.

References

[1] Ahmad, A.; Kumam, P.; Harbau, M. H. Convergence theorems for common solutions of nonlinear problems and applications. *Carpathian J. Math.* **40** (2024), 207–241.
[2] Buakird, A.; Nimana, N.; Petrot, N.; Promsinchai, P. A Halpern's mean subgradient extragradient method for solving variational inequalities. *Nonlinear Convex Anal. Optim.* **1** (2022), 177–191.
[3] Censor, Y.; Gibali, A.; Reich, S. Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space. *Optim. Methods Softw.* **26** (2011), 827–845.
[4] Censor, Y.; Gibali, A.; Reich, S. Extensions of Korpelevich's extragradient method for the variational inequality problem in Euclidean space. *Optimization* **61** (2012), 1119–1132.
[5] Dong, Q. L.; Yang, J.; Yuan, H. B. The projection and contraction algorithm for solving variational inequality problems in Hilbert spaces. *J. Nonlinear Convex Anal.* **20** (2019), 111–122.
[6] Nagurney, A.; Nagurney, L. S. Medical nuclear supply chain design: A tractable network model and computational approach. *Int. J. Prod. Econ.* **140** (2012), 865–874.
[7] Gibali, A.; Hieu, D. V. A new inertial double-projection method for solving variational inequalities. *J. Fixed Point Theory Appl.* **21** (2019), 97.
[8] Iiduka, H. Three-term conjugate gradient method for the convex optimization problem over the fixed point set of a nonexpansive mapping. *Appl. Math. Comput.* **217** (2011), 6315–6327.
[9] Iiduka, H. Fixed point optimization algorithm and its application to network bandwidth allocation. *J. Comput. Appl. Math.* **236** (2012), 1733–1742.
[10] Iiduka, H. Fixed point optimization algorithm and its application to power control in CDMA data networks. *Math. Program.* **133** (2012), 227–242.
[11] Iiduka, H.; Uchida, M. Fixed point optimization algorithms for network bandwidth allocation problems with compoundable constraints. *IEEE Commun. Lett.* **15** (2011), 596–598.
[12] Iusem, A. N.; Jofré, A.; Oliveira, R. I.; Thompson, P. Extragradient method with variance reduction for stochastic variational inequalities. *SIAM J. Optim.* **27** (2017), 686–724.
[13] Konnov, I. V. On systems of variational inequalities. *Russ. Math.* **41** (1997), 77–86.
[14] Korpelevich, G. M. The extragradient method for finding saddle points and other problems. *Matecon* **12** (1976), 747–756.
[15] Lyashko, S. I.; Semenov, V. V.; Voitova, T. A. Low-cost modification of Korpelevich's methods for monotone equilibrium problems. *Cybern. Syst. Anal.* **47** (2011), 631–639.
[16] Malitsky, Y. V.; Semenov, V. V. An extragradient algorithm for monotone variational inequalities. *Cybern. Syst. Anal.* **50** (2014), 271–277.
[17] Opial, Z. Weak convergence of the sequence of successive approximations for nonexpansive mappings. *Bull. Amer. Math. Soc.* **73** (1967), 591–597.
[18] Popov, L. D. A modification of the Arrow-Hurwicz method for search of saddle points. *Math. Notes USSR* **28** (1980), 845–848.
[19] Takahashi, W.; Toyoda, M. Weak convergence theorems for nonexpansive mappings and monotone mappings. *J. Optim. Theory Appl.* **118** (2003), 417–428.

[20] Tan, K. K.; Xu, H. K. Approximating fixed points of non-expansive mappings by the Ishikawa iteration process. *J. Math. Anal. Appl.* **178** (1993), 301–308.
[21] Thammasiri, P.; Ungchittrakool, K. Accelerated hybrid Mann-type algorithm for fixed point and variational inequality problems. *Nonlinear Convex Anal. Optim.* **1** (2022), 97–111.
[22] Tseng, P. A modified forward-backward splitting method for maximal monotone mappings. *SIAM J. Control Optim.* **38** (2000), 431–446.
[23] Hieu, D. V. Halpern subgradient extragradient method extended to equilibrium problems. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math.* **111** (2017), 823–840.
[24] Voitova, T. A.; Denisov, S. V.; Semenov, V. V. A strongly convergent modified version of the Korpelevich method for equilibrium programming problems. *Zh. Obchisl. Prykl. Mat.* **1** (2022), 104.
[25] Zykina, A. V.; Melen'chuk, N. V. A two-step extragradient method for variational inequalities. *Russ. Math.* **54** (2010), 71–73.
[26] Kravchuk, A. S.; Neittaanmäki, P. J. *Variational and Quasi-Variational Inequalities in Mechanics*, vol. 147. Springer, New York, 2007.
[27] Nagurney, A. *Network Economics: A Variational Inequality Approach*, vol. 10. Springer, New York, 2013.
[28] Saberi, S. *Network Game Theory Models of Services and Quality Competition with Applications to Future Internet Architectures and Supply Chains*. University of Massachusetts, Amherst, 2016.
[29] Solodov, M. V.; Svaiter, B. F. A new projection method for variational inequality problems. *SIAM J. Control Optim.* **37** (1999), 765–776.
[30] Shehu, Y., Vuong, P. T., and Zemkoho, A. An inertial extrapolation method for convex simple bilevel optimization. *Optim. Methods Softw.* **36**, (2021), no. 1, 1–19.
[31] Sabach, S., and Shtern, S. A first-order method for solving convex bilevel optimization problems. *SIAM J. Optim.* **27**, (2017), no. 2, 640–660.

[1] Center of Excellence in Theoretical and Computational Science (TaCS-CoE) KMUTT Fixed Point Research Laboratory, Science Laboratory Building, Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi, 126 Pracha Uthit Rd., Bang Mod, Thung Khru ,10140, Bangkok, Thailand

Email address: ibrahim.arzuk@kmutt.ac.th
Email address: parin.cha@kmutt.ac.th

[2] Department of Mathematical Sciences, Faculty of Sciences, Bauchi State University, Gadau. Nigeria