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Generalized metric spaces having the fixed point property with respect to contractions

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ABSTRACT. Let X be a nonempty set and $d : X \times X \to \mathbb{R}_+$ a mapping (i.e., a *distance*) and $f : X \to X$ a contraction. In this paper we study the following problem:

Under which conditions on (X, d) do we have that

$$F_f := \{x \in X | f(x) = x\} = \{x^*\}\,?$$

Similar problems for $\mathbb{R}^m_+$-distances, K-distances, $s(\mathbb{R}_+)$-distances, and for extending distances are investigated. Applications to contractions on generalized (dislocated, quasi-, partial, ultra-) metric spaces, are also given. In order to study these problems we introduce the notion of *suitable distance space for contractions*. The paper [Berinde V., Păcurar M., Rus I.A. *Some classes of distance spaces as generalized metric spaces: terminology, mappings, fixed points and applications in Theoretical Informatics*, Creat. Math. Inform. 34 (2025), no. 2, 155–174] is an heuristic introduction to the present one.

Our results open new perspectives in the fixed point theory and theoretical computer science and have important applicability in denotational semantics, as semantic operators in most programming language paradigms satisfy the requirements of fixed point principles for contractions on generalized metric spaces.

1. INTRODUCTION AND PRELIMINARIES

Let X be a nonempty set and $f : X \to X$ a mapping. By the notation $F_f = \{x^*\}$, we understand that f has a unique fixed point, denoted by x^*.

By a *distance* on X we shall understand a mapping $d : X \times X \to \mathbb{R}_+$.

A distance d on X is a *metric* if the following conditions hold:

(I) $d(x, x) = 0$, for all $x \in X$.
(II) $x, y \in X$, $d(x, y) = d(y, x) = 0 \Rightarrow x = y$.
(III) $d(x, y) = d(y, x)$, for all $x, y \in X$.
(IV) $d(x, z) \le d(x, y) + d(y, z)$, for all $x, y, z \in X$.

A mapping $f : X \to X$ on a distance space (X, d) is called an *l-contraction* if there exists a constant l, $0 < l < 1$, such that

$$d(f(x), f(y)) \le l d(x, y),\ \forall x, y \in X.$$

There exist various contraction principles on generalized metric spaces, related mainly to some fundamental applications in theoretical computer science, like denotational semantics, to several paradigms in logic programming and non-monotonic reasoning as well as to converting logic programs into artificial neural networks, see Matthews [27], Hitzler [22], Bukatin et al. [14], Seda and Hitzler [52], Dudley [17], Agarwal et al. [3], Berinde et al. [9], Berinde and Choban [8], Rus [40], Samet et al. [48], Rus [43], Flagg and Kopperman [20], Seda [51], Zabrejko [55], Ilkhan and Kara [23], Reilly and Vamanamurthy [35],...

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These fixed point principles are given in terms of the following three basic concepts: Cauchy sequence, convergent sequence and completeness of (X,d). The definitions of these concepts depend on the axioms satisfied by each class of distances: metric, quasimetric, dislocated metric, partial metrical, ultrametric and others, and therefore they influence the statements of the fixed point principles formulated for contraction type mappings. For an heuristic introduction to such results we refer to Berinde et al. [9].

Based on these facts, our main aim in this paper is to unify such fixed point results, from the point of view of the fixed point theory, on the one hand, and from that of its significant applications to theoretical computer science, on the other hand. We start by considering the following general problem:

Problem. *Let (X,d) be a distance space and $f : X \to X$ a contraction. Under which conditions on (X,d) do we have that*

$$F_f = \{x^*\}\,?$$

For an answer to this question in the case of a metric space, see Rus [39], where various important results from literature are presented.

The structure of the present paper is as follows:

2. Suitable distance spaces for contractions
3. Complete dislocated metric spaces
4. Complete quasimetric spaces
5. Complete partial metric spaces as dislocated metric spaces
6. Partial metric spaces with associated metric
7. Quasimetric spaces with associated order relation
8. Distance spaces with distance in $[0,+\infty]$
9. Some research directions

For the basic notations and concepts in ordered sets and metric spaces used in this paper, see Brown and Pearcy [11], Păcurar and Rus [28], Rus [38] and Berinde et al. [10].

2. Suitable distance space for contractions

Let (X,d) be a distance space and $f : X \to X$ an l-contraction on (X,d). Denote by f^n the n-th iterate of f, i.e.,

$$f^0 := 1_X, f^n := f^{n-1} \circ f, \text{ for } n \geq 1.$$

We mention the following important properties of an l-contraction f on a distance space (X,d).

Remark 2.1. *If $x, y \in X$, then*

$$d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty.$$

Remark 2.2. *We have* $\inf\{d(x,x) | x \in X\} = 0$. *So, if*

$$\inf\{d(x,x) | x \in X\} > 0,$$

then there are no contractions on (X,d).

Remark 2.3. *If $x^* \in F_f$, then $d(x^*,x^*) = 0$. Therefore, if $d(x,x) \neq 0$, for all $x \in X$, then any contraction on (X,d) has no fixed point.*

Remark 2.4. *If $x^*, y^* \in F_f$, then $d(x^*,y^*) = d(y^*,x^*) = 0$.*

Remark 2.5.

$$\sum_{n=0}^{\infty} d(f^n(x), f^{n+1}(x)) < \infty, \forall x \in X.$$

Remark 2.6. *If $x^* \in F_f$, then*

$$d(f^n(x), x^*) \to 0 \text{ as } n \to \infty, \forall x \in X.$$

Remarks 2.1-2.6 and the heuristic considerations in Berinde et al. [9] give rise to the following notion.

Definition 2.1. *A distance space (X, d) is said to be a* suitable distance space for contractions *if d satisfies:*

(II) $x, y \in X$, $d(x, y) = d(y, x) = 0 \Rightarrow x = y$.

(CCC) $(x_n)_{n\in\mathbb{N}} \subset X$, $\sum\limits_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty \Rightarrow$ *there exists a unique* $\overline{x} \in X$ *such that* $d(x_n, \overline{x}) \to 0$ as $n \to \infty$.

Remark 2.7. *Any complete metric space (X, d) is a suitable distance space for contractions, see Rus* [38].

As for the incomplete metric spaces, among them there exist some which do have the fixed point property for contractions, see Subrahmanyam [53], *but there are as well incomplete metric spaces which are not suitable for contractions, as shown by the next example, adapted after the one in Anisiu and Anisiu* [7].

Example 2.1. *Consider the space l^0 of all real sequences with a finite number of zero terms, endowed with the* sup *distance, i.e.,*

$$d((x_1, x_2, \dots), (y_1, y_2, \dots)) = \sup_{i\in\{1,2,\dots\}} \{|x_i - y_i|\},$$

and let $s : l^0 \to l^0$ be the shift operator, defined by

$$s(x_1, x_2, \dots) = (0, x_1, x_2, \dots),\ \forall (x_1, x_2, \dots) \in l^0.$$

For $k \in (0, 1)$ and $e_1 = (1, 0, 0, \dots)$ consider the mapping $f : l^0 \to l^0$ defined by

$$f(x) = ks(x) + e_1,\ \forall x \in l^0.$$

It is easily seen that f is a k-contraction, but $F_f = \emptyset$, as $f(x^*) = x^*$ implies $x^* = (1, k, k^2, k^3, \dots) \notin l^0$.

The reason is that (l^0, d) is not a suitable distance space for contractions, as it does not have property (CCC). To see that, we consider the sequence $(x_n)_{n\in\mathbb{N}} \subset l^0$ given by

$$x_n = \left(\underbrace{0, 0, \dots, 0}_{n \text{ terms}}, \frac{1}{(n+1)^2}, \frac{1}{(n+2)^2}, \dots\right),\ n = 1, 2, \dots$$

Then $\sum\limits_{n=1}^{\infty} d(x_n, x_{n+1}) = \sum\limits_{n=1}^{\infty} \dfrac{1}{(n+1)^2} < \infty$ *but* $d(x_n, \overline{x}) \to 0$ as $n \to \infty$ *holds only for* $\overline{x} = (0, 0, \dots) \notin l^0$. *In fact (l^0, d) is not complete.*

Lemma 2.1. *Let (X, d) be a suitable distance space for contractions, $(x_n)_{n\in\mathbb{N}}$ a sequence in X and $\overline{x} \in X$ such that*

(i) $\sum\limits_{n=0}^{\infty} d(x_n, x_{n+1}) < +\infty$;

(ii) $d(x_n, \overline{x}) \to 0$ *as* $n \to \infty$.

Let also $(x_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$ such that

$$\sum_{k=0}^{\infty} d(x_{n_k}, x_{n_{k+1}}) < +\infty$$

and

$$d(x_{n_k}, \overline{y}) \to 0 \text{ as } k \to \infty.$$

for some $\overline{y} \in X$.

Then $\overline{y} = \overline{x}$.

Corollary 2.1. *Let* (X, d) *be a suitable distance space for contractions and* $(x_n)_{n\in\mathbb{N}}$ *be a sequence in* X *such that*

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < +\infty$$

and $d(x_n, \overline{x}) \to 0$ *as* $n \to \infty$ *for some* $\overline{x} \in X$.

Let $n_0 \in \mathbb{N}^*$ *and the subsequence* $(x_{n+n_0})_{n\in\mathbb{N}}$ *be such that*

$$d(x_{n+n_0}, \overline{y}) \to 0 \text{ as } n \to \infty,$$

for some $\overline{y} \in X$.

Then $\overline{y} = \overline{x}$.

The basic result of this paper is the following.

Theorem 2.1. *Let* (X, d) *be a suitable distance space for contractions and* $f : X \to X$ *an* l*-contraction. Then we have that:*

(i) $F_f = \{x^*\}$ *and* $d(x^*, x^*) = 0$;

(ii) $d(f^n(x), x^*) \to 0$ *as* $n \to \infty$.

Proof. Let $x \in X$. Then

$$\sum_{n=0}^{\infty} d(f^n(x), f^{n+1}(x)) \leq \sum_{n=0}^{\infty} l^n d(x, f(x)) = \frac{1}{1-l} d(x, f(x)) < \infty, \forall x \in X.$$

By (CCC), for any $x \in X$, there exists a unique $x^* = x^*(x)$ such that

$$d(f^n(x), x^*(x)) \to 0 \text{ as } n \to \infty.$$

But the subsequence $(f^{n+1}(x))_{n\in\mathbb{N}}$ satisfies the condition

$$d(f^{n+1}(x), f(x^*(x))) \leq l d(f^n(x), x^*(x)) \to 0 \text{ as } n \to \infty.$$

Hence, by Corollary 2.1 it follows that $f(x^*(x)) = x^*(x)$, i.e., $F_f \neq \emptyset$.

Now, let $x^*, y^* \in F_f$. By Remark 2.4, it follows that

$$d(x^*, y^*) = d(y^*, x^*) = 0,$$

which, by condition (II), implies $x^* = y^*$. So, $F_f = \{x^*\}$.

Conclusion (ii) is in fact Remark 2.6. □

Remark 2.8. *Various fixed point results in literature, see for example Rus* [38]*, can be obtained as corollaries of Theorem 2.1.*

Remark 2.9. *Similar problems can be formulated in case we consider a distance with values in a suitable set* Γ *instead of* $\mathbb{R}_+$.

For example, let $(\Gamma, +, \mathbb{R}, \leq, \xrightarrow{F})$ *be an ordered linear L-space, see Berinde et al.* [9]*, Dudley* [17]*, De Pascale et al.* [16]*, Hitzler* [22]*, Rus* [41]*, Rus and Şerban* [46]*, Seda and Hitzler* [52] *and Zabrejko* [55].

A Γ_+*-distance on a set* X *is a mapping*

$$d : X \times X \to \Gamma_+, \text{ where } \Gamma_+ := \{\gamma \in \Gamma | \gamma \geq 0\}.$$

In a similar way to the case of an $\mathbb{R}_+$*-distance one can define notions like* Γ_+*-metric, dislocated* Γ_+*-metric,* Γ_+*-quasimetric, partial* Γ_+*-metric and so on.*

A general problem that can be formulated is the following one

Problem. *Give an appropriate definition for a contraction in all the above mentioned settings, i.e., Γ_+-metric space, dislocated Γ_+-metric space, Γ_+-quasimetric space, partial Γ_+-metric space, etc.*

In the next sections of the paper we give various examples of suitable distance spaces for contractions, along with some applications of Theorem 2.1.

3. Complete dislocated metric spaces

A distance $d : X \times X \to \mathbb{R}_+$ is by definition a *dislocated metric* if it satisfies the metric axioms (II), (III) and (IV). Then (X, d) is called *a dislocated metric space*. Note that in Matthews [26] a dislocated metric space was called a *metric domain*, while later in Matthews [27] it was renamed as dislocated metric space.

In a dislocated metric space the notions of convergent sequence, Cauchy sequence and completeness of (X, d) are defined in a standard way (i.e., like in the case of a metric space) but these concepts do not possess the same properties as in the case of a metric space.

For example, the convergence structure in a dislocated metric space is, in general, not an L-space convergence structure, see Berinde et al. [9], Dudley [17], Filip [19], Brown and Pearcy [11], Hitzler [22], Seda and Hitzler [52].

Remark 3.10. *A complete dislocated metric space is a suitable distance space for contractions, as it satisfies condition (CCC). Indeed, let $(x_n)_{n\in\mathbb{N}}$ be such that*

$$\text{(W)} \qquad \sum_{n=0}^{\infty} d(x_n, x_{n+1}) < +\infty.$$

From condition (W) *it follows that*

$$\sum_{k=n}^{n+p} d(x_k, x_{k+1}) \to 0 \text{ as } n \to \infty \text{ and } n + p \to \infty$$

and hence, by axiom (IV), we have

$$d(x_n, x_{n+p}) \leq \sum_{k=n}^{n+p} d(x_k, x_{k+1}) \to 0 \text{ as } n \to \infty.$$

Since (X, d) is complete, there exists a unique $\overline{x} \in X$ such that $d(x_n, \overline{x}) \to 0$ as $n \to \infty$, so (CCC) holds.

Having in view Remark 3.10 and Theorem 2.1, we obtain the following result, which extends Mathews' fixed point theorem, in the sense that assertion (i) in the next theorem is assembling Theorems 4 and 5 in Matthews [26].

Theorem 3.2. *Let (X, d) be a complete dislocated metric space and $f : X \to X$ an l-contraction. Then we have that:*

(i) $F_f = \{x^*\}$ and $d(x^*, x^*) = 0$;

(ii) $d(f^n(x), x^*) \to 0$ as $n \to \infty$, $\forall x \in X$;

(iii) $d(x, x^*) \leq \dfrac{1}{1-l} d(x, f(x))$, $\forall x \in X$.

Proof. Conclusions (i) and (ii) follow from Theorem 2.1 and Remark 3.10. Conclusion (iii) follows from axiom (IV). Indeed, we have

$$d(x, x^*) \leq d(x, f(x)) + d(f(x), f^2(x)) + \cdots + d(f^n(x), f^{n+1}(x)) + d(f^{n+1}(x), x^*)$$

$$\leq \frac{1}{1-l}d(x,f(x)) + d(f^{n+1}(x),x^*) \to \frac{1}{1-l}d(x,f(x)) \text{ as } n \to \infty.$$

□

Remark 3.11. *In Matthews* [26]*, conclusion $d(x^*,x^*) = 0$ from Theorem 3.2 is interpreted as expressing the fact that the fixed point x^* is* complete.

Remark 3.12. *It is possible to give similar results, along with the corresponding notions, for other instances of the set Γ. Such an approach is presented in the sequel for the case $\Gamma = \mathbb{R}^m_+$.*

Let (X,d) be a $\mathbb{R}^m_+$-distance space. Then one can define the notions of contraction and suitable space for contractions as follows:

Definition 3.2. *A mapping $f : X \to X$ is a* contraction *if there exists a matrix $S \in \mathbb{R}^{m\times m}_+$ convergent to* 0 *(i.e., $S^n \to 0$ as $n \to \infty$ in the term-wise convergence) such that*

$$d(f(x),f(y)) \leq Sd(x,y),\ \forall x,y \in X.$$

Definition 3.3. *An $\mathbb{R}^m_+$-distance space (X,d) is a* suitable distance space for contractions *if the following conditions hold:*

(II) $x,y \in X$, $d(x,y) = d(y,x) = 0 \Rightarrow x = y$.

(CCC) $(x_n)_{n\in\mathbb{N}} \subset X$, $\sum\limits_{n=0}^{\infty} d(x_n,x_{n+1}) < \infty \Rightarrow$
there exists a unique $\overline{x} \in X$ such that

$$d(x_n,\overline{x}) \to 0 \text{ as } n \to \infty.$$

Similarly to the case of an $\mathbb{R}_+$-distance, we have the following results.

Theorem 3.3. *Let (X,d) be a $\mathbb{R}^m_+$-distance space that is suitable for contractions and $f : X \to X$ be an S-contraction. Then:*

(i) $F_f = \{x^*\}$ and $d(x^*,x^*) = 0$;

(ii) $d(f^n(x),x^*) \to 0$ as $n \to \infty$, $\forall x \in X$.

In view of this result and of the fact that a complete dislocated $\mathbb{R}^m_+$-metric space is suitable for contractions, the next theorem is immediate.

Theorem 3.4. *Let (X,d) be a complete dislocated $\mathbb{R}^m_+$-metric space and $f : X \to X$ an S-contraction. Then:*

(i) $F_f = \{x^*\}$ and $d(x^*,x^*) = 0$;

(ii) $d(f^n(x),x^*) \to 0$ as $n \to \infty$, $\forall x \in X$;

(iii) $d(x,x^*) \leq (I-S)^{-1}d(x,f(x))$, $\forall x \in X$, where I is the identity matrix in $\mathbb{R}^{m\times m}_+$.

A more general result similar to Theorem 3.4 above can be formulated by taking $\Gamma := K$, the cone of positive elements in an ordered Banach space.

Let $(\mathbb{B},+,\mathbb{R},\|\cdot\|,\leq)$ be an ordered Banach space (Zabrejko [55], Berinde et al. [9]) and let $K := \{y \in \mathbb{B} | y \geq 0\}$ be the cone of positive elements in $\mathbb{B}$.

Let (X,d) be a *dislocated K-metric space*. A mapping $f : X \to X$ is said to be a *contraction* if there exists an increasing linear operator S on $\mathbb{B}$ such that $S^n \to 0$ as $n \to \infty$ and

$$d(f(x),f(y)) \leq Sd(x,y),\ \forall x,y \in X.$$

Problem. *The problem is to identify conditions on the linear operator S that lead to results similar to those established in the other settings above.*

4. Complete quasimetric spaces

A distance space (X, d) is called a *quasimetric space*, see Wilson [54], if it satisfies the metric axioms (I), (II) and (IV).

In the case of a quasimetric space there exist distinct notions of Cauchy sequence (forward Cauchy, backward Cauchy,...), of convergent sequence (forward convergent, backward convergent,...) and of completeness (forward, backward,...), see for example Seda and Hitzler [52], Hitzler [22], Seda [51], Reilly and Vamanamurthy [35], Ilkhan and Kara [23], Romaguera and Schellekens [36], Acar and Ö; Erdoğan [1], Altun et al. [5], Eroğlu [18].

We introduce the following notions from the point of view of the contraction mapping principle.

Definition 4.4. *A quasimetric space (X, d) is said to satisfy* condition (C) *if the following implication holds:*
$(x_n)_{n\in\mathbb{N}} \subset X, \overline{x} \in X, d(x_n, \overline{x}) \to 0$ *as* $n \to \infty \Rightarrow$
$d(x_n, y) \to d(\overline{x}, y)$ *as* $n \to \infty, \forall y \in X.$

Definition 4.5. *A sequence $(x_n)_{n\in\mathbb{N}}$ in a quasimetric space (X, d) is said to be a* Cauchy sequence *if*

$$d(x_n, x_m) \to 0 \text{ as } m \geq n, m, n \to \infty.$$

Definition 4.6. *A Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in a quasimetric space (X, d) is said to be* convergent *if there exists $\overline{x} \in X$ such that*

$$d(x_n, \overline{x}) \to 0 \text{ as } n \to \infty.$$

Definition 4.7. *A quasimetric space (X, d) is said to be* complete *if any Cauchy sequence in X is a convergent sequence.*

Lemma 4.2. *In a quasimetric space with property (C), the element $\overline{x} \in X$ in Definition 4.6, i.e., the limit of the sequence $(x_n)_{n\in\mathbb{N}}$, is unique.*

Proof. Let $\overline{x}, \overline{y} \in X$ be such that

$$d(x_n, \overline{x}) \to 0 \text{ and } d(x_n, \overline{y}) \to 0 \text{ as } n \to \infty.$$

By condition (C) we have that

$$d(x_n, \overline{y}) \to d(\overline{x}, \overline{y}) \text{ and } d(x_n, \overline{x}) \to d(\overline{x}, \overline{x})$$

and, respectively

$$d(x_n, \overline{x}) \to d(\overline{y}, \overline{x}) \text{ and } d(x_n, \overline{y}) \to d(\overline{y}, \overline{y}).$$

It follows on the one hand that $d(\overline{y}, \overline{x}) = d(\overline{x}, \overline{x})$ and on the other hand that $d(\overline{x}, \overline{y}) = d(\overline{y}, \overline{y})$. This implies, by (I), that $d(\overline{y}, \overline{x}) = d(\overline{x}, \overline{y}) = 0$. So by (II) we get $\overline{x} = \overline{y}$. □

Lemma 4.3. *A complete quasimetric space (X, d) satisfying condition (C) is a suitable distance space for contractions.*

Proof. Let us prove that a complete quasimetric space (X, d) satisfying condition (C) also satisfies condition (CCC).

Let $(x_n)_{n\in\mathbb{N}} \subset X$ satisfy condition (W) above, that is

$$\text{(W)} \qquad \sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty.$$

From (W) and axiom (IV) we have

$$d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + \cdots + d(x_{n+p-1}, x_{n+p}) \to 0$$

as $n \to \infty$ and $n + p \to \infty$.

Since (X, d) is complete, it follows that $(x_n)_{n\in\mathbb{N}}$ is convergent, so there exists $\overline{x} \in X$ such that $d(x_n, \overline{x}) \to 0$ as $n \to \infty$. But X satisfies condition (C), so by Lemma 4.2 this limit is unique. Therefore (X, d) satisfies condition (CCC).

But d also satisfies the metric axiom (II), so (X, d) is a suitable space for contractions. □

From Theorem 2.1 and (IV) we get the following fixed point result, which is a generalization of Banach's fixed point principle, from the case of metric spaces to that of quasimetric spaces.

Theorem 4.5. *Let (X, d) be a complete quasimetric space satisfying condition (C). If $f : X \to X$ is an l-contraction, then:*
(i) $F_f = \{x^*\}$;
(ii) $d(f^n(x), x^*) \to 0$ *as* $n \to \infty$;
(iii) $d(x, x^*) \leq \dfrac{1}{1-l} d(x, f(x))$, $\forall x \in X$.

It is possible to define the notions of contraction and of a space suitable for contractions in the case of $\mathbb{R}_+$-quasimetric spaces, K-quasimetric spaces and $s(\mathbb{R}_+)$-quasimetric spaces as well, similarly to Definitions 3.2 and 3.3.

5. Partial metric spaces as dislocated metric spaces

By definition, a distance space (X, d) is a *partial metric space*, see Matthews [27], if the distance d satisfies the following conditions:

(I′) $d(x, x) \leq d(x, y)$ and $d(y, y) \leq d(x, y)$, for all $x, y \in X$;
(II′) $x, y \in X$, $d(x, x) = d(x, y) = d(y, y) \Rightarrow x = y$ for all $x, y \in X$;
(III) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(IV′) $d(x, z) \leq d(x, y) + d(y, z) - d(y, y)$, for all $x, y, z \in X$.

First of all, we note that each partial metric space is a dislocated metric space. Indeed, (IV′) implies (IV). Let $x, y \in X$ with $d(x, y) = d(y, x) = 0$. From (I′) it follows that $d(x, x) \leq d(x, y) = 0$ and $d(y, y) \leq d(x, y) = 0$. Hence, by (II′) we have $x = y$.

In terms of the notions used in Section 3 for the case of a dislocated metric space, from Theorem 3.2 one obtains the following result, which extends the partial metric contraction principle (Theorem 5.3, in Matthews [27]).

Theorem 5.6. *Let (X, d) be a partial metric space and $f : X \to X$ an l-contraction. If (X, d) is complete as a dislocated metric space, then:*
(i) $F_f = \{x^*\}$ *and* $d(x^*, x^*) = 0$;
(ii) $d(f^n(x), x^*) \to 0$ *as* $n \to \infty$, $\forall x \in X$;
(iii) $d(x, x^*) \leq \dfrac{1}{1-l} d(x, f(x))$, $\forall x \in X$.

For more considerations on fixed points in partial metric spaces and on their relevance to non-operational semantics for well behaved programming languages, see for example Matthews [27], Bukatin et al. [14], Agarwal et al. [3], Berinde et al. [9], Bugajewski et al. [13], Filip [19], Hitzler [22], Pasicki [30], Rus [40], Samet et al. [48], Schellekens [49], Seda and Hitzler [52].

6. Partial metric spaces with associated metric

Let (X, d) be a partial metric space. Then, the following distance functions are metrics on X:

$$d_0(x,y) := \begin{cases} d(x,y), \text{ for } x \neq y \\ 0, \text{ for } x = y \end{cases}$$
$$d^s(x,y) := 2d(x,y) - d(x,x) - d(y,y)$$
$$d^m(x,y) := \max\{d(x,y) - d(x,x), d(x,y) - d(y,y)\}.$$

Starting with Matthews [27], the basic notions in a partial metric space are defined in terms of d^s. So, by definition, a sequence in (X,d) is *convergent* if it is convergent in the metric space (X,d^s); it is Cauchy in (X,d) if it is a *Cauchy* sequence in (X,d^s) and (X,d) is *complete* if (X,d^s) is complete.

Let (X,d) be a partial metric space and denote by

$$Z(d) := \{x \in X | d(x,x) = 0\}$$

the set of all *complete points*, according to the terminology in Matthews [26]. If $Z(d) = \emptyset$, then any contraction on (X,d) has no fixed points, see Section 2 and also Rus [40].

The following variant of Matthews' theorem has a simple proof.

Theorem 6.7. *Let (X,d) be a partial metric space with associated metric d^s and $f : X \to X$ a mapping. We suppose that*

(1) f is an l-contraction with respect to d;

(2) (X,d^s) is a complete metric space;

(3) $Z(d) \neq \emptyset$.

Then:

(i) $F_f = \{x^*\}$ and $d(x^*,x^*) = 0$;

(ii) $d(f^n(x),x^*) \to 0$ as $n \to \infty$, $\forall x \in X$;

(iii) $d(x,x^*) \leq \dfrac{1}{1-l} d(x,f(x))$, $\forall x \in X$.

Proof. First, we note that

$$d^s(x,y) = d(x,y), \forall x,y \in Z(d),$$

i.e., d is a metric on $Z(d)$. Taking $x_n \in Z(d)$ such that $x_n \overset{d^s}{\longrightarrow} x$ implies $x \in Z(d)$, so $(Z(d),d)$ is a complete metric space.

From the contraction mapping principle in a complete metric space we have that

$$F_f \cap Z(d) = \{x^*\}.$$

Hence, by Remarks 2.5 and 2.6 we obtain (i) and (ii).

Conclusion (iii) also follows, as axiom (IV') implies (IV). □

7. Quasimetric spaces with associated order relation

Let (X,d) be a distance space. On (X,d) we consider the binary relation $\leq_d$ defined by

$$x \leq_d y \Leftrightarrow d(x,y) = 0.$$

We call it the *associated binary relation corresponding to the distance d*. If d satisfies the axiom (I), then $\leq_d$ is reflexive; if d satisfies axiom (II), then $\leq_d$ is antisymmetric; the transitivity of $\leq_d$ follows from axiom (IV).

Therefore, if d is a quasimetric, then the associated binary relation is an order relation.

Theorem 7.8. *Let (X,d) be a quasimetric space satisfying condition (C) and $f : X \to X$ a mapping. We assume that:*

(1) (X,d) is complete;

(2) f is Lipschitz;
(3) there exists $x_0 \in X$ such that $x_0 \leq_d f(x_0)$.

Then f has a minimum fixed point x^* above x_0, i.e., x^* is the minimum element in $(F_f \cap \{x \in X | x_0 \leq_d x\}, \leq_d)$.

Proof. Let $x_0 \in X$ such that $x_0 \leq_d f(x_0)$. This is equivalent to $d(x_0, f(x_0)) = 0$ and leads to

$$d(x_0, f(x_0)) = d(f(x_0), f^2(x_0)) = \cdots = d(f^n(x_0), f^{n+1}(x_0)) = 0, n \in \mathbb{N}.$$

By axiom (IV) it follows that the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is Cauchy. But (X, d) satisfies condition (C) and so by assumption (1) it follows that there exists a unique x^* such that

$$d(f^n(x_0), x^*) \to 0 \text{ as } n \to \infty.$$

On the other hand, from (2), it follows that

$$d(f^{n+1}(x_0), f(x^*)) \to 0 \text{ as } n \to \infty.$$

Since $(f^{n+1}(x_0))_{n\in\mathbb{N}}$ is Cauchy this implies that $x^* = f(x^*)$.

Now let $y^* \in F_f$ such that $x_0 \leq_d y^*$, that is, $d(x_0, y^*) = 0$. From (2) it follows that f is increasing and by (3) we obtain that

$$f^n(x_0) \leq_d y^*, \forall n \in \mathbb{N},$$

i.e., $d(f^n(x_0), y^*) = 0, \forall n \in \mathbb{N}$.

Now, by condition (C) it follows that $d(x^*, y^*) = 0$, i.e.

$$x^* \leq_d y^*$$

as required. □

Remark 7.13. *Theorem 7.8 is a consistent generalization of a Rutten theorem, see Rutten* [47], *Hitzler* [22], *Seda and Hitzler* [52] *and Berinde et al.* [9].

Remark 7.14. *It is possible to give similar results for other types of quasimetric spaces. Such an approach is presented in the sequel for the case $\mathbb{R}^m_+$-quasimetric spaces.*

Let (X, d) be a $\mathbb{R}^m_+$-distance space. A mapping $f : X \to X$ is *Lipschitz* if there exists a matrix $L \in \mathbb{R}^{m\times m}_+$ such that

$$d(f(x), f(y)) \leq Ld(x, y), \forall x, y \in X.$$

In a $\mathbb{R}^m_+$-quasimetric space, the result corresponding to Theorem 7.8 is the following one.

Theorem 7.9. *Let (X, d) be a $\mathbb{R}^m_+$-quasimetric space with condition (C) and $f : X \to X$ a mapping. We assume that:*

(1) (X, d) is complete;
(2) f is Lipschitz;
(3) there exists $x_0 \in X$ such that $x_0 \leq_d f(x_0)$.

Then f has a minimum fixed point x^* above x_0.

We end this Section by formulating the following

Problem. *Give similar results to Theorems 7.8 and 7.9 in the case of K-quasimetric spaces and of $s(\mathbb{R}_+)$-quasimetric spaces.*

8. Extending distance spaces

Let (X, d) be a Γ-distance space with $\Gamma := \overline{\mathbb{R}}_+$, where $\overline{\mathbb{R}}_+ = [0, \infty]$. A $\overline{\mathbb{R}}_+$-distance space is called an *extending distance space*. For extending metric spaces and contractions on such spaces (due to Luxemburg 1958, Diaz and Margalis 1968, Jung 1969), see Petruşel et al. [32], Berinde et al. [9]. In an extending distance space (X, d), a mapping $f : X \to X$ is an l-contraction, see Berinde et al. [9], if $0 < l < 1$ and the following implication holds:

$$x, y \in X, d(x, y) < +\infty \;\Rightarrow\; d(f(x), f(y)) \le l d(x, y).$$

By definition an extending distance space (X, d) is a *strong extending distance space* if d satisfies the following conditions:

(a) $d(x, x) < +\infty, \forall x \in X$;
(b) $x, y \in X$, $d(x, y) < \infty \Rightarrow d(y, x) < \infty$;
(c) $x, y, z \in X, d(x, z) < \infty, d(z, y) < \infty \Rightarrow d(x, y) < \infty$.

Definition 8.8. *A strong extending distance space (X, d) is called a* suitable distance space for contractions *if it satisfies the following conditions:*

(II) $x, y \in X, d(x, y) = d(y, x) = 0 \Rightarrow x = y$;
(CCC) $(x_n)_{n\in\mathbb{N}} \subset X$, $\sum\limits_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty$ *implies that there exists a unique* $\overline{x} \in X$ *such that* $d(x_n, \overline{x}) \to 0$ *as* $n \to \infty$.

Remark 8.15. *Let (X, d) be a strong extending distance space. Then there exists a partition $X = \bigcup\limits_{\lambda\in\Lambda} X_\lambda$ of X such that $d_\lambda := d|_{X_\lambda \times X_\lambda}$ is a distance, that is, $d_\lambda(x, y) \in \mathbb{R}_+$, for each $\lambda \in \Lambda$.*

Indeed, we note that the binary relation

$$x \sim y \Leftrightarrow d(x, y) < \infty$$

is an equivalence relation on X. The above partition of X will be called the canonical partition *of a strong extending distance space.*

Remark 8.16. *A strong extending distance space with the canonical partition $X = \bigcup\limits_{\lambda\in\Lambda} X_\lambda$ is a suitable distance space for contractions if and only if (X_λ, d_λ) is a suitable distance space, for all $\lambda \in \Lambda$.*

Remark 8.17. *Let (X, d) be a strong extending distance space, $X = \bigcup\limits_{\lambda\in\Lambda} X_\lambda$ be the canonical partition of (X, d) and $f : X \to X$ an l-contraction.*

If for some λ there exists $x \in X_\lambda$ such that $d(x, f(x)) < \infty$, then $f(X_\lambda) \subset X_\lambda$.

By Theorem 2.1 we obtain the following result:

Theorem 8.10. *Let (X, d) be a strong extending distance space suitable for contractions, $f : X \to X$ an l-contraction and $X = \bigcup\limits_{\lambda\in\Lambda} X_\lambda$ the canonical partition of (X, d). Then the following hold:*

(i) If $\{x \in X | d(x, f(x)) < \infty\} \neq \emptyset$, then $F_f \neq \emptyset$.
(ii) If for some λ there exists $x \in X_\lambda$ such that $d(x, f(x)) < \infty$, then
  (a) $F_f \cap X_\lambda = \{x^*_\lambda\}$ and $d(x^*_\lambda, x^*_\lambda) = 0$;
  (b) $d(f^n(x), x^*_\lambda) \to 0$ as $n \to \infty, \forall x \in X_\lambda$.

Similar results can be formulated if instead of a strong extending distance space one considers strong extending *dislocated metric* spaces and, respectively, strong extending *quasimetric* spaces.

Remark 8.18. *Let (X, d) be a strong extending dislocated metric space. If in addition (X, d) is complete, then it is a suitable distance space for contractions.*

The next result is a version of Theorem 8.10 in the case of strong extending dislocated metric spaces.

Theorem 8.11. *Let (X,d) be a complete strong extending dislocated metric space, $f : X \to X$ an l-contraction and $X = \underset{\lambda\in\Lambda}{\cup} X_\lambda$ the canonical partition of (X,d). Then the following hold:*

(i) If $\{x \in X | d(x,f(x)) < \infty\} \neq \emptyset$, then $F_f \neq \emptyset$.

(ii) If for some λ there exists $x \in X_\lambda$ such that $d(x,f(x)) < \infty$, then

(a) $F_f \cap X_\lambda = \{x^*_\lambda\}$ and $d(x^*_\lambda, x^*_\lambda) = 0$;

(b) $d(f^n(x), x^*_\lambda) \to 0$ as $n \to \infty$, $\forall x \in X_\lambda$.

Remark 8.19. *Let (X,d) be a complete strong extending quasimetric space satisfying condition (C). Then (X,d) is suitable for contractions.*

The next result is a version of Theorem 8.10 in the case of strong extending quasimetric spaces.

Theorem 8.12. *Let (X,d) be a complete strong extending quasimetric space satisfying condition (C), $f : X \to X$ an l-contraction and $X = \underset{\lambda\in\Lambda}{\cup} X_\lambda$ the canonical partition of (X,d). Then the following hold:*

(i) If $\{x \in X | d(x,f(x)) < \infty\} \neq \emptyset$, then $F_f \neq \emptyset$.

(ii) If for some λ there exists $x \in X_\lambda$ such that $d(x,f(x)) < \infty$, then

(a) $F_f \cap X_\lambda = \{x^*_\lambda\}$ and $d(x^*_\lambda, x^*_\lambda) = 0$;

(b) $d(f^n(x), x^*_\lambda) \to 0$ as $n \to \infty$, $\forall x \in X_\lambda$.

Remark 8.20. *For other considerations on extending generalized metric spaces see Petruşel et al.* [32], *Berinde et al.* [9], *Hitzler* [22], *Seda* [51], *Seda and Hitzler* [52].

9. Some research directions

In this last section we aim to identify some research directions in the line of the results presented in Sections 2-8, that still need to be explored.

9.1. **Towards a good definition for the notion of contractions on $s(\mathbb{R}_+)$-distance spaces.** Let $s(\mathbb{R}) := \{(x_n)_{n\in\mathbb{N}} | x_n \in \mathbb{R}, n \in \mathbb{N}\}$ and $\mathbb{M}(\mathbb{R}) := \{(x_{ij})_{i,j\in\mathbb{N}} | x_{ij} \in \mathbb{R}, i,j \in \mathbb{N}\}$, where

$$(x_{ij})_{i,j\in\mathbb{N}} = \begin{pmatrix} x_{00} & x_{01} & x_{02} & \dots \\ x_{10} & x_{11} & x_{12} & \dots \\ x_{20} & x_{21} & x_{22} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

is an infinite matrix. Then $(s(\mathbb{R}), +, \mathbb{R}, \leq, \to)$ and $(\mathbb{M}(\mathbb{R}), +, \mathbb{R}, \leq, \to)$ are ordered linear L-spaces, where $\to$ is the termwise convergence.

A matrix $S \in \mathbb{M}(\mathbb{R})$ is called *row-column-finite* if it contains only a finite number of nonzero elements in each row and in each column.

Let (X,d) be an $s(\mathbb{R}_+)$-metric space, i.e., d is a $s(\mathbb{R}_+)$-distance and it satisfies the metric axioms (I), (II), (III) and (IV).

By definition, a mapping $f : X \to X$ is said to be *S-Lipschitz* if $S \in \mathbb{M}(\mathbb{R})$, S is row-column-finite and

$$d(f(x), f(y)) \leq S d(x,y), \forall x, y \in X.$$

Then the following question arises:

Problem. *Under which conditions on (X,d) and on S do we have for an S-Lipschitz mapping f that*

(i) $F_f = \{x^*\}$;
(ii) $d(f^n(x), x^*) \to 0$ *as* $n \to \infty$?

For further reading on this topic we refer to Rus [41], Cooke [15], Frigon [21], Berinde et al. [9], Angelov [6], Berinde et al. [10], De Pascale et al. [16], Rus [38].

9.2. **Towards a good definition for the notion of contraction on K-distance spaces.** Let $(\mathbb{B}, +, \mathbb{R}, \|\cdot\|, \leq)$ be an ordered Banach space and K the cone of positive elements in $\mathbb{B}$. Let (X, d) be a K-distance space. By definition a mapping $f : X \to X$ is said to be *S-Lipschitz* if $S : \mathbb{B} \to \mathbb{B}$ is an increasing linear operator such that

$$d(f(x), f(y)) \leq S(d(x, y)), \forall x, y \in X.$$

Problem. *If (X, d) is a K-metric space, under which conditions on (X, d) and on S do we have for an S-Lipschitz mapping that*
(i) $F_f = \{x^*\}$;
(ii) $d(f^n(x), x^*) \to 0$ *as* $n \to \infty$?

For further reading on this topic we refer to Zabrejko [55], Zabrejko and Makarevich [56], Rus [38], Proinov [34], Rus [43], Berinde et al. [10].

9.3. **Two problems on contractive mappings on a distance space.**

Problem 9.1 (Edelstein's Problem). *Let (X, d) be a distance space, $f : X \to X$ a contractive mapping and $x \in X$. Under which conditions on (X, d) and f does the following implication hold: there exist $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ and $x^* \in X$ for which*
$d(f^{n_k}(x), x^*) \to 0$ *as* $k \to \infty$
implies
$d(f^n(x), x^*) \to 0$ *as* $n \to \infty$ *and* $F_f = \{x^*\}$?

Problem 9.2 (Nadler's Problem). *Let (X, d) be a distance space and $f : X \to X$ a contractive mapping with $F_f = \{x^*\}$. Under which conditions on (X, d) does the following hold:*

$$d(f^n(x), x^*) \to 0 \text{ as } n \to \infty, \forall x \in X?$$

For further reading on this problems due to M. Edelstein (1962) and S.B. Nadler (1972) in the case of a metric space we refer to Bryant and Guseman [12] and the references therein.

9.4. **Distances with value in an ordered set with minimum element 0: nonexpansive and contractive mappings.**

Let X be a nonempty set, Γ an ordered set with minimum element denoted by 0 (or $\perp$) and $d : X \times X \to \Gamma$ a Γ-distance. By definition, the triple (X, d, Γ) is called *Γ-ultrametric space* if it satisfies the following axioms:

(I) $d(x, x) = 0, \forall x \in X$;
(II) $x, y \in X$, $d(x, y) = d(y, x) = 0 \Rightarrow x = y$;
(III) $d(x, y) = d(y, x), \forall x, y \in X$;
(IV)$_u$ $x, y, z \in X, \gamma \in \Gamma, d(x, y) \leq \gamma, d(y, z) \leq \gamma \Rightarrow d(x, z) \leq \gamma$.

In a similar way to the case of $\mathbb{R}_+$-distance spaces one can define the following notions: *dislocated Γ-ultrametric space*, if axiom (I) is dropped; *pseudo-Γ-ultrametric space*, if axiom (II) is dropped; *quasi-Γ-ultrametric space*, if axiom (III) is dropped, etc.

There are various fixed point results for contractive mappings on a Γ-ultrametric space.

Problem. *The general problem that can be formulated in this case is: assuming* (X, d) *is a* Γ*-distance space with axioms (II) and* $(IV)_u$*, and* $f : X \to X$ *is a contractive mapping, then what additional conditions on* (X, d) *are required in order to ensure that* $F_f = \{x^*\}$*?*

For further reading on this topic we refer to Hitzler [22], Prieß- Crampe and Rienboim [33], Seda and Hitzler [52], Rutten [47], Ackerman [2], Brown and Pearcy [11], Păcurar and Rus [28], Berinde et al. [10].

9.5. **Multivalued mappings.**

Let (X, d) be a distance space and $T : X \to P(X)$ a multivalued mapping. We denote by $F_T := \{x \in X | x \in T(x)\}$ the fixed point set of T and by $(SF)_T := \{x \in X | T(x) = \{x\}\}$ the set of strict fixed points of T (see Petruşel et al [31], Rus [38]).

By definition, $T : X \to P(X)$ is called *l-contraction*, where $0 < l < 1$, if for every $x \in X$, for every $y \in X$ and for each $u \in T(x)$ there exists $v \in T(y)$ such that

$$d(u, v) \leq l d(x, y).$$

According to the above definition we have the following:

Remark 9.21. *Let* (X, d) *be a distance space and* $T : X \to P(X)$ *an l-contraction. Then there exists a sequence* $(x_n)_{n\in\mathbb{N}} \subset X$ *such that*

(a) $x_{n+1} \in T(x_n), \forall n \in \mathbb{N}$;

(b) $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leq \frac{1}{1-l} d(x_0, x_1)$.

This remark suggests the following general problems.

Problem. *Let* (X, d) *be a distance space and* $T : X \to P(X)$ *a multivalued l-contraction.*

(A) Under which conditions on (X, d) *and* T *is the following guaranteed: for each* $x_0 \in X$ *and each* $x_1 \in T(x_0)$ *there exist a sequence* $(x_n)_{n\in\mathbb{N}}$ *and* $x^*(x_0, x_1) \in F_T$ *such that*

$$d(x_n, x^*(x_0, x_1)) \to 0 \text{ as } n \to \infty?$$

(B) Under which conditions on (X, d) *and* T *do we have that*

$$F_T = (SF)_T = \{x^*\}?$$

For further reading on this topic we refer to Hitzler [22], Petruşel et al. [31], Rus [45], [38], [44].

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