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Operational almost periodic alpha unpredictability

MARAT AKHMET[1] AND AKYLBEK ZHAMANSHIN[2]

ABSTRACT. Modern science is continually seeking new types of functions to meet industrial demands. These needs have expanded to include the analysis of brain activity and advancements in artificial intelligence, particularly in the realms of machine learning and deep learning. Our research is centered on this area of functional analysis.

This paper introduces advanced concepts known as operational almost periodic alpha unpredictable functions. To develop this idea, we present a new operational research method for analyzing qualitative dynamic features. In simple terms, this method effectively distinguishes between regularity and irregularity in the dynamics of chaotic processes. Specifically, we identify almost periodic and alpha unpredictable components within the solutions of linear and quasilinear differential equations. Together, these components produce a functional dependence that could significantly contribute to research on complexity in today's challenging problems. The theoretical results are illustrated through both analytical and graphical representations of specific dynamics.

1. INTRODUCTION

Oscillations are a fundamental characteristic of many processes found in nature. Of significant theoretical and practical importance are oscillatory motions described by differential equations. The literature provides numerous results regarding periodic, quasiperiodic, and almost periodic solutions of differential equations, attributable to established mathematical methods and their important applications [16, 18, 19, 21, 22, 25, 28]. Additionally, recurrent and Poisson stable solutions are essential to the theory of differential equations [12, 23, 24, 26, 27, 28].

The pioneers of the theory of nonlinear oscillations, H. Poincaré and A.M. Lyapunov, developed a mathematical framework appropriate for studying nonlinear systems. Initially, the theory of nonlinear dynamics primarily focused on periodic motions. The first functions that could still be considered "periodic" and sufficiently defined for rigorous mathematical analysis were quasiperiodic functions, which were independently introduced and studied by P. Bohl [13] and E. Esclangon [20]. H. Bohr's key papers laid the foundation for the theory of almost periodic functions, which are now referred to as H. Bohr almost periodic functions [14]. Subsequently, various approaches to almost periodicity were explored by N.N. Bogolyubov, A.S. Besicovitch, S. Bochner, V.V. Stepanov, and others [11, 15, 18, 22, 25]. Almost periodic functions play a crucial role in the advancement of harmonic analysis on groups, as well as Fourier series and integrals on groups. The first paper detailing the existence of almost periodic solutions was written by H. Bohr and O. Neugebauer, and today, the theory of almost periodic equations has been further developed in conjunction with problems related to differential equations, stability theory,

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Corresponding author: Marat Akhmet; marat@metu.edu.tr

and dynamical systems. This theory has seen applications not only in ordinary differential equations and classical dynamical systems but also across various classes of partial differential equations and equations in Banach spaces.

The study of dynamics in various processes greatly benefits from recurrence functions, especially those that are Poisson stable [12, 23, 24, 26, 27, 28]. In several texts and papers, the exploration of recurrency has expanded to include alpha unpredictable functions. In integrated style the results are given in [1, 3, 4]. The concept of compartmental alpha unpredictable functions was introduced in a research paper [6], which combines elements of recurrence and chaos to give more constructive details of sophisticated dynamics [1, 2, 4].

The notions of recurrent motions and Poisson stable points are central to the qualitative theory of motion in dynamical systems. H. Poincaré regarded Poisson stable points as fundamental in describing complexity within celestial dynamics [26, 27].

A few years ago, we introduced the concepts of unpredictable points and functions, which significantly expanded upon the classical theory of dynamical systems established by H. Poincaré and G. Birkhoff [9, 10]. In this and subsequent research, we will refer to these as *alpha unpredictable points and functions*. This terminology is intended to distinguish our concepts from existing ones and to avoid confusing the reader.

An alpha unpredictable point can be seen as an evolution of the Poisson stable one. Our research demonstrated that the quasi-minimal set is chaotic if the Poisson stable point possesses the alpha unpredictability property. Consequently, the existence of chaos in a dynamic system can depend on the presence of just one point—an alpha unpredictable point.

Alpha unpredictable functions were defined as points within the functional space of the dynamical system, with elements shifted along the time coordinate, and in the context of the topology of convergence on compact sets of the time axis. This approach has significantly eased the challenge of proving the existence of alpha unpredictable solutions for differential equations, allowing us to remain firmly within the realm of differential equations without necessarily relying on related results from dynamical systems or chaos theory.

The primary innovation of this paper is the introduction of a new method for defining functions. In its most general sense, a function is understood as an assignment of a single member from one set to an element of another set. In this study, we focus on functions that are defined operationally in a general context and specifically through integral operators. To highlight the unique construction method, we have chosen to retain the term "operational" for the algorithm.

This approach is essential for researching complex dynamics, where both regular and irregular inputs are combined within a single process. Despite the intricate relationships involved, our simulations demonstrate that this method is effective. The impact of these dynamic components is reflected in the output solutions of the models. We believe that more detailed investigations of the functions obtained through this process will be a priority in the near future.

2. Preliminaries

Now, let us present the main definitions of the research. Throughout the paper, $\mathbb{R}$ and $\mathbb{N}$ will stand for the set of real and natural numbers, respectively, and the Euclidean norm will be used.

A set of numbers is said to be *relatively dense*, if there is a positive number l, such that any interval of length l contains a member of the set.

Definition 2.1. [22] *A continuous function* $f(t) : \mathbb{R} \to \mathbb{R}^n$ *is said to be almost periodic if, for any positive* ϵ, *the set* $S(f, \epsilon) = \{\tau : \|f(t+\tau) - f(t)\| < \epsilon$ *for all* $t \in \mathbb{R}\}$ *is relatively dense.*

The number τ is called an $\epsilon-$almost period of function $f(t)$.

The following property of almost periodic functions will be used in the main part of the paper.

Remark 2.1. [25] *Let the functions* $f_1(t)$, $f_2(t)$,..., $f_n(t)$ *be almost periodic functions. Then for any* $\epsilon > 0$, *all the functions* $f_1(t)$, $f_2(t)$,..., $f_n(t)$ *have a common relatively dense set of* $\epsilon-$*almost periods.*

Definition 2.2. [1] *A uniformly continuous and bounded function* $g(t) : \mathbb{R} \to \mathbb{R}^n$ *is alpha unpredictable if there exist positive numbers* ϵ_0, δ *and sequences* t_k, s_k *both of which diverge to infinity such that* $\|g(t+t_k) - g(t)\| \to 0$ *as* $k \to \infty$ *uniformly on compact subsets of* $\mathbb{R}$ *and* $\|g(t+t_k) - g(t)\| > \epsilon_0$ *for each* $t \in [s_k - \delta, s_k + \delta]$ *and* $k \in \mathbb{N}$.

An illustrative example of alpha unpredictable function. We will now demonstrate how to constructively determine alpha-unpredictable functions. This method relies entirely on discrete and differential equations. While this approach is not original, it is effective. For a simple analogy, remember that the trigonometric functions $\sin$ and $\cos$ can be derived as solutions to a harmonic equation. Furthermore, we will use asymptotic presentation to visualize alpha unpredictability.

Let $\psi_i, i \in \mathbb{Z}$, be a solution of the logistic difference equation

$$\lambda_{i+1} = \mu\lambda_i(1 - \lambda_i), \tag{2.1}$$

where $i \in \mathbb{Z}$, and $\mu \in [3 + (2/3)^{1/2}, 4)$ is a fixed parameter. The section $[0, 1]$ is invariant with respect to (2.1) for the considered values of μ.

Now, consider the following integral function

$$\Theta(t) = \int_{-\infty}^{t} e^{-3(t-s)}\Omega(s)ds, \tag{2.2}$$

where $\Omega(t)$ is a piecewise constant function defined on the real axis through the equation $\Omega(t) = \psi_i$ for $t \in [i, i+1)$, $i \in \mathbb{Z}$. It is worth noting that $\Theta(t)$ is bounded on the whole real axis such that $\sup\limits_{t\in\mathbb{R}} |\Theta(t)| \le \dfrac{1}{3}$, and it satisfies the differential equation,

$$\theta'(t) = -3\theta(t) + \Omega(t). \tag{2.3}$$

In [9], it was demonstrated that the function $\Theta(t)$ is alpha unpredictable. Figure 1 illustrates the graph of the solution $\theta(t)$ for the system described in 2.3, with the initial condition $\theta(0) = 0.5$. This solution was obtained for $\mu = 3.85$ and the initial condition $\lambda_0 = 0.4$ in equation (2.1). The graph asymptotically approaches the graph of the function $\Theta(t)$, and this is clearly observable for $t > 30$.

Remark 2.2. *Periodic and quasi-periodic functions are in the class of almost periodic functions, as well as periodic, quasi-periodic and almost periodic functions are Poisson stable functions. Alpha unpredictability implies Poisson stability, but not vice versa.*

The sequence $t_k, k = 1, 2, \ldots$, is said to be the *convergence sequence* of the function $g(t)$. We call the uniform convergence on compact subsets of $\mathbb{R}$, *the convergence or Poisson property*, and the existence of the sequence s_k and positive numbers ϵ_0, δ is called *the separation property*.

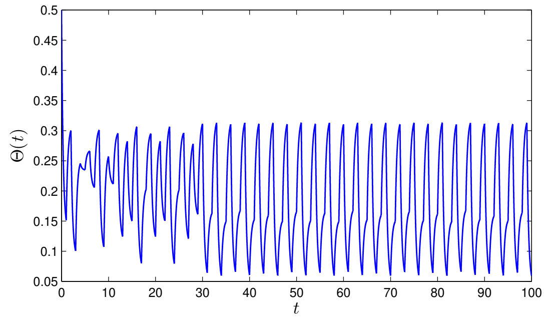

FIGURE 1. An asymptotic approximation of the alpha unpredictable function $\Theta(t)$ is vividly seen, if $t > 30$.

Definition 2.3. [1, 6] *A function $g(t) : \mathbb{R} \to \mathbb{R}^n$ is said to be compartmental almost periodic alpha unpredictable function, if $g(t) = G(t,t)$, where $G(u,v)$ is a continuous bounded function, almost periodic in u uniformly with respect to v, and alpha unpredictable in v uniformly with respect to u, i.e.,*

(i) for any positive ϵ, the set $S(G,\epsilon) = \{\tau : \|G(u+\tau,v) - G(u,v)\| < \epsilon$ for all $u \in \mathbb{R}\}$ is relatively dense;

(ii) there exist a sequence t_k which diverges to infinity and $\sup_{u\in\mathbb{R}} \|G(u,v+t_k) - G(u,v)\| \to 0$ as $k \to \infty$ uniformly on bounded intervals of v;

(iii) there exist positive numbers ϵ_0, δ and sequences t_k, s_k, both of which diverge to infinity, such that $\|G(u,v+t_k) - G(u,v)\| > \epsilon_0$ for $v \in [s_k - \delta, s_k + \delta]$, $u \in \mathbb{R}$ and $k \in \mathbb{N}$.

The last definition has not been used in the theory of differential equations because no sufficient conditions for alpha unpredictability of solutions have been established. For this reason, the present research proposes a new type of function that includes an almost periodic component, which is more applicable and constructive at this time.

Definition 2.4. *A function $z(t) : \mathbb{R} \to \mathbb{R}^n$ is said to be operational almost periodic alpha unpredictable function, if $z(t) = \Psi g(t)$, where $g(t) = G(t,t)$, is a compartmental almost periodic alpha unpredictable function, Ψ is an operator on the space of bounded and continuous on $\mathbb{R}$ functions such that $\Psi G(t,v)$ is an almost periodic function in t uniformly with respect to v, and $\Psi G(u,t)$ is alpha unpredictable function uniformly with respect to u.*

It can be of a scientific interest to consider the following definitions of operational irregularity.

Definition 2.5. *A function $z(t) : \mathbb{R} \to \mathbb{R}^n$ is said to be operational periodic alpha unpredictable function, if $z(t) = \Psi g(t)$, where $g(t) = G(t,t)$, is a compartmental periodic alpha unpredictable function, Ψ is an operator on the space of bounded and continuous on $\mathbb{R}$ functions such that $\Psi G(t,v)$ is an periodic function in t uniformly with respect to v, and $\Psi G(u,t)$ is alpha unpredictable function uniformly with respect to u.*

Definition 2.6. *A function $z(t) : \mathbb{R} \to \mathbb{R}^n$ is said to be operational quasi-periodic alpha unpredictable function, if $z(t) = \Psi g(t)$, where $g(t) = G(t,t)$, is a compartmental quasi-periodic alpha unpredictable function, Ψ is an operator on the space of bounded and continuous on $\mathbb{R}$ functions such that $\Psi G(t,v)$ is an quasi-periodic function in t uniformly with respect to v, and $\Psi G(u,t)$ is alpha unpredictable function uniformly with respect to u.*

The last proposal can be of effective use for constructive realizations of Definition 2.4.

It is clear that the operational dependence on arguments can be viewed as a composition function. However, to highlight the development of new relationships and to focus on the algorithms used to approve these properties, we have chosen to use the term "operational".

Remark 2.3. *Consider an alpha unpredictable function $\psi(t)$ with convergence sequence t_k. For a fixed number $\omega > 0$ there exist a subsequence t_{k_l} and a number τ_ω such that $t_{k_l} \to \tau_\omega (mod\ \omega)$ as $l \to \infty$. In what follows, we shall call the number τ_ω as the Poisson shift for the function $\psi(t)$ with respect to the ω. The set of Poisson shifts $\mathcal{T}_\omega$ is not empty, in general case, it can consist of several or even an infinite number of elements. The number $\kappa_\omega = inf\ \mathcal{T}_\omega, 0 \le \kappa_\omega < \omega$, is said to be Poisson number for the function $\psi(t)$ with respect to the number ω. We say that the sequence t_k satisfies kappa property with respect to the number ω if $\kappa_\omega = 0$.*

In [1, 6] *it has been proved that under the kappa property there are quasilinear systems, which operational periodic alpha unpredictable solutions are compartmental periodic alpha unpredictable ones, if the ω is the period. It is of strong interest to find conditions, similar to kappa property, such that operational quasi or almost periodic solutions are alpha unpredictable.*

3. Linear systems

The main object of the present section is the system of linear differential equations,

$$x'(t) = A(t)x + g(t), \tag{3.4}$$

where $t \in \mathbb{R}, x \in \mathbb{R}^n$, n is a fixed natural number; $A(t)$ is $n-$dimensional square matrix; $g(t) : \mathbb{R} \to \mathbb{R}^n$ is uniformly continuous and bounded function.

We assume that the following conditions are satisfied.

- **(C1)** the matrix $A(t)$ is an $\omega-$periodic;
- **(C2)** the function $g(t)$ is compartmental almost periodic alpha unpredictable in the sense of Definition 2.3, $g(t) = G(t,t)$, and $\sup_{t\in\mathbb{R}} \|G(t,t)\| = M$;
- **(C3)** the convergence sequence t_k of function $g(t)$ satisfies the kappa property with respect to the period ω.

Consider the homogeneous system, associated with (3.4),

$$x'(t) = A(t)x(t). \tag{3.5}$$

Let $X(t)$, $t \in \mathbb{R}$, be the fundamental matrix of the system (3.5) such that $X(0) = I$, and I is the $n \times n$ identical matrix. Moreover, $X(t,s)$ is the transition matrix, such that $X(t + \omega, s + \omega) = X(t,s)$ for all $t, s \in \mathbb{R}$.

Assume that the following assumption is valid.

- **(C4)** The eigenvalues $\rho_i, i = 1, 2, \ldots, n$, of the monodromy matrix $X(\omega)$ in modulus are less than one.

It follows from the last condition that there exist positive numbers K and α such that

$$\|X(t,s)\| \le Ke^{-\alpha(t-s)}, \tag{3.6}$$

for $t \ge s$ [19].

The next lemma is necessary for further reasoning.

Lemma 3.1. [4, 5] *If the inequality (3.6) is satisfied, then the following estimation is correct*

$$\|X(t+\tau, s+\tau) - X(t,s)\| \le \max_{t\in\mathbb{R}} \|A(t+\tau) - A(t)\| \frac{2K^2}{\alpha e} e^{-\frac{\alpha}{2}(t-s)}, \tag{3.7}$$

for $t \ge s$ and arbitrary real number τ.

Proof. Since

$$\frac{dX(t+\tau,s+\tau)}{dt} = A(t)X(t+\tau,s+\tau) + (A(t+\tau)-A(t))X(t+\tau,s+\tau),$$

we have that

$$X(t+\tau,s+\tau) = X(t,s) + \int_s^t X(t,u)(A(u+\tau)-A(u))X(u+\tau,s+\tau)du.$$

That is why,

$$\begin{aligned}
&\|X(t+\tau,s+\tau)-X(t,s)\| \le \\
&\int_s^t \|X(t,u)\|\|A(u+\tau)-A(u)\|\|X(u+\tau,s+\tau)\|du \le \\
&\max_{t\in\mathbb{R}}\|A(t+\tau)-A(t)\|\int_s^t K^2 e^{-\alpha(t-s)}du = \\
&\max_{t\in\mathbb{R}}\|A(t+\tau)-A(t)\|K^2 e^{-\alpha(t-s)}(t-s) = \\
&\max_{t\in\mathbb{R}}\|A(t+\tau)-A(t)\|K^2 e^{-\frac{\alpha}{2}(t-s)}e^{-\frac{\alpha}{2}(t-s)}(t-s).
\end{aligned}$$

Since $\sup_{u\ge 0} e^{-\frac{\alpha}{2}u}u = \dfrac{2}{\alpha e}$, the lemma is proved. □

Theorem 3.1. *If conditions (C1)-(C4) are valid, then the system (4.22) possesses a unique asymptotically stable operational almost periodic alpha unpredictable solution.*

Proof. There exists the unique asymptotically stable bounded solution for system (3.4) as the following integral expression [16, 18, 21, 22, 25],

$$z(t) = \int_{-\infty}^t X(t,s)g(s)ds,\ t\in\mathbb{R}. \tag{3.8}$$

Introduce the following functional operator, to stay in the circumstances of Definition 2.4,

$$\Psi h(t) = \int_{-\infty}^t X(t,s)h(s)ds,\ t\in\mathbb{R}, \tag{3.9}$$

where $h(t)$ is a continuous bounded function.

In what follows, to prove that the solution $z(t)$ in (3.8) is operational almost periodic alpha unpredictable function. We verify that

$$z_1(t,v) = \Psi G(t,v) = \int_{-\infty}^t X(t,s)G(s,v)ds,\ t\in\mathbb{R}, \tag{3.10}$$

is almost periodic function in t uniformly in v, and the function

$$z_2(u,t) = \Psi G(u,t) = \int_{-\infty}^t X(t,s)G(u,s)ds,\ t\in\mathbb{R}, \tag{3.11}$$

is alpha unpredictable in t, uniformly in u.

Summarizing the last two properties, the solution $z(t)$ is operational almost periodic alpha unpredictable function in the sense of Definition 2.4.

a). Firstly, let us show that the function $z_1(t,v)$ is almost periodic. According to Remark 2.1, for a fixed positive ϵ the matrix $A(t)$ and function $G(t,v)$ have a common relatively dense set of ϵ-almost periods, uniformly for all v. Fix τ, one of the almost periods, to get

$$\|z_1(t+\tau,v)-z_1(t,v)\| = \|\int_{-\infty}^{t} X(t+\tau,s+\tau)G(s+\tau,v) - \int_{-\infty}^{t} X(t,s)G(s,v)ds\| \leq$$
$$\int_{-\infty}^{t}\Big(\|X(t+\tau,s+\tau)-X(t,s)\|G(s+\tau,v)\| + \|X(t,s)\|\|G(s+\tau,v)-G(s,v)\|\Big)ds <$$
$$\int_{-\infty}^{t}\Big(\epsilon\frac{2K^2}{\alpha e}e^{-\frac{\alpha}{2}(t-s)}M + Ke^{-\alpha(t-s)}\epsilon\Big)ds \leq \epsilon\frac{4K^2}{\alpha^2 e}M + \frac{K\epsilon}{\alpha} \leq \epsilon\Big(\frac{4K^2}{\alpha^2 e}M + \frac{K}{\alpha}\Big),$$

for all $t \in \mathbb{R}$. Therefore, $z_1(t,v)$ is almost periodic function.

b). Next, we will prove that the function $z_2(u,t)$ is alpha unpredictable in t. Applying the method of included intervals [7], we will approve that $z_2(u,t+t_k) \to z_2(u,t)$ as $k \to \infty$, uniformly on compact subsets of $\mathbb{R}$ and all v.

Let us fix a positive ϵ and an interval $[a,b]$, $-\infty < a < b < \infty$. There exist numbers c, ξ such that $c < a$ and $\xi > 0$, which satisfy the following inequalities:

$$\frac{4K^2\xi}{\alpha^2 e}M < \frac{\epsilon}{3}, \tag{3.12}$$

$$\frac{2K}{\alpha}Me^{-\alpha(a-c)} < \frac{\epsilon}{3}, \tag{3.13}$$

and

$$\frac{K\xi}{\alpha}[1-e^{-\alpha(b-c)}] < \frac{\epsilon}{3}. \tag{3.14}$$

According to condition (C3), one can assume without lost of generality that $\|A(t+t_k) - A(t)\| < \xi$, and $\|G(u,t+t_k) - G(u,t)\| < \xi$ for $t \in [c,b]$ for sufficiently large k. Now, by applying Lemma 3.1, we get that

$$\|z_2(u,t+t_k)-z_2(u,t)\| =$$
$$\|\int_{-\infty}^{t} X(t+t_k,s+t_k)G(u,s+t_k) - \int_{-\infty}^{t} X(t,s)G(u,s)ds\| \leq$$
$$\int_{-\infty}^{t}\Big(\|X(t+t_k,s+t_k)-X(t,s)\|G(u,s+t_k)\|ds + \|X(t,s)\|\|G(u,s+t_k)-G(u,s)\|\Big)ds \leq$$
$$\int_{-\infty}^{t}\|X(t+t_k,s+t_k)-X(t,s)\|G(u,s+t_k)\|ds + \int_{-\infty}^{c}\|X(t,s)\|\|G(u,s+t_k)-G(u,s)\|ds +$$
$$\int_{c}^{t}\|X(t,s)\|\|G(u,s+t_k)-G(u,s)\|ds \leq \int_{-\infty}^{t}\frac{2K^2\xi}{\alpha e}e^{-\frac{\alpha}{2}(t-s)}Mds +$$
$$\int_{-\infty}^{c}2Ke^{-\alpha(t-s)}Mds + \int_{c}^{t}Ke^{-\alpha(t-s)}\xi ds \leq$$
$$\frac{4K^2\xi}{\alpha^2 e}M + \frac{2K}{\alpha}Me^{-\alpha(a-c)} + \frac{K\xi}{\alpha}[1-e^{-\alpha(b-c)}],$$

for all $t \in [a,b]$. From inequalities (3.12) to (3.14) it follows that $\|z_2(u,t+t_k)-z_2(u,t)\| < \epsilon$ for $t \in [a,b]$. Therefore, $z_2(u,t+t_k)$ uniformly converges to $z_2(u,t)$ on bounded intervals of $\mathbb{R}$. Thus, the convergence property of function $z_2(u,t)$ is shown.

Now, let us prove the separation property for the function. According to condition (C2) $G(u,t)$ is alpha unpredictable in t, and therefore exist positive numbers δ and ϵ_0 such that

$\|G(u,t+t_k) - G(u,t)\| > \epsilon_0$ for $t \in [s_k - \delta, s_k + \delta]$. Denote $K_0 = \inf\{\|X(t,s)\| : t \geq s\}$. The number K_0 is positive, since of the periodicity and uniqueness of solutions true for the homogeneous system. One can find numbers $l \in \mathbb{N}$ and $\delta_1 > 0$ which satisfy the following inequalities

$$\delta_1 < \delta, \tag{3.15}$$

$$\|X(t+t_k, s+t_k) - X(t,s)\| \leq \frac{\epsilon_0}{l}, t \in \mathbb{R}, |s| < \delta_1, \tag{3.16}$$

$$\delta_1 (K_0 - \frac{M}{l}) > \frac{3}{2l}, \tag{3.17}$$

$$\|z_2(u,t+s) - z_2(u,t)\| \leq \frac{\epsilon_0}{4l}, t \in \mathbb{R}, |s| < \delta_1, \tag{3.18}$$

Assume that the numbers l, δ_1 and $k \in \mathbb{N}$ are fixed. Denote by Δ the value of $\|z_2(u, s_k + t_k) - z_2(u, s_k)\|$, and consider two alternative cases: (1) $\Delta \geq \frac{\epsilon_0}{l}$, and (2) $\Delta < \frac{\epsilon_0}{l}$.

(1) If $\Delta \geq \frac{\epsilon_0}{l}$, it is easily find, from (3.18), that

$$\|z_2(u,t+t_k) - z_2(u,t)\| \geq \|z_2(u,s_k+t_k) - z_2(u,s_k)\| - \|z_2(u,s_k) - z_2(u,t)\| -$$
$$\|z_2(u,t+t_k) - z_2(u,s_k+t_k)\| > \frac{\epsilon_0}{l} - \frac{\epsilon_0}{4l} - \frac{\epsilon_0}{4l} = \frac{\epsilon_0}{2l},$$

for $t \in [s_k - \delta_1, s_k + \delta_1]$.

(2) Applying the relation

$$z_2(u,t+t_k) - z_2(u,t) = z_2(u,s_k+t_k) - z_2(u,s_k) +$$
$$\int_{s_k}^{t} X(t+t_k, s+t_k) G(u, s+t_k) - \int_{s_k}^{t} X(t,s) G(u,s) ds = z(s_k + t_k) - z(s_k) +$$
$$\int_{s_k}^{t} [X(t+t_k, s+t_k) - X(t,s)] G(u, s+t_k) ds +$$
$$\int_{s_k}^{t} X(t,s) [G(u,s+t_k) - G(u,s)] ds,$$

and inequalities (3.15)-(3.18), one can obtain that

$$\|z_2(u,t+t_k) - z_2(u,t)\| \geq \int_{s_k}^{t} \|X(t,s)\| \|G(u,s+t_k) - G(u,s)\| ds -$$
$$\|z_2(u,s_k+t_k) - z_2(u,s_k)\| - \int_{s_k}^{t} \|X(t+t_k,s+t_k) - X(t,s)\| \|G(u,s+t_k)\| ds \geq$$
$$\int_{s_k}^{t} K_0 \epsilon_0 ds - \frac{\epsilon_0}{l} - \int_{s_k}^{t} \frac{\epsilon_0}{l} M ds \geq \delta_1 K_0 \epsilon_0 - \frac{\epsilon_0}{l} - \delta_1 \frac{\epsilon_0}{l} M > \frac{\epsilon_0}{2l},$$

for each $t \in [s_k, s_k + \delta_1]$. Thus, the function $z_2(u,t)$ is alpha unpredictable. The results in parts a) and b) imply that the bounded solution $z(t)$ is asymptotically stable operational almost periodic alpha unpredictable solution, which satisfies Definition 2.4.

□

3.1. **Numerical examples.** In this part of the paper, we shall provide several concrete linear systems, which satisfy the conditions of Theorem 3.1. Solutions of the models are operational almost periodic alpha unpredictable functions. In fact, they are operational quasi-periodic solutions, but it is known that the last one almost periodic functions. This is why, one can accept that the present suggestions, indeed support the theoretical observations of this section.

In what follows the piecewise constant function $\Omega(t)$, which described in Section 2, will be defined for $t \in [hi, h(i+1))$, where $i \in \mathbb{Z}$, and h is a positive real number. The number h is said to be *the length of step* of the functions $\Omega(t)$ and $\Theta(t)$. The ratio of the period and the length of step, $\nabla = \omega/h$, we call *the degree of periodicity*. In the book [4] and papers [5, 6, 8], the dependence of the dynamics of the *compartmental periodic alpha unpredictable* functions on the degree of periodicity was considered. Now, we will show how the degree of periodicity can affect shape of the graphs trajectory of *operational almost periodic alpha unpredictable solutions*.

Example 3.1. *Consider the following system,*

$$\begin{aligned} x_1' &= (-2+\sin(t))x_1 + 0.24\Theta(t)(0.2\sin(2t)+\sin(\sqrt{2}t)) \\ x_2' &= (-4+\cos(2t))x_2 + 0.25\Theta(t)(0.1\cos(t)-\sin(\sqrt{3}t)), \end{aligned} \tag{3.19}$$

where $\Theta(t)$ is the alpha unpredictable function with the step of length $h = 6\pi$, described above in (2.2),

$$A(t) = \begin{pmatrix} -2+\sin(t) & 0 \\ 0 & -4+\cos(2t) \end{pmatrix},$$

$$g(t) = \begin{pmatrix} 0.24\Theta(t)(0.2\sin(2t)+\sin(\sqrt{2}t)) \\ 0.25\Theta(t)(0.1\cos(t)-\sin(\sqrt{3}t)) \end{pmatrix}.$$

The matrix $A(t)$ is $2\pi-$periodic, and the degree of periodicity ∇ is equal to $1/3$. Condition (C4) holds with multipliers $\rho_1 = e^{-4\pi}$ and $\rho_2 = e^{-8\pi}$. Figure 2 demonstrates the time series of the coordinates $x_1(t)$, $x_2(t)$ of the solution $x(t)$ of the system (3.19) with initial values $x_1(0) = 0$, $x_2(0) = 0.1$, which asymptotically approximates the unique operational almost periodic alpha unpredictable solution of the system. Since the degree is less than one, it is evident that alpha unpredictability dominates over alpha periodicity in the solution. This confirms that the operational relations of these properties are consistent with those observed in our previous papers on compartmental solutions.

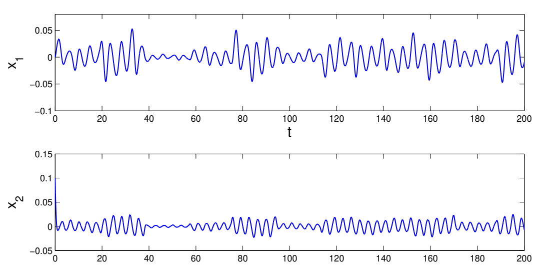

FIGURE 2. The coordinates of the solution $x(t)$ of system (3.19), which asymptotically converge to those of the operational almost periodic alpha unpredictable solution.

Example 3.2. *Consider the system, where matrix $A(t)$ is $2-$periodic*

$$\begin{aligned} x_1' &= (-2+\sin(\pi t))x_1 + 0.24\Theta(t)(0.2\sin(2\pi t)+\sin(\sqrt{2}\pi t)) \\ x_2' &= (-4+\cos(2\pi t))x_2 + 0.25\Theta(t)(0.1\cos(\pi t)-\sin(\sqrt{3}\pi t)), \end{aligned} \tag{3.20}$$

and alpha unpredictable function $\Theta(t)$ with the step of length $h = 2$. The degree of periodicity is equals to 1. The multipliers $\rho_1 = e^{-2}$ and $\rho_2 = e^{-8}$ of the homogeneous system, associated *with (3.20), are satisfy to the condition (C4). In Figure 3 shown the coordinates $x_1(t)$, $x_2(t)$ of the solution $x(t)$ of the system (3.20) with initial data $x_1(0) = 0$, $x_2(0) = 0.1$. Since the degree is less than one, it is clear that alpha unpredictability takes precedence over alpha periodicity in the solution. In other words, one can not recognize almost periodicity through the graph. This confirms that the operational relationships of these properties align with the compartmental ones observed in our previous papers* [6].

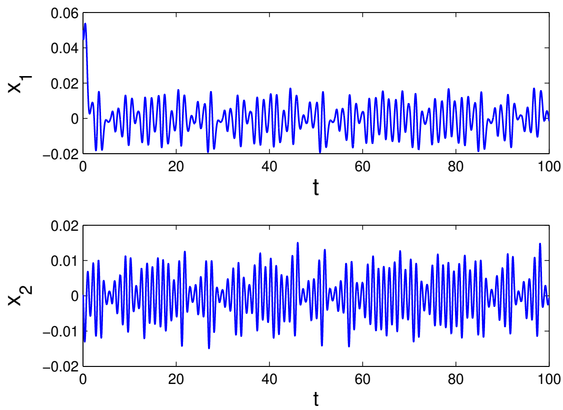

FIGURE 3. The coordinates of the solution $x(t)$ of system (3.20), when the degree of periodicity is 1, and there is no domination, neither alpha unpredictability nor almost periodicity.

Example 3.3. *In the following system*

$$\begin{aligned} x_1' &= (-2 + \sin(0.1t))x_1 + 0.24\Theta(t)(0.2\sin(0.2t) + \sin(0.2\sqrt{2}t)) \\ x_2' &= (-4 + \cos(0.2t))x_2 + 0.25\Theta(t)(0.1\cos(0.1t) - \sin(0.2\sqrt{3}t)), \end{aligned} \tag{3.21}$$

the matrix $A(t)$ is $20-$periodic and alpha unpredictable function $\Theta(t)$ with the step of length $h = 0.1\pi$. The degree of periodicity $\nabla = 200/\pi$. The coordinates of the solution $x(t)$ of the system (3.21) with initial data $x_1(0) = 0$, $x_2(0) = 0.1$ are illustrated in Figure 4. As the degree is large one can expect that almost periodicity will be evidently seen in the visual presentation of the dynamics, and the alpha unpredictability could not mask the almost periodicity.

4. QUASILINEAR SYSTEMS

Let us commence with the following definitions.

Definition 4.7. [7] *A bounded function $f(t,x) : \mathbb{R} \times D \to \mathbb{R}^n$, $D \subset \mathbb{R}^n$ is a domain, is said to be alpha unpredictable in t, uniformly, with respect to $x \in D$, if there exist positive numbers ϵ_0, δ and sequences t_k, s_k, both of which diverge to infinity, such that $\sup_D \|f(t + t_k, x) - f(t,x)\| \to 0$ as $k \to \infty$ uniformly on bounded intervals of t and $x \in D$, and $\|f(t + t_k, x) - f(t,x)\| > \epsilon_0$ for $t \in [s_k - \delta, s_k + \delta]$, $x \in D$ and $k \in \mathbb{N}$.*

Definition 4.8. [8] *A function $f(t,x) : \mathbb{R} \times D \to \mathbb{R}^n$, $D \subset \mathbb{R}^n$ is an open and bounded set, is said to be compartmental almost periodic alpha unpredictable in t uniformly for x function, if $f(t,x) = G(t,t,x)$, where $G(u,v,x)$ is a continuous bounded function, almost periodic in u uniformly with respect to arguments v, x, and alpha unpredictable in v uniformly with respect to u and x.*

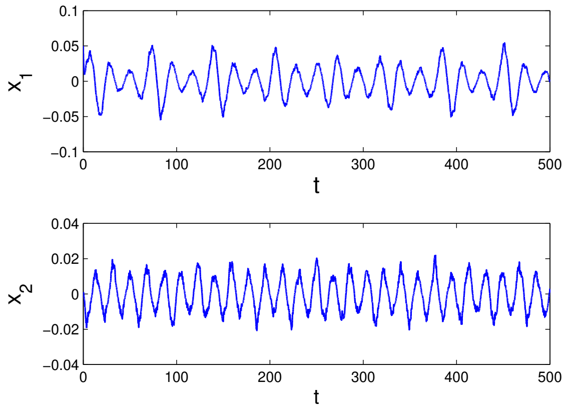

FIGURE 4. The coordinates of the solution $x(t)$ of system (3.21), which asymptotically approximates those of the operational almost periodic alpha unpredictable solution.

In this section we consider quasilinear differential equation

$$x'(t) = A(t)x + g(t, x), \tag{4.22}$$

where $t \in \mathbb{R}, x \in \mathbb{R}^n$, n is a fixed natural number; $A(t)$ is $n-$dimensional square matrix; $g : \mathbb{R} \times D \to \mathbb{R}^n$, $D = \{x \in \mathbb{R}^n, \|x\| < H\}$, where H is a fixed positive number.

The following conditions are needed

- **(C5)** the function $g(t, x)$ is compartmental almost periodic alpha unpredictable in the sense of Definition 4.8, such that $g(t, x) = G(t, t, x)$;
- **(C6)** The sequence of convergence $t_k, k = 1, 2, \dots,$ admits kappa property with respect to the period ω of matrix $A(t)$;
- **(C7)** $\sup\limits_{\mathbb{R}\times\mathbb{R}\times D} \|G(t, s, x)\| = M$;
- **(C8)** there exists a positive constant L such that $\|G(t, s, x_1) - G(t, s, x_2)\| \le L \|x_1 - x_2\|$ for all $t, s \in \mathbb{R}, x_1, x_2 \in D$;
- **(C9)** $\dfrac{KM}{H} < \alpha$;
- **(C10)** $KL < \alpha$,

According to [18, 19, 21, 22], a bounded on the real axis function $y(t)$ is a solution of (4.22), if and only if it satisfies the equation

$$y(t) = \int_{-\infty}^{t} X(t, s)g(s, y(s))ds,\ t \in \mathbb{R}, \tag{4.23}$$

where $X(t, s)$ is the transition matrix of the system (3.5).

In what follows, we denote $\mathcal{C}(\mathcal{R})$ the set of all uniformly continuous and bounded on $\mathbb{R}$ functions with the norm $\|\psi(t)\|_0 = \sup_{t\in\mathbb{R}} \|\psi(t)\|$.

Lemma 4.2. *Suppose that conditions(C1), (C4)-(C10) are valid, then the system (4.22) possesses a unique bounded asymptotically stable solution.*

Proof. Define on $\mathcal{C}(\mathcal{R})$ an operator $\Psi g(t, \phi(t))$ such that

$$\Psi g(t, \phi(t)) = \int_{-\infty}^{t} X(t, s)g(s, \phi(s))ds,\ t \in \mathbb{R}. \tag{4.24}$$

The function $\Psi g(t, \phi(t))$ is a uniformly continuous since its derivative is a bounded on the real axis. Fix an arbitrary function $\phi(t)$ that belongs to $\mathcal{C}(\mathcal{R})$. For all $t \in \mathbb{R}$, we have

$$\|\Psi g(t, \phi(t))\| \leq \int_{-\infty}^{t} \|X(t,s)\| \|g(s, \phi(s))\| ds \leq \int_{-\infty}^{t} K e^{-\alpha(t-s)} M ds \leq \frac{KM}{\alpha}.$$

Therefore, according to condition (C9) it is true that $\|\Psi g(t, \phi(t))\| < H, t \in \mathbb{R}$. Summarizing the above discussion, the set $\mathcal{C}(\mathcal{R})$ is invariant for the operator.

We proceed to show that the operator $\Psi g(t, \phi(t)) : \mathcal{C}(\mathcal{R}) \to \mathcal{C}(\mathcal{R})$ is contractive. Let $u(t)$ and $v(t)$ be members of $\mathcal{C}(\mathcal{R})$. Then, we obtain that

$$\|\Psi g(t, u(t)) - \Psi g(t, v(t))\| \leq \int_{-\infty}^{t} \|X(t,s)\| \|g(s, u(s)) - g(s, v(s))\| ds =$$

$$\int_{-\infty}^{t} K e^{-\alpha(t-s)} L \|u(s) - v(s)\| ds \leq \frac{KL}{\alpha} \|u(t) - v(t)\|_0,$$

for all $t \in \mathbb{R}$. Thus, $\|\Psi g(t, u(t)) - \Psi g(t, v(t))\|_0 < \frac{KL}{\alpha} \|u - v\|_0$, and by condition (C10) the operator $\Psi g(t, \phi(t)) : \mathcal{C}(\mathcal{R}) \to \mathcal{C}(\mathcal{R})$ is contractive.

According to the contraction mapping theorem there exists the unique fixed point, $\psi(t) \in \mathcal{C}(\mathcal{R})$, of the operator Ψ, which is the unique bounded solution of the system (4.22).

Finally, we will consider asymptotic stability of the solution $\psi(t)$ of system (4.22). It is true that

$$\psi(t) = X(t, t_0)\psi(t_0) + \int_{t_0}^{t} X(t,s) g(s, \psi(s))) ds,$$

for $t \geq t_0$.

Let φ be another solution of system (4.22). One can write that

$$\varphi(t) = X(t, t_0)\varphi(t_0) + \int_{t_0}^{t} X(t,s) g(s, \varphi(s)) ds.$$

Subtraction of two solutions gives us that

$$\|\psi(t) - \varphi(t)\| = \|X(t, t_0)\| \|\psi(t_0) - \varphi(t_0)\| + \int_{t_0}^{t} \|X(t,s)\| \|g(s, \psi(s)) - g(s, \varphi(s))\| ds \leq$$

$$K e^{-\alpha(t-t_0)} \|\psi(t_0) - \varphi(t_0)\| + \int_{t_0}^{t} K e^{-\alpha(t-s)} L \|\psi(s) - \varphi(s)\| ds.$$

Applying Gronwall-Bellman Lemma, one can get that

$$\|\psi(t) - \varphi(t)\| \leq K e^{-(\alpha - KL)(t-t_0)} \|\psi(t_0) - \varphi(t_0)\|, t \geq t_0.$$

The last inequality and condition (C10) confirm that the solution $\psi(t)$ is asymptotically stable. The lemma is proved. □

In what follows, we shall consider conditions for the solution $y(t)$ to be an operational almost periodic alpha unpredictable function.

For this reason, let us adapt Definition 2.4 to the quasilinear case.

Definition 4.9. *A function $z(t) : \mathbb{R} \to \mathbb{R}^n$ is said to be operational almost periodic alpha unpredictable function, if it uniquely satisfies $z(t) = \Psi g(t, z(t))$, where $g(t, x) = G(t, t, x)$, is a compartmental almost periodic alpha unpredictable in t function, Ψ is an operator on the space of bounded and continuous on $\mathbb{R}$ functions such that there exists a unique function $z(t, v)$ which satisfies $z(t, v) = \Psi G(t, v, z_1(t, v))$, and almost periodic in t uniformly with respect to v, x, and a unique function $z(u, t)$ which satisfies $z(u, t) = \Psi G(u, t, z_2(u, t))$, and alpha unpredictable uniformly with respect to u, x.*

Theorem 4.2. *If conditions (C1),(C4)-(C10) are valid, then the system (4.22) possesses a unique asymptotically stable operational almost periodic alpha unpredictable solution.*

Proof. Denote by $\mathcal{A}$ the set of almost periodic functions $\phi(t,v) : \mathbb{R} \to \mathbb{R}^n$, with common $\epsilon-$ periods τ. Fix a function $y(t,v)$ that belongs to $\mathcal{B}$.

Define on $\mathcal{A}$ the operator $\Psi\phi(t,v)$ such that

$$\Psi G(t,v,\phi(t,v)) = \int_{-\infty}^{t} X(t,s)G(s,v,\phi(s,v))ds,\ t \in \mathbb{R}. \tag{4.25}$$

Conditions of the theorem and Lemma 4.2 imply that there exists a unique bounded fixed point, $z(t,v)$, of the operator. To prove that the function is almost periodic, it is sufficient to show that the $\Psi G(t,v,y(t,v))$ is almost periodic in t for each function $y(t,v)$ in $\mathcal{A}$.

Using inequalities (3.6),(3.7) and condition (C8) we obtain that

$$\begin{aligned}
&\|\Psi G(t,v,y(t+\tau,v)) - \Psi G(t,v,y(t,v))\| = \\
&\| \int_{-\infty}^{t} X(t+\tau,s+\tau)G(s+\tau,v,y(s+\tau,v)) - \int_{-\infty}^{t} X(t,s)G(s,v,y(s,v))ds\| \le \\
&\int_{-\infty}^{t} \Big(\|X(t+\tau,s+\tau) - X(t,s)\|G(s+\tau,v,y(s+\tau,v))\| + \\
&\|X(t,s)\|\|G(s+\tau,v,y(s+\tau,v)) - G(s+\tau,v,y(s,v))\| + \\
&\|X(t,s)\|\|G(s+\tau,v,y(s,v)) - G(s,v,y(s,v))\|\Big)ds \le \\
&\int_{-\infty}^{t} \Big(\epsilon\frac{2K^2}{\alpha e}e^{-\frac{\alpha}{2}(t-s)}M_g + Ke^{-\alpha(t-s)}L\epsilon + Ke^{-\alpha(t-s)}\epsilon\Big)ds \\
&\le \epsilon\frac{4K^2}{\alpha^2 e}M_g + \frac{KL\epsilon}{\alpha} + \frac{K\epsilon}{\alpha} \le \epsilon\Big(\frac{4K^2}{\alpha^2 e}M_g + \frac{KL}{\alpha} + \frac{K}{\alpha}\Big).
\end{aligned}$$

Thus the function $\Psi y(t,v)$ is almost periodic in t.

Let t_k is the convergence sequence of the function $G(u,t,x)$. We denote by $\mathcal{B}$ the set of functions $\psi(u,t) : \mathbb{R} \to \mathbb{R}^n$, satisfy convergence property with common sequence t_k, and $\|\psi\|_0 < H$.

Define the operator $\Psi G(u,t,\psi(u,t))$ on $\mathcal{B}$ such that

$$\Psi G(u,t,\psi(u,t)) = \int_{-\infty}^{t} X(t,s)G(u,s,\psi(u,s))ds,\ t \in \mathbb{R}. \tag{4.26}$$

Duo to conditions of the theorem and Lemma 4.2 there exists a unique bounded fixed point, $y(t)$, for the operator. To prove that it is a Poisson stable function, it is sufficient to check that $\Psi G(u,t,y(u,t))$ is from $\mathcal{B}$ provided $y(u,t)$ belongs to the set.

Let us fix a positive number ϵ and an interval $[a,b]$, $-\infty < a < b < \infty$.

There exist numbers c, ξ such that $c < a$ and $\xi > 0$, which satisfy the following inequalities:

$$\Big(\frac{4K^2}{\alpha^2 e}M_g + \frac{2KM_g}{\alpha}\Big)e^{-\alpha(a-c)} < \frac{\epsilon}{2}, \tag{4.27}$$

and

$$\Big(\frac{4K^2}{\alpha^2 e}\xi + \frac{K\xi(1+L)}{\alpha}\Big)e^{-\alpha(b-c)} < \frac{\epsilon}{2}. \tag{4.28}$$

According to condition (C5), it is true that $\|A(t+t_k)-A(t)\| < \xi$, and $\|G(u,t+t_k)-G(u,t)\| < \xi$ for $t \in [c,b]$ for sufficiently large k. Next, we have that

$$\|\Psi G(u,t,y(u,t+t_k)) - \Psi G(u,t,y(u,t))\| =$$
$$\|\int_{-\infty}^{t} X(t+t_k,s+t_k)G(u,s+t_k,y(u,s+t_k))ds - \int_{-\infty}^{t} X(t,s)G(u,s,y(s))ds\| \leq$$
$$\int_{-\infty}^{t}\Big(\|X(t+t_k,s+t_k)-X(t,s)\|\|G(u,s+t_k,y(u,s+t_k))\| +$$
$$\|X(t,s)\|\|G(u,s+t_k,y(u,s+t_k)) - G(u,s+t_k,y(u,s))\| +$$
$$\|X(t,s)\|\|G(u,s+t_k,y(u,s)) - G(u,s,y(u,s))\|\Big)ds.$$

Consider the last integral on two intervals $(-\infty,c)$ and $[c,t]$. Using inequalities (4.27) and (4.28) we get that

$$I_1 = \int_{-\infty}^{c}\Big(\|X(t+t_k,s+t_k)-X(t,s)\|\|G(u,s+t_k,y(u,s+t_k))\| +$$
$$\|X(t,s)\|\|G(u,s+t_k,y(u,s+t_k)) - G(u,s+t_k,y(u,s))\| +$$
$$\|X(t,s)\|\|G(u,s+t_k,y(u,s)) - G(u,s,y(u,s))\|\Big)ds \leq$$
$$\int_{-\infty}^{c}\Big(\frac{2K^2}{\alpha e}e^{-\frac{\alpha}{2}(t-s)}M_g + Ke^{-\alpha(t-s)}(2M_g+2M_g)\Big)ds \leq$$
$$\Big(\frac{4K^2}{\alpha^2 e}M_g + \frac{2KM_g}{\alpha}\Big)e^{-\alpha(a-c)} < \frac{\epsilon}{2},$$

and

$$I_2 = \int_{c}^{t}\Big(\|X(t+t_k,s+t_k)-X(t,s)\|\|G(u,s+t_k,y(u,s+t_k))\| +$$
$$\|X(t,s)\|\|G(u,s+t_k,y(u,s+t_k)) - G(u,s+t_k,y(u,s))\| +$$
$$\|X(t,s)\|\|G(u,s+t_k,y(u,s)) - G(u,s,y(u,s))\|\Big)ds \leq$$
$$\int_{c}^{t}\Big(\frac{2K^2}{\alpha e}e^{-\frac{\alpha}{2}(t-s)}\xi + Ke^{-\alpha(t-s)}(\xi+L\xi)\Big)ds \leq$$
$$\Big(\frac{4K^2}{\alpha^2 e}\xi + \frac{K\xi(1+L)}{\alpha}\Big)e^{-\alpha(b-c)} < \frac{\epsilon}{2},$$

for all $t \in [a,b]$. From inequalities (4.27) to (4.28) it follows that $\|\Psi G(u,t,y(u,t+t_k)) - \Psi G(u,t,y(u,t))\| < \epsilon$ for $t \in [a,b]$. Therefore, $\Psi G(u,t,y(u,t+t_k))$ uniformly converges to $\Psi G(u,t,y(u,t))$ on the bounded interval of $\mathbb{R}$.

Let us prove the separation property of the function $y(u,t)$. According to condition (C5), the function $G(u,t,x)$ is unpredictable in t, and there exists positive numbers δ and ϵ_0 such that $\|G(u,t+t_k,x) - G(u,t,x)\| > \epsilon_0$ for $t \in [s_k-\delta, s_k+\delta]$. Denote $K_0 = \inf\{\|X(t,s) : t > s\}\|$. One can find numbers $l \in \mathbb{N}$ and $\delta_1 > 0$ which satisfy the following inequalities

$$\delta_1 < \delta, \tag{4.29}$$

$$\|X(t+t_k,s+t_k)-X(t,s)\| \leq \frac{\epsilon_0}{l}, t\in\mathbb{R}, |s| < \delta_1, \tag{4.30}$$

$$\delta_1(K_0 - \frac{M_g}{l} - \frac{KL}{\alpha}) > \frac{3}{2l}, \tag{4.31}$$

$$\|y(u,t+s)-y(u,t)\| \le \frac{\epsilon_0}{4l}, t\in\mathbb{R}, |s|<\delta_1 y \tag{4.32}$$

Assume that the numbers l, δ_1 and $k \in \mathbb{N}$ are fixed. Denote by Δ the value of $\|y(u,s_k+t_k)-y(u,s_k)\|$, and consider two alternative cases: (1) $\Delta \ge \frac{\epsilon_0}{l}$, and (2) $\Delta < \frac{\epsilon_0}{l}$.

(1) If $\Delta \ge \frac{\epsilon_0}{l}$, it is easily find, from (4.32), that

$$\|y(u,t+t_k)-y(u,t)\| \ge \|y(u,s_k+t_k)-y(u,s_k)\| - \|y(u,s_k)-y(u,t)\| - \|y(u,t+t_k)-y(u,s_k+t_k)\| > \frac{\epsilon_0}{l} - \frac{\epsilon_0}{4l} - \frac{\epsilon_0}{4l} = \frac{\epsilon_0}{2l},$$

for $t \in [s_k-\delta_1, s_k+\delta_1]$.

(2) Applying the relation

$$\begin{aligned}
&y(u,t+t_k)-y(u,t) = y(u,s_k+t_k)-y(u,s_k)+\\
&\int_{s_k}^{t} X(t+t_k,s+t_k)G(u,s+t_k,y(u,s+t_k))ds-\\
&\int_{s_k}^{t} X(t,s)G(u,s,y(u,s))ds = y(u,s_k+t_k)-y(u,s_k)+\\
&\int_{s_k}^{t} [X(t+t_k,s+t_k)-X(t,s)]G(u,s+t_k,y(u,s+t_k))ds+\\
&\int_{s_k}^{t} X(t,s)[G(u,s+t_k,y(u,s+t_k))-G(u,s,y(u,s+t_k))]ds+\\
&\int_{s_k}^{t} X(t,s)[G(u,s,y(u,s+t_k))-G(u,s,y(u,s))]ds,
\end{aligned}$$

and inequalities (4.29)-(4.32), one can obtain that

$$\begin{aligned}
&\|y(u,t+t_k)-y(u,t)\| \ge \int_{s_k}^{t} \|X(t,s)\|\|G(u,s+t_k,y(u,s+t_k))-G(u,s,y(u,s+t_k))\|ds-\\
&\int_{s_k}^{t} \|X(t+t_k,s+t_k)-X(t,s)\|\|G(u,s+t_k,y(u,s+t_k))\|ds-\\
&\int_{s_k}^{t} \|X(t,s)[G(u,s,y(u,s+t_k))-G(u,s,y(u,s))]\|ds - \|y(u,s_k+t_k)-y(u,s_k)\| >\\
&\int_{s_k}^{t} K_0\epsilon_0 ds - \int_{s_k}^{t} \frac{M_g\epsilon_0}{l}ds - \int_{s_k}^{t} Ke^{-\alpha(t-s)}L\epsilon_0 ds - \frac{\epsilon_0}{l} \ge\\
&\delta_1 K_0\epsilon_0 - \delta_1\frac{M_g\epsilon_0}{l} - \delta_1\frac{KL\epsilon_0}{\alpha} - \frac{\epsilon_0}{l} > \frac{\epsilon_0}{2l},
\end{aligned}$$

for each $t \in [s_k, s_k+\delta_1]$. Thus, the function $y(t)$ satisfies the separation property and is alpha unpredictable. Following the Definition 4.9 one can conclude that the solution $y(t)$ of the system (4.22) is operational almost periodic alpha unpredictable function. □

4.1. **Numerical examples.**

Example 4.4. *Consider the following quasilinear system,*

$$\begin{aligned}
x_1' &= (-4+\sin(2t))x_1 + 0.4\Theta(t)(0.2\cos(t)+0.3\sin(\sqrt{5}t)) + \arctan(x_2)\\
x_2' &= (-3+\cos(t))x_2 + 0.6\Theta(t)(0.1\cos(2t)-\cos(\sqrt{2}t)) + 2\arctan(x_1),
\end{aligned} \tag{4.33}$$

where $\Theta(t)$ is the alpha unpredictable function $\Theta(t)$ with the step of length $h = 8\pi$, described above in (2.2),

$$A(t) = \begin{pmatrix} -4+\sin(2t) & 0 \\ 0 & -3+\cos(t) \end{pmatrix},$$

$$g(t) = \begin{pmatrix} 0.4\Theta(t)(0.2\cos(t) + 0.3\sin(\sqrt{5}t)) + \arctan(x_2) \\ 0.6\Theta(t)(0.1\cos(2t) - \cos(\sqrt{2}t)) + 2\arctan(x_1) \end{pmatrix}.$$

The matrix $A(t)$ is $2\pi-$periodic, the degree of periodicity $\nabla = 0.25$. All conditions of the Theorem 4.2 are satisfied with $L = 2$, $K = 1$, $\rho_1 = e^{-8\pi}$, $\rho_2 = e^{-4\pi}$, $\alpha = 4\pi$, $M = 2.02$ and $H = 0.5$. The coordinates of the solution $x(t)$ for the system (4.33) with the initial conditions $x_1(0) = 0$ and $x_2(0) = 0$ are illustrated in Figure 5. The degree of unpredictability is less than one, which supports the claim made in [6] *that alpha unpredictability dominates over regularity, specifically with regard to almost periodicity. In other words, the concept of characteristic implementation, which observes the contributions of regularity and chaos in individual motions, proves to be effective not only in the periodic case but also for almost periodic components.*

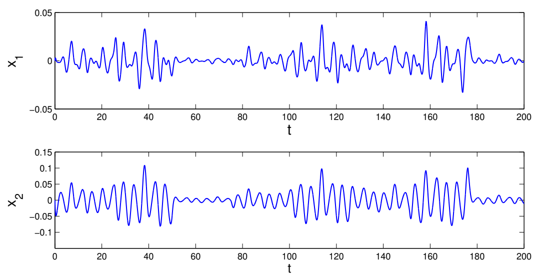

FIGURE 5. The coordinates of the solution $x(t)$ of system (4.33), which asymptotically converge to the coordinates of the operational almost periodic alpha unpredictable solution.

Example 4.5. *In the next quasilinear system,*

$$\begin{aligned} x_1' &= (-4+\sin(2\pi t))x_1 + 0.4\Theta(t)(0.2\cos(\pi t) + 0.3\sin(\sqrt{5}\pi t)) + \arctan(x_2) \\ x_2' &= (-3+\cos(\pi t))x_2 + 0.6\Theta(t)(0.1\cos(2\pi t) - \cos(\sqrt{2}\pi t)) + 2\arctan(x_1), \end{aligned} \tag{4.34}$$

the matrix $A(t)$ is $2\pi-$periodic, the degree of periodicity $\nabla = 1$. The coordinates of the solution $x(t)$ of the system (4.33) with the initial conditions $x_1(0) = 0$ and $x_2(0) = 0$, which asymptotically converge to the coordinates of operational almost periodic alpha unpredictable solution are illustrated in Figure 6.

Example 4.6. *In the following system,*

$$\begin{aligned} x_1' &= (-4+\sin(0.2t))x_1 + 0.4\Theta(t)(0.2\cos(0.1t) + 0.3\sin(0.1\sqrt{5}t)) + \arctan(x_2) \\ x_2' &= (-3+\cos(0.1t))x_2 + 0.6\Theta(t)(0.1\cos(0.2t) - \cos(0.2\sqrt{2}t)) + 2\arctan(x_1), \end{aligned} \tag{4.35}$$

The matrix $A(t)$ is 20π-periodic, with a periodicity degree of $\nabla = 200$. Figure 7 shows the time series of the coordinates $x_1(t)$ and $x_2(t)$ for the solution $x(t)$ of the system outlined in (3.19), given the initial conditions $x_1(0) = 0$ and $x_2(0) = 0$. Due to the high degree of periodicity, the almost periodicity (or quasiperiodicity) remains evident and is not overshadowed by chaotic behaviour.

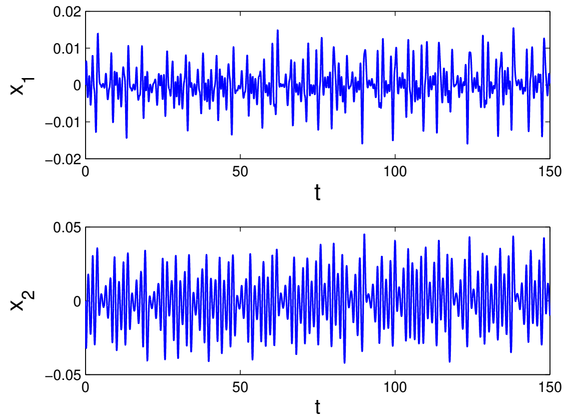

FIGURE 6. The coordinates of the solution $x(t)$ of system (4.34).

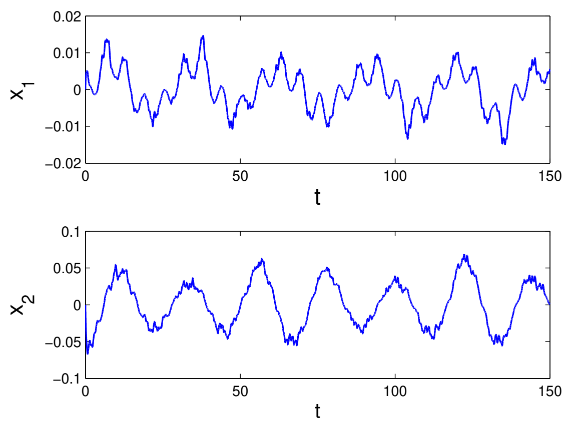

FIGURE 7. The time series of the coordinates $x_1(t)$ and $x_2(t)$ of the solution $x(t)$ of system (4.35), which asymptotically converge to the coordinates of operational almost periodic alpha unpredictable solution.

5. CONCLUSION

The demands of modern science and technology can be addressed through functions that describe the motions of models, combining constructive properties with sophisticated behaviors. These needs have expanded to include the analysis of brain activity and advancements in artificial intelligence, particularly in machine learning and deep learning.

This study focuses on operational definitions of functions, emphasizing that the foundational aspects of activity depend on a detailed structural description of algorithms. The approach is centered around almost periodic and alpha unpredictable functions, defined operationally. It utilizes the experience of quasi-periodicity derived by diagonalization of periodicity on several arguments. Accordingly, one can be guided by this in the further development of the theory.

To say concretely, operational almost periodic and alpha unpredictable components within the solutions of linear and quasi-linear differential equations have been identified. These components create a functional dependence on the diagonal that could significantly enhance research on complexity in many challenging problems. The theoretical novelties are presented analytically and illustrated graphically and numerically by specific dynamics.

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[1] Middle East Technical University
Department of Mathematics
06800, Ankara, Turkey
Email address: marat@metu.edu.tr
Email address: akylbek78@mail.ru

[2] K.Zhubanov Aktobe Regional University
Department of Mathematics
030000, Aktobe, Kazakhstan