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Iterative criteria for oscillation of third-order linear delay dynamic equations

G. N. CHHATRIA[1], S. R. GRACE[2], AND I. JADLOVSKÁ[3]

ABSTRACT. We establish new oscillation criteria for a linear third-order functional dynamic equation on time scales. Our approach employs an iterative process combined with comparison principles to improve and generalize existing results. We demonstrate our findings through illustrative examples.

1. INTRODUCTION

The calculus on time scales, introduced by Stefan Hilger in his seminal 1988 dissertation and subsequently published in 1990 [23], provides a unified framework for both continuous and discrete analysis, opening the space for additional discrete analogues in different less standard calculi. The monographs by Bohner and Peterson [7, 8] comprehensively summarize and organize much of the foundational theory on time scales.

In this work, we investigate the oscillatory and asymptotic behavior of solutions to the third-order linear delay dynamic equation

$$\left(r_2(\nu)\left(r_1(\nu)w^{\Delta}(\nu)\right)^{\Delta}\right)^{\Delta}+q(\nu)w(\tau(\nu))=0. \tag{1.1}$$

Let $\mathbb{T}$ be a time scale, i.e., a nonempty closed subset of reals, with $\sup\mathbb{T}=\infty$, and consider $\nu\in[\nu_0,\infty)_{\mathbb{T}}:=[\nu_0,\infty)\cap\mathbb{T}$ for some $\nu_0\in\mathbb{T}$. A function $w(\nu)\in \mathrm{C}_{rd}([\nu_0,\infty)_{\mathbb{T}})$ is said to be a solution of (1.1) if $r_1(\nu)w^{\Delta}(\nu)\in \mathrm{C}^1_{rd}([\nu_0,\infty)_{\mathbb{T}})$, $r_2(\nu)(r_1(\nu)w^{\Delta}(\nu))\in \mathrm{C}^1_{rd}([\nu_0,\infty)_{\mathbb{T}})$, and w satisfies (1.1) on $[T_w,\infty)_{\mathbb{T}}$ for some $T_w>\nu_0$. The solution is termed *oscillatory* if it is neither eventually positive nor eventually negative; otherwise, it is classified as *nonoscillatory*.

Throughout the paper, we impose the following conditions:

($\mathcal{A}_1$) $r_i(\nu), q(\nu)\in \mathrm{C}_{rd}\left([\nu_0,\infty)_{\mathbb{T}},\mathbb{R}_+\right)$ for $i=1,2$ and

$$\int_{\nu_0}^{\infty}\frac{\Delta s}{r_i(s)}=\infty,\quad i=1,2; \tag{1.2}$$

($\mathcal{A}_2$) $\tau\in \mathrm{C}^1_{rd}\left([\nu_0,\infty)_{\mathbb{T}},\mathbb{T}\right)$ with $\tau(\nu)\le\nu$, $\tau^{\Delta}(\nu)\ge 0$ and $\lim_{\nu\to\infty}\tau(\nu)=\infty$.

Remark 1.1. *For simplicity, we put*

$$\mathcal{L}_0(w(\nu))=w(\nu),\quad \mathcal{L}_i(w(\nu))=r_i(\nu)\Big(\mathcal{L}_{i-1}(w(\nu))\Big)^{\Delta}\quad \textit{for } i=1,2,$$

and

$$\mathcal{L}_3(w(\nu))=\Big(\mathcal{L}_2(w(\nu))\Big)^{\Delta},$$

so that (1.1) *can be rewritten in the more compact form*

$$\mathcal{L}_3(w(\nu))+q(\nu)w(\tau(\nu))=0\quad \textit{for } \nu\ge\nu_0. \tag{1.3}$$

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Corresponding author: I. Jadlovská; jadlovska@saske.sk

To establish a basis for the concepts discussed in this paper, we first present the relevant background. Third-order dynamic equations are fundamental in modeling a wide range of phenomena in applied science. These equations find applications in fluid dynamics [21], thermal explosion theory [27], and many other areas, as outlined in the monographs [6, 13, 28, 31] and research articles [26]. In recent years, significant progress has been made in investigating the oscillatory behavior and asymptotic properties of third-order dynamic equations, as evidenced by the expanding literature [1, 3, 10, 11, 12, 15, 18, 19, 22, 25, 30, 32].

In [29], Saker et al. investigated third-order nonlinear delay difference equations of the form

$$\Delta\left(r_2(\nu)\Delta\left(r_1(\nu)\Delta w(\nu)\right)^{\gamma}\right)+q(\nu)f(w(\nu-k)))=0,\ \nu\geq\nu_0, \tag{1.4}$$

where $\gamma>0$ is a quotient of odd positive integers, the functions $r_i(\nu)$ (for $i=1,2$) satisfy (1.2), $k\in\mathbb{N}$ and $\frac{f(u)}{u^{\gamma}}\geq K>0$ for $u\neq 0$. By employing a generalized Riccati transformation technique, they established that every solution $w(\nu)$ of (1.5) either oscillates or converges to zero.

Subsequently, Han et al. [20] considered the third-order nonlinear delay dynamic equation

$$\left(r_2(\nu)\left[\left(r_1(\nu)\left(w^{\Delta}(\nu)\right)\right)^{\Delta}\right]\right)^{\Delta}+q(\nu)f(w(\tau(\nu)))=0,\ \nu\geq\nu_0, \tag{1.5}$$

where the functions $r_i(\nu)$ (for $i=1,2$) again satisfy (1.2) and $\frac{f(u)}{u}\geq K>0$ for $u\neq 0$. Utilizing both the Riccati transformation and integral averaging techniques, they proved the following result.

Theorem A. *Assume $r_1^{\Delta}(\nu)\leq 0$. If there exists a function $\delta(\nu)\in C^1_{rd}([\nu_0,\infty)_{\mathbb{T}},\mathbb{R}_+)$ such that*

$$\limsup_{\nu\to\infty}\int_{\nu_0}^{\nu}\left(\delta(\sigma(s))q(s)\frac{\tau(s)}{\sigma(s)}-\frac{r_1(s)\sigma(s)(\delta^{\Delta}(s))^2}{4Ks\delta(\sigma(s))\rho(s)}\right)\Delta s=\infty$$

and

$$\int_{\nu_0}^{\infty}\frac{1}{r_1(\nu)}\int_{\nu_0}^{\nu}\frac{1}{r_2(s)}\int_{\nu_0}^{s}q(u)\Delta u\Delta s\Delta\nu=\infty,$$

where $\rho(\nu)=\int_{\nu_0}^{\nu}\frac{1}{r_2(s)}\Delta s$, then every solution $w(\nu)$ of (1.5) *either oscillates or converges to zero.*

In [14], Grace and Chhatria investigated third-order nonlinear delay dynamic equations of the form

$$\left(r(\nu)\left(w^{\Delta\Delta}(\nu)\right)^{\alpha}\right)^{\Delta}+q(\nu)w^{\alpha}(\tau(\nu))=0,\ \nu\geq\nu_0, \tag{1.6}$$

where $\alpha\geq 1$ is a quotient of positive odd integers. By employing linearisation, the comparison method, and the Riccati transformation technique, they established the following result.

Theorem B. *Assume there exists a nondecreasing function $\eta(\nu)\in C_{rd}([\nu_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that*

$$\eta(\nu)>\nu,\ \eta(\tau(\nu))>\tau(\nu)\ \text{and}\ \eta(\eta(\tau(\nu)))<\nu. \tag{1.7}$$

If the first-order delay dynamic equations

$$\mathcal{Y}^{\Delta}(\nu)+\frac{1}{\alpha}q(\nu)\left(\int_{\nu_1}^{\tau(\nu)}\mathcal{R}(s,\nu_1)\Delta s\right)^{\alpha}\mathcal{Y}(\tau(\nu))=0$$

and

$$\mathcal{W}^{\Delta}(\nu)+\frac{1}{\alpha}q(\nu)\left(\int_{\tau(\nu)}^{\eta(\tau(\nu))}\mathcal{R}(\eta(s),s)\Delta s\right)^{\alpha}\mathcal{W}(\eta(\eta(\tau(\nu))))=0$$

are oscillatory for $\nu \geq \nu_1 > \nu_0$, *where* $\mathcal{R}(\nu,\nu_1)=\int_{\nu_1}^{\nu}\frac{\Delta s}{r^{1/\alpha}(s)}$ *and* $\mathcal{R}(\eta(\nu),\nu)=\int_{\nu}^{\eta(\nu)}\frac{\Delta s}{r^{1/\alpha}(s)}$, *then every solution of* (1.6) *is oscillatory.*

We make several observations related to (1.1):

(1) Let $\mathbb{T}=\mathbb{Z}$, so that $w^{\Delta}(\nu)=\Delta w(\nu)$. Setting $\tau(\nu)=\nu-k$ for $k\in\mathbb{N}$, equation (1.1) coincides with (1.4) for $\gamma=1$ and $f(u)=u$. In our context, [29, Lemma 5] plays an important role, showing that

$$\mathcal{L}_1(w(\nu-k))\geq\mathcal{L}_2(w(\nu))\left(\sum_{s=\nu_0}^{\nu-k-1}\frac{1}{r_2(s)}\right)$$

for $w\in C_2$.

It is not difficult to see from the preceding inequality that

$$w(\nu-k)\geq\mathcal{L}_2(w(\nu-k))\left(\sum_{t=\nu_0}^{\nu-1}\frac{1}{r_1(t)}\sum_{s=\nu_0}^{t}\frac{1}{r_2(s)}\right). \tag{1.8}$$

(2) When $f(u)=u$, (1.5) and (1.1) coincide. In this case, the results crucially rely on a lower bound

$$w^{\Delta}(\nu)\geq\mathcal{L}_2(w(\nu))\frac{1}{r_1(\nu)}\int_{\nu_1}^{\nu}\frac{1}{r_2(s)}\Delta s,$$

which further implies

$$w(\nu)\geq\mathcal{L}_2(w(\nu))\int_{\nu_*}^{\nu}\frac{1}{r_1(t)}\int_{\nu_*}^{t}\frac{1}{r_2(s)}\Delta s\Delta t \tag{1.9}$$

for $w\in C_2$.

(3) When $\alpha=1$, then (1.6) is a particular case of (1.1). We observe that the analysis of the authors relies on the crucial estimates

$$w(\nu)\geq(r(\nu)w^{\Delta\Delta}(\nu))\int_{\nu_1}^{\nu}\mathcal{R}(s,\nu_1)\Delta s \tag{1.10}$$

for $w\in C_2$ and

$$w(\nu)\geq(r(\eta(\eta(\nu)))w^{\Delta\Delta}(\eta(\eta(\nu))))\int_{\nu}^{\eta(\nu)}\mathcal{R}(\eta(s),s)\Delta s \tag{1.11}$$

for $w\in C_0$.

The purpose of this paper is to employ iterative techniques to sharpen previously established estimates such as those in (1.9) and (1.10) for the case $w\in C_2$ and those in (1.11) when $w\in C_0$ in the context of (1.1). These improvements, combined with suitable comparison principles, allow us to formulate oscillation criteria for (1.1) on arbitrary time scales. Our methodology builds on the approach proposed in [10], where the authors investigated the continuous analogue $\mathbb{T}=\mathbb{R}$ of the studied problem, i.e., the third-order delay differential equation. However, a notable gap remains in the literature regarding the discrete analogues and dynamic extensions of these results. Here, we address this gap by establishing sufficient conditions for the oscillation and asymptotic behavior of all solutions to (1.1) on arbitrary time scales, thereby improving and generalizing earlier results.

Remark 1.2. *In the sequel, we make the assumption that $\nu \geq \nu_{**}$ for some sufficiently large $\nu > \nu_*$ means there exists $\nu_{**} > \nu_*$ such that $\nu \in [\nu_{**}, \infty)_{\mathbb{T}} \subset [\nu_*, \infty)_{\mathbb{T}}$.*

2. Auxiliary Lemmas

In this section, we present several definitions and preliminary results from the calculus on time scales that will be used in the subsequent analysis. For further details, we refer the reader to [7, 8, 13].

Definition 2.1. *A function $p : \mathbb{T} \to \mathbb{R}$ is said to be* regressive *(denoted by $p \in \mathcal{R}$) provided that*

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all } t \in \mathbb{T}.$$

Furthermore, $p : \mathbb{T} \to \mathbb{R}$ is called positive regressive *(denoted by $p \in \mathcal{R}^+$) if it is rd-continuous and satisfies*

$$1 + \mu(t)p(t) > 0 \quad \text{for all } t \in \mathbb{T}.$$

Definition 2.2. *If $p \in \mathcal{R}$, then the exponential function $e_p(t, s)$ is defined by*

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(u)}\big(p(u)\big)\Delta u\right),$$

where the cylindrical transformation $\xi_h(z)$ is given by

$$\xi_h(z) = \begin{cases} z, & \text{if } h = 0, \\ \dfrac{1}{h}\log(1 + zh), & \text{if } h > 0. \end{cases}$$

In the following, we provide some specific examples of generalized exponential functions on different time scales:

(1) **The real numbers.** If $\mathbb{T} = \mathbb{R}$ and p is continuous, then

$$e_p(t, s) = \exp\left\{\int_s^t p(u)\,du\right\} \quad \text{and} \quad e_a(t, s) = e^{a(t-s)} \quad \text{for constant } a.$$

(2) **The integers.** If $\mathbb{T} = \mathbb{Z}$ and $p(t) \neq -1$, then

$$e_p(t, s) = \begin{cases} \displaystyle\prod_{u=s}^{t-1}[1 + p(u)], & \text{if } t > s, \\ 1, & \text{if } t = s, \\ \dfrac{1}{\prod_{u=t}^{s-1}[1 + p(u)]}, & \text{if } t < s. \end{cases}$$

For a constant $a \neq -1$, we have

$$e_a(t, s) = (1 + a)^{t-s}.$$

(3) **The q-numbers.** If $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, then

$$e_p(q^k, 1) = \prod_{u=0}^{k-1}\Big[1 + (q-1)q^u p(q^u)\Big] \quad \text{for all } k \in \mathbb{N}.$$

(4) **A general time scale.** Let $\{a_k\}_{k\in\mathbb{N}}$ be a sequence of positive real numbers and let $t_0 \in \mathbb{R}$. Define

$$t_k = t_0 + \sum_{u=1}^{k} a_u \quad \text{for } k \in \mathbb{N}.$$

Then

$$e_p(t_k, t_0) = \prod_{u=1}^{k} \Big[1 + a_u p(t_{u-1})\Big] \quad \text{for all } k \in \mathbb{N}_0.$$

Lemma 2.1 (See [9, Lemma 1]). *Let p be rd-continuous and nonnegative such that $-p \in \mathcal{R}^+$, and let $\nu_0, \nu \in \mathbb{T}$ with $\nu \geq \nu_0$. If*

$$w^{\Delta}(\nu) + p(\nu)\, w(\nu) \leq 0,$$

then

$$w(\nu) \leq e_{-p}(\nu, \nu_0) w(\nu_0).$$

Lemma 2.2 (See [15, Lemma 1]). *If w is an eventually positive solution of Eq.* (1.1) *for $\nu \geq \nu_0$, then there exists $\nu_1 > \nu_0$ such that*

$$w \in C_0 \cup C_2,$$

where

$$C_0 = \{w : \mathcal{L}_1(w(\nu)) < 0,\ \mathcal{L}_2(w(\nu)) > 0,\ \mathcal{L}_3(w(\nu)) < 0 \text{ for } \nu \geq \nu_1\},$$
$$C_2 = \{w : \mathcal{L}_1(w(\nu)) > 0,\ \mathcal{L}_2(w(\nu)) > 0,\ \mathcal{L}_3(w(\nu)) < 0 \text{ for } \nu \geq \nu_1\}.$$

We are now ready to present our first result, which establishes the connection between w and $\mathcal{L}_2(w)$ under the assumption that the solution belongs to the class C_2. For this purpose, we define

$$R_{m+1}(\nu) = \int_{\nu_*}^{\nu} \frac{1}{r_1(t)} \int_{\nu_*}^{t} \frac{e_{-\widehat{q}_m}(s, \nu)}{r_2(s)} \Delta s \Delta t \tag{2.12}$$

with

$$R_1(\nu) = \int_{\nu_*}^{\nu} \frac{1}{r_1(t)} \int_{\nu_*}^{t} \frac{1}{r_2(s)} \Delta s \Delta t \quad \text{and} \quad \widehat{q}_m(\nu) = q(\nu) R_m(\tau(\nu))$$

for $\nu \geq \nu_*$ and $m \in \mathbb{N}$.

Lemma 2.3. *Let* (2.12) *hold. If w is a solution of* (1.1) *satisfying (C_2) of Lemma 2.2, then*

$$w(\tau(\nu)) \geq R_m(\tau(\nu)) \mathcal{L}_2(w(\tau(\nu))) \tag{2.13}$$

for sufficiently large $\nu \geq \nu_*$.

Proof. Suppose that w is a nonoscillatory solution of Eq. (1.1), which belongs to (C_2) for $\nu \geq \nu_1$ for some $\nu_1 > \nu_0$. We will prove (2.13) by induction on m. Indeed, observe that

$$\mathcal{L}_1(w(\nu)) - \mathcal{L}_1(w(\nu_1)) = \int_{\nu_1}^{\nu} (\mathcal{L}_1(w(s)))^{\Delta} \Delta s = \int_{\nu_1}^{\nu} \frac{\mathcal{L}_2(w(s))}{r_2(s)} \Delta s, \tag{2.14}$$

that is,

$$r_1(\nu) w^{\Delta}(\nu) := \mathcal{L}_1(w(\nu) \geq \mathcal{L}_2(w(\nu)) \int_{\nu_1}^{\nu} \frac{1}{r_2(s)} \Delta s,$$

so that

$$w^{\Delta}(\nu) \geq \mathcal{L}_2(w(\nu)) \frac{1}{r_1(\nu)} \int_{\nu_1}^{\nu} \frac{1}{r_2(s)} \Delta s.$$

Integrating the above inequality from ν_1 to ν yields

$$w(\nu) \geq w(\nu) - w(\nu_1) \geq \mathcal{L}_2(w(\nu)) \int_{\nu_1}^{\nu} \frac{1}{r_1(u)} \int_{\nu_1}^{u} \frac{1}{r_2(s)} \Delta s \Delta u \geq R_1(\nu) \mathcal{L}_2(w(\nu)),$$

or, equivalently,

$$w(\tau(\nu)) \geq R_1(\tau(\nu)) \mathcal{L}_2(w(\tau(\nu))). \tag{2.15}$$

Thus, (2.13) holds for $m = 1$.

Next, assume that (2.13) holds for some $m > 1$; that is,

$$w(\tau(\nu)) \geq R_m(\tau(\nu))\mathcal{L}_2(w(\tau(\nu))) \text{ for } \nu \geq \nu_2 > \nu_1. \tag{2.16}$$

Using (2.16) in Eq. (1.3), we obtain

$$\mathcal{L}_3(w(\nu)) + q(\nu)R_m(\tau(\nu))\mathcal{L}_2(w(\tau(\nu)) \leq 0.$$

Setting $Z(\nu) = \mathcal{L}_2(w(\nu))$ and using the fact $Z(\nu)$ is non-increasing, we get

$$Z^{\Delta}(\nu) + \widehat{q}_m(\nu)Z(\nu) \leq 0. \tag{2.17}$$

Note that

$$\begin{aligned} 0 \geq Z^{\Delta}(\nu) + \widehat{q}_m(\nu)Z(\nu) &= Z(\sigma(\nu)) - Z(\nu) + \mu(\nu)\widehat{q}_m(\nu)Z(\nu) \\ &> -Z(\nu) + \mu(\nu)\widehat{q}_m(\nu)Z(\nu) = -Z(\nu)\left[1 - \mu(\nu)\widehat{q}_m(\nu)\right]. \end{aligned}$$

Consequently, the condition $1 - \mu(\nu)\widehat{q}_m(\nu) > 0$ ensures that $-\widehat{q}_m(\nu) \in \mathcal{R}^+$. Thus, applying Lemma 2.1 to (2.17) yields that

$$Z(\nu) \leq e_{-\widehat{q}_m}(\nu, s)Z(s) \quad \text{for } \nu \geq s > \nu_2,$$

that is,

$$Z(s) \geq Z(\nu)e_{-\widehat{q}_m}(s, \nu) \quad \text{for } \nu \geq s > \nu_2,$$

or,

$$\mathcal{L}_2(w(s)) \geq \mathcal{L}_2(w(\nu))e_{-\widehat{q}_m}(s, \nu). \tag{2.18}$$

Using (2.18) in (2.14), we get

$$\mathcal{L}_1(w(\nu)) \geq \mathcal{L}_2(w(\nu)) \int_{\nu_2}^{\nu} \frac{e_{-\widehat{q}_m}(s, \nu)}{r_2(s)} \Delta s,$$

or, equivalently,

$$w^{\Delta}(\nu) \geq \frac{\mathcal{L}_2(w(\nu))}{r_1(\nu)} \int_{\nu_2}^{\nu} \frac{e_{-\widehat{q}_m}(s, \nu)}{r_2(s)} \Delta s.$$

Integrating the above inequality from ν_2 to ν and using (2.18), we get

$$\begin{aligned} w(\nu) &\geq \int_{\nu_2}^{\nu} \frac{\mathcal{L}_2(w(t))}{r_1(t)} \int_{\nu_2}^{t} \frac{e_{-\widehat{q}_m}(s, t)}{r_2(s)} \Delta s \Delta t \\ &\geq \mathcal{L}_2(w(\nu)) \int_{\nu_2}^{\nu} \frac{e_{-\widehat{q}_m}(t, \nu)}{r_1(t)} \int_{\nu_2}^{t} \frac{e_{-\widehat{q}_m}(s, t)}{r_2(s)} \Delta s \Delta t \\ &= \mathcal{L}_2(w(\nu)) \int_{\nu_2}^{\nu} \frac{1}{r_1(t)} \int_{\nu_2}^{t} \frac{e_{-\widehat{q}_m}(s, \nu)}{r_2(s)} \Delta s \Delta t. \end{aligned}$$

Thus, there exists $\nu_3 > \nu_2$ such that $\nu \geq \nu_3$ and

$$w(\tau(\nu)) \geq \mathcal{L}_2(w(\tau(\nu))) \int_{\nu_3}^{\tau(\nu)} \frac{1}{r_1(t)} \int_{\nu_3}^{t} \frac{e_{-\widehat{q}_m}(s, \tau(\nu))}{r_2(s)} \Delta s \Delta t.$$

Hence,

$$w(\tau(\nu)) \geq \mathcal{L}_2(w(\tau(\nu)))R_{m+1}(\tau(\nu)).$$

This completes the induction step and thus the proof. □

Remark 2.3. *We note that for $m = 1$, inequality* (2.13) *simplifies to inequality* (1.9).

In the next result, we establish the connection between w and $\mathcal{L}_2(w)$ under the assumption that the solution lies within the class C_0. For this purpose, we define

$$\widehat{R}_{n+1}(v,u) = \int_u^v \frac{1}{r_1(t)} \int_t^v \frac{e_{-\widehat{q}_n^*}(s,v)}{r_2(s)} \Delta s \Delta t \tag{2.19}$$

with

$$\widehat{R}_1(v,u) = \int_u^v \frac{1}{r_1(t)} \int_t^v \frac{1}{r_2(s)} \Delta s \Delta t \quad \text{and} \quad \widehat{q}_n^*(\nu) = q(\nu)\widehat{R}_n(\nu,\tau(\nu))$$

for $v \geq u \geq \nu_*$ and $n \in \mathbb{N}$.

Lemma 2.4. *Let* (2.19) *hold. If $w(\nu)$ is any nonoscillatory solution of* (1.1) *satisfying (C_0) of Lemma 2.2, then*

$$w(u) \geq \mathcal{L}_2(w(v))\widehat{R}_n(v,u) \tag{2.20}$$

for sufficiently large $v \geq u \geq \nu_*$.

Proof. Suppose that w is a nonoscillatory solution of Eq. (1.1) which belongs to (C_0) for $\nu \geq \nu_1$ for some $\nu_1 > \nu_0$. To prove the desired result, we adopt the method of induction. Indeed, for $v \geq u \geq \nu_1$, we have

$$\mathcal{L}_1(w(v)) - \mathcal{L}_1(w(u)) = \int_u^v \left(\mathcal{L}_1(w(s))\right)^\Delta \Delta s = \int_u^v \frac{\mathcal{L}_2(w(s))}{r_2(s)} \Delta s, \tag{2.21}$$

that is,

$$-r_1(u)w^\Delta(u) = -\mathcal{L}_1(w(u) \geq \mathcal{L}_2(w(v)) \int_u^v \frac{1}{r_2(s)} \Delta s,$$

or, equivalently,

$$-w^\Delta(u) \geq \mathcal{L}_2(w(v)) \frac{1}{r_1(u)} \int_u^v \frac{1}{r_2(s)} \Delta s.$$

By integrating the above inequality from u to v, we get

$$w(u) \geq w(u) - w(v) \geq \mathcal{L}_2(w(v)) \int_u^v \frac{1}{r_1(t)} \int_t^v \frac{1}{r_2(s)} \Delta s \Delta t,$$

or, equivalently,

$$w(u) \geq \mathcal{L}_2(w(v))\widehat{R}_1(v,u). \tag{2.22}$$

Thus, the desired inequality (2.20) holds for $n = 1$. Next, assume that (2.20) holds for $n > 1$, that is,

$$w(u) \geq \mathcal{L}_2(w(v))\widehat{R}_n(v,u) \text{ for } v \geq u \geq \nu_2 > \nu_1. \tag{2.23}$$

Using (2.23) with $u = \tau(\nu)$ and $v = \nu$ in (1.1), we get

$$\mathcal{L}_3(w(\nu)) + \widehat{q}_n^*(\nu)\mathcal{L}_2(w(\nu)) \leq 0.$$

Following the reasoning in the proof of Lemma 2.3, we deduce that

$$\mathcal{L}_2(w(s)) \geq \mathcal{L}_2(w(v))e_{-\widehat{q}_n^*}(s,v) \quad \text{for } v \geq s \geq \nu_2. \tag{2.24}$$

Substituting (2.24) into (2.21) yields

$$-\mathcal{L}_1(w(u)) \geq \int_u^v \frac{\mathcal{L}_2(w(v))}{r_2(s)} e_{-\widehat{q}_n^*}(s,v)\Delta s \geq \mathcal{L}_2(w(v)) \int_u^v \frac{e_{-\widehat{q}_n^*}(s,v)}{r_2(s)} \Delta s,$$

or, equivalently,

$$-w^\Delta(u) \geq \frac{\mathcal{L}_2(w(v))}{r_1(u)} \int_u^v \frac{e_{-\widehat{q}_n^*}(s,v)}{r_2(s)} \Delta s. \tag{2.25}$$

Integrating (2.25) from u to v gives

$$w(u) \geq \mathcal{L}_2(w(v)) \int_u^v \frac{1}{r_1(t)} \int_t^v \frac{e_{-\widehat{q}_n^*}(s,v)}{r_2(s)} \Delta s \Delta t.$$

Thus, we obtain

$$w(u) \geq \mathcal{L}_2(w(v))\, \widehat{R}_{n+1}(v,u),$$

which completes the induction step and thus the proof. □

3. Oscillation Results

Theorem 3.1. *Let $R_m(\nu)$ be defined by* (2.12) *for $m \in \mathbb{N}$. Assume that*

$$\int_{\nu_*}^{\infty} \frac{1}{r_1(t)} \int_t^{\infty} \frac{1}{r_2(v)} \int_v^{\infty} q(s) \Delta s \Delta v \Delta t = \infty. \tag{3.26}$$

If the first-order delay dynamic equation

$$X^{\Delta}(\nu) + q(\nu) R_m(\tau(\nu)) X(\tau(\nu)) = 0 \tag{3.27}$$

is oscillatory, then every solution of (1.1) *either oscillates or satisfies* $\lim_{\nu\to\infty} w(\nu) = 0$.

Proof. Let w be a nonoscillatory solution of Eq. (1.1) such that $w(\nu) > 0$, $w(\tau(\nu)) > 0$ for $\nu \geq \nu_1$ for some $\nu_1 > \nu_0$. By Lemma 2.2, we have $w \in C_0 \cup C_2$.

Case C_0: This case follows from [20, Lemma 2.4]; hence, the details are omitted.

Case C_2: Following the line of proof of Lemma 2.3, we obtain (2.13) for $\nu \geq \nu_2 > \nu_1$. Now, using (2.13) in Eq. (1.3), we deduce that

$$\mathcal{L}_3(w(\nu)) + q(\nu) R_m(\tau(\nu)) \mathcal{L}_2(w(\tau(\nu))) \leq 0. \tag{3.28}$$

Setting $X(\nu) = \mathcal{L}_2(w(\nu))$ in (3.28) yields

$$X^{\Delta}(\nu) + q(\nu) R_m(\tau(\nu)) X(\tau(\nu)) \leq 0.$$

Following [15, Lemma 4 (I)], it follows that the corresponding dynamic equation (3.27) possesses a positive solution, which is a contradiction. This completes the proof. □

By employing the known oscillation criteria for (3.27), we can directly derive an oscillation criterion applicable to Eq. (1.1). The following result, due to Agarwal and Bohner [2, Theorem 1], ensures the oscillatory behavior of (1.1).

Corollary 3.1. *Let* (3.26) *hold and $R_m(\nu)$ be defined by* (2.12) *for $m \in \mathbb{N}$. If*

$$\liminf_{\nu\to\infty} \int_{\tau(\nu)}^{\nu} q(s) R_m(\tau(s)) \Delta s > \frac{1}{e}, \tag{3.29}$$

then every solution of (1.1) *either oscillates or $w(\nu) \to 0$ as $\nu \to \infty$.*

The following result, attributed to Karpuz [24, Theorem 1], formalizes this approach by providing a specific criterion that guarantees the oscillatory behavior of (1.1).

Corollary 3.2. *Let* (3.26) *hold and $R_m(\nu)$ be defined by* (2.12) *for $m \in \mathbb{N}$. If*

$$\liminf_{\nu\to\infty} \inf_{\lambda \in [1,\infty)_{\mathbb{R}}} \left\{ \frac{1}{\lambda} e_{\lambda \widehat{q}_m}(\sigma(\nu), \tau(\nu)) \right\} > 1, \tag{3.30}$$

where $\widehat{q}_m(\nu) = q(\nu) R_m(\tau(\nu))$, then every solution of (1.1) *either oscillates or $w(\nu) \to 0$ as $\nu \to \infty$.*

Theorem 3.2. *Let $R_m(\nu)$ for $m \in \mathbb{N}$ and $\widehat{R}_n(v,u)$ for $n \in \mathbb{N}$ be defined by* (2.12) *and* (2.19)*, respectively. Assume that there exists a constant $\eta \in (1,\infty)_{\mathbb{R}}$ such that $\eta\tau(\nu) \leq \nu$ for $\nu \geq \nu_1$ for sufficiently large $\nu_1 > \nu_0$. If the first-order delay dynamic equations* (3.27) *and*

$$\widehat{X}^{\Delta}(\nu) + q(\nu)\widehat{R}_n(\eta\tau(\nu),\tau(\nu))\widehat{X}(\eta\tau(\nu)) = 0 \tag{3.31}$$

are oscillatory, then every solution of (1.1) *oscillates.*

Proof. Let w be a nonoscillatory solution of Eq. (1.1) such that $w(\nu) > 0$, $w(\tau(\nu)) > 0$ for $\nu \geq \nu_1$ for some $\nu_1 > \nu_0$. By Lemma 2.2, we have $w \in C_0 \cup C_2$.

Case C_0: Following the line of proof of Lemma 2.4, we obtain inequality (2.20) for $\nu \geq \nu_2 > \nu_1$. Now, using (2.20) with $u = \tau(\nu)$ and $v = \eta\tau(\nu)$ in (1.3), we deduce that

$$\mathcal{L}_3(w(\nu)) + q(\nu)\widehat{R}_n(\eta\tau(\nu),\tau(\nu))\mathcal{L}_2(w(\eta\tau(\nu))) \leq 0. \tag{3.32}$$

Setting $\widehat{X}(\nu) = \mathcal{L}_2(w(\nu))$ in (3.32), we have

$$\widehat{X}^{\Delta}(\nu) + q(\nu)\widehat{R}_n(\eta\tau(\nu),\tau(\nu))\widehat{X}(\eta\tau(\nu)) \leq 0.$$

Then, by [15, Lemma 4 (I)], the corresponding dynamic equation (3.31) possesses a positive solution, which is a contradiction.

Case C_2: This case follows from the proof of Theorem 3.1, hence the details are omitted. The proof is complete. □

Theorem 3.3. *Let $R_m(\nu)$ for $m \in \mathbb{N}$ and $\widehat{R}_n(v,u)$ for $n \in \mathbb{N}$ be defined by* (2.12) *and* (2.19)*, respectively. If the first-order delay dynamic equation* (3.27) *is oscillatory, and*

$$\limsup_{\nu\to\infty} \int_{\tau(\nu)}^{\nu} q(s)\widehat{R}_n(\tau(\nu),\tau(s))\Delta s > 1, \tag{3.33}$$

then (1.1) *is oscillatory.*

Proof. Let w be a nonoscillatory solution of Eq. (1.1) such that $w(\nu) > 0$, $w(\tau(\nu)) > 0$ for $\nu \geq \nu_1$ for some $\nu_1 > \nu_0$. By Lemma 2.2, we have $w \in C_0 \cup C_2$.

Case C_0: Integrating (1.1) from $\tau(\nu)$ to ν, we get

$$\mathcal{L}_2(w(\tau(\nu))) \geq \mathcal{L}_2(w(\tau(\nu))) - \mathcal{L}_2(w(\nu)) = \int_{\tau(\nu)}^{\nu} q(s)w(\tau(s))\Delta s.$$

Using (2.20) with $u = \tau(s)$ and $v = \tau(\nu)$ in the preceding inequality, we get

$$\mathcal{L}_2(w(\tau(\nu))) \geq \mathcal{L}_2(w(\tau(\nu))) \int_{\tau(\nu)}^{\nu} q(s)\widehat{R}_n(\tau(\nu),\tau(s))\Delta s,$$

which implies

$$\int_{\tau(\nu)}^{\nu} q(s)\widehat{R}_n(\tau(\nu),\tau(s))\Delta s \leq 1,$$

a contradiction to (3.33).

Case C_2: This case follows from the proof of Theorem 3.1, hence the details are omitted. The proof is complete. □

By choosing particular time scales such as $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{Z}$, and $\mathbb{T}=q^{\mathbb{N}}$, the preceding results yield corresponding oscillation criteria for third-order differential, difference, and q-difference equations, respectively.

We will illustrate the process in the case $\mathbb{T}=\mathbb{Z}$. Then $\sigma(\ell)=\ell+1$, and the delta-integral reduces to a summation:

$$\int_a^b w(s)\Delta s=\sum_{s=a}^{b-1} w(s).$$

Under this setting, (1.1) takes the discrete form:

$$\Delta\left(r_2(\nu)\Delta\left(r_1(\nu)\Delta w(\nu)\right)\right)+q(\nu)w(\tau(\nu))=0, \tag{4.34}$$

so that Lemma 2.3 reduces to the following result:

Lemma 4.5. *If w is any nonoscillatory solution of* (1.1) *satisfying (C_2) of Lemma 2.2, then for sufficiently large $\nu\geq\nu_*$*

$$w(\tau(\nu))\geq R_m(\tau(\nu))\mathcal{L}_2(w(\tau(\nu))), \tag{4.35}$$

where

$$R_{m+1}(\nu)=\sum_{t=\nu_*}^{\nu-1}\frac{1}{r_1(t)}\sum_{s=\nu_*}^{t-1}\frac{1}{r_2(s)}\left(\prod_{u=s}^{\nu-1}[1+\widehat{q}_m(u)]\right)\quad \text{for}\quad \widehat{q}_m(u)\neq-1 \tag{4.36}$$

with

$$R_1(\nu)=\sum_{t=\nu_*}^{\nu-1}\frac{1}{r_1(t)}\left(\sum_{s=\nu_*}^{t-1}\frac{1}{r_2(s)}\right)\quad \text{and}\quad \widehat{q}_m(\nu)=q(\nu)R_m(\tau(\nu))$$

for $\nu\geq\nu_*$ and $m\in\mathbb{N}$.

Remark 4.4. *We may note that when $\tau(\nu)=n-k$ for $k\in\mathbb{N}$ and $m=1$, inequality* (4.35) *reduces to inequality* (1.8). *Obviously, for $m>1$, the inequality* (4.35) *improves* (1.8).

Similarly, we state the discrete version of Lemma 2.4.

Lemma 4.6. *If $w(\nu)$ is any nonoscillatory solution of* (1.1) *satisfying (C_0) of Lemma 2.2, then for sufficiently large $v\geq u\geq\nu_*$*

$$w(u)\geq\mathcal{L}_2(w(v))\widehat{R}_n(v,u), \tag{4.37}$$

where

$$\widehat{R}_{n+1}(v,u)=\sum_{t=u}^{v-1}\frac{1}{r_1(t)}\sum_{s=t}^{v-1}\frac{1}{r_2(s)}\left(\prod_{l=s}^{v-1}[1+\widehat{q}^*_n(l)]\right)\quad \text{for}\quad \widehat{q}^*_n(l)\neq-1 \tag{4.38}$$

with

$$\widehat{R}_1(v,u)=\sum_{t=u}^{v-1}\frac{1}{r_1(t)}\left(\sum_{s=t}^{v-1}\frac{1}{r_2(s)}\right)\quad \text{and}\quad \widehat{q}^*_n(\nu)=q(\nu)\widehat{R}_n(\nu,\tau(\nu))$$

The following are discrete versions of the main results in this paper.

Theorem 4.4. *Let $R_m(\nu)$ be defined by* (4.36) *for $m\in\mathbb{N}$. Assume that*

$$\sum_{t=\nu_*}^{\infty}\frac{1}{r_1(t)}\sum_{v=t}^{\infty}\frac{1}{r_2(v)}\sum_{s=v}^{\infty}q(s)=\infty. \tag{4.39}$$

If the first-order delay difference equation

$$\Delta X(\nu)+q(\nu)R_m(\tau(\nu))X(\tau(\nu))=0 \tag{4.40}$$

is oscillatory, then every solution of (1.1) *either oscillates or satisfies* $\lim_{\nu\to\infty} w(\nu) = 0$.

Corollary 4.3. *Let* (4.39) *hold and* $R_m(\nu)$ *be defined by* (4.36) *for* $m \in \mathbb{N}$. *If*

$$\liminf_{\nu\to\infty} \sum_{s=\tau(\nu)}^{\nu-1} q(s)R_m(\tau(s)) > \frac{1}{e}, \tag{4.41}$$

then every solution of (1.1) *either oscillates or* $w(\nu) \to 0$ *as* $\nu \to \infty$.

Theorem 4.5. *Let* $R_m(\nu)$ *for* $m \in \mathbb{N}$ *and* $\widehat{R}_n(v,u)$ *for* $n \in \mathbb{N}$ *be defined by* (4.36) *and* (4.38), *respectively. Assume that there exists a constant* $\eta \in (1,\infty)$ *such that* $\eta\tau(\nu) \le \nu$ *for* $\nu \ge \nu_1$ *for sufficiently large* $\nu_1 > \nu_0$. *If the first-order delay difference equations* (4.40) *and*

$$\Delta\widehat{X}(\nu) + q(\nu)\widehat{R}_n(\eta\tau(\nu),\tau(\nu))\widehat{X}(\eta\tau(\nu)) = 0 \tag{4.42}$$

are oscillatory, then every solution of (1.1) *oscillates.*

Theorem 4.6. *Let* $R_m(\nu)$ *for* $m \in \mathbb{N}$ *and* $\widehat{R}_n(v,u)$ *for* $n \in \mathbb{N}$ *be defined by* (4.36) *and* (4.38), *respectively. If the first-order delay difference equation* (4.40) *is oscillatory, and*

$$\limsup_{\nu\to\infty} \sum_{s=\tau(\nu)}^{\nu-1} q(s)\widehat{R}_n(\tau(\nu),\tau(s)) > 1, \tag{4.43}$$

then (1.1) *is oscillatory.*

The following example illustrates the applicability of our results in the discrete case.

Example 4.1. *Consider the difference equation of the form*

$$\Delta^3 w(\nu) + q(\nu)Y(\nu-3) = 0, \quad \nu \ge \nu_* = 4, \tag{4.44}$$

where $r_1(\nu) = 1 = r_2(\nu)$, $q(\nu) = \frac{1}{9\nu^2}$ *and* $\tau(\nu) = \nu - 3$. *Here,*

$$\begin{aligned} R_1(\tau(\nu)) &= \sum_{t=\nu_*}^{\nu-3} \frac{1}{r_1(t)} \left(\sum_{s=\nu_*}^{t-1} \frac{1}{r_2(s)} \right) = \sum_{t=4}^{\nu-3} \left(\sum_{s=4}^{t-1} 1 \right) \\ &= \sum_{t=4}^{\nu-3} (t-3) = \frac{(\nu-6)(\nu-5)}{2}. \end{aligned}$$

Now, from (4.41), *we have*

$$\liminf_{\nu\to\infty} \sum_{s=\tau(\nu)}^{\nu-1} q(s)R_1(\tau(s)) = \liminf_{\nu\to\infty} \sum_{s=\nu-5}^{\nu-1} \frac{1}{9s^2} \frac{(s-6)(s-5)}{2} \approx \frac{5}{18} \approx 0.27777 < \frac{1}{e}.$$

Using $\widehat{q}_1(\nu) = q(\nu)R_1(\tau(\nu)) = \frac{(\nu-6)(\nu-5)}{18\nu^2}$, *we get*

$$R_2(\tau(\nu)) = \sum_{t=\nu_*}^{\nu-3} \sum_{s=\nu_*}^{t-1} \left(\prod_{u=s}^{\nu-3} [1+\widehat{q}_1(u)] \right) = \sum_{t=4}^{\nu-3} \sum_{s=4}^{t-1} \left(\prod_{u=s}^{\nu-3} \left[1 + \frac{(u-6)(u-5)}{18u^2} \right] \right)$$

Setting $n = \nu - 3$ *and* $\theta(s,n) = \prod_{u=s}^{n}\left[1 + \frac{(u-6)(u-5)}{18u^2}\right]$, *we have*

$$\begin{aligned}
R_2(\tau(\nu)) &= \sum_{t=4}^{n}\sum_{s=4}^{t-1}\theta(s,n) = \sum_{s=4}^{n-1}\theta(s,n)\left(\sum_{t=s+1}^{n} 1\right) \\
&= \sum_{s=4}^{n-1}(n-s)\theta(s,n) = \sum_{s=4}^{n-1}(n-s)\prod_{u=s}^{n}\left[1 + \frac{(u-6)(u-5)}{18u^2}\right] \\
&= \sum_{s=4}^{n-1}(n-s)\prod_{u=s}^{n}\left[\frac{19u^2 - 11u + 30}{18u^2}\right] \\
&\approx \sum_{s=4}^{n-1}(n-s)\prod_{u=s}^{n}\left[\frac{19u^2 - 11u + 30}{18u^2}\right] \\
&= \sum_{s=4}^{\nu-4}(\nu - 3 - s)\prod_{u=s}^{\nu-3}\left[\frac{19u^2 - 11u + 30}{18u^2}\right].
\end{aligned}$$

It is obvious to observe that

$$\liminf_{\nu\to\infty}\sum_{s=\tau(\nu)}^{\nu-1} q(s)R_2(\tau(s)) > \frac{1}{e}.$$

Therefore, (4.41) *holds for* $m = 2$. *Also, it is easy to check* (4.39) *holds. So, by Corollary 4.3, every solution of* (4.44) *either oscillates or or* $w(\nu) \to 0$ *as* $\nu \to \infty$.

Our second example shows the progress of present results over the existing ones on q-difference equations.

Example 4.2. *Consider the dynamic equation*

$$w^{\Delta^3}(\nu) + \frac{3}{\nu^3} w\left(\frac{\nu}{2}\right) = 0, \tag{4.45}$$

where $\nu \in \mathbb{T} = 2^{\mathbb{N}_0}$ *with* $\nu_* = 1 = q^0$, $r_1(\nu) = r_2(\nu) = 1$, $q(\nu) = \frac{4}{\nu^3}$ *and* $\tau(\nu) = \frac{\nu}{2}$. *If* $\nu = q^K$ *and* $t = q^k$, *then*

$$\begin{aligned}
R_1(\nu) = \int_{\nu_*}^{\nu}\frac{1}{r_1(t)}\int_{\nu_*}^{t}\frac{1}{r_2(s)}\Delta_q s\Delta_q t &= (q-1)^2\sum_{k=0}^{K-1} q^k \sum_{j=0}^{k-1} q^j = (q-1)^2 \sum_{k=k_*}^{K-1} q^k \frac{q^k - 1}{q-1} \\
&= (q-1)\sum_{k=0}^{K-1} q^k\left(q^k - 1\right) = (q-1)\sum_{k=k_*}^{K-1} q^{2k} - q^k \\
&= (q-1)\left[\frac{q^{2K}-1}{q^2-1} - \frac{q^K - 1}{q-1}\right],
\end{aligned}$$

so

$$R_1(\tau(\nu)) = R_1\left(\frac{\nu}{q}\right) = R_1(q^{K-1}) = (q-1)\left[\frac{q^{2K-2}-1}{q^2-1} - \frac{q^{K-1}-1}{q-1}\right].$$

Now, from (3.29)

$$\begin{aligned}\int_{\tau(\nu)}^{\nu} q(s)R_1(\tau(s))\Delta_q s &= \int_{q^{K-1}}^{q^K} \frac{4}{s^3} R_1\left(\frac{s}{q}\right)\Delta_q s\\ &= (q-1)q^{K-1}\frac{4}{q^{3(K-1)}}R_1(q^{K-2})\\ &= 4(q-1)^2 q^{-2K+2}\left[\frac{q^{2K-4}-1}{q^2-1}-\frac{q^{K-2}-1}{q-1}\right]\\ &\geq 4(q-1)^2 q^{-2K+2}\left[\frac{q^{2K-4}}{q^2-1}-\frac{q^{K-2}}{q-1}\right].\end{aligned}$$

In particular, if we choose $q = 2 > 1$*, then*

$$\liminf_{\nu\to\infty}\int_{\tau(\nu)}^{\nu} q(s)R_1(\tau(s))\Delta_q s = \liminf_{K\to\infty} 4(q-1)^2 q^{-2K+2}\left[\frac{q^{2K-4}}{q^2-1}-\frac{q^{K-2}}{q-1}\right] = \frac{1}{3} < \frac{1}{e}.$$

So, neither Theorem A nor B is applicable to (4.45) *with* $q = 2$*. Using*

$$\widehat{q}_1(\nu) = q(\nu)R_1(\tau(\nu)) = \frac{4(q-1)}{q^{3K}}\left[\frac{q^{2K-2}-1}{q^2-1}-\frac{q^{K-1}-1}{q-1}\right],$$

we get

$$R_2(\tau(\nu)) = \int_{\nu_*}^{\nu}\int_{\nu_*}^{t} e_{-\widehat{q}_1}(s,\nu)\Delta_q s\Delta_q t = (q-1)^2\sum_{k=0}^{K-1} q^k \sum_{j=0}^{k-1} q^j e_{-\widehat{q}_1}(q^j, q^K),$$

where

$$e_{-\widehat{q}_1}(q^j, q^K) = \prod_{u=j}^{K-1}\left[1+(q-1)q^u\widehat{q}_1(q^u)\right]^{-1}.$$

Now,

$$\begin{aligned}q^u\widehat{q}_1(q^u) &= q^u\frac{4(q-1)}{q^{3u}}\left[\frac{q^{2u-2}-1}{q^2-1}-\frac{q^{u-1}-1}{q-1}\right] = \frac{4(q-1)}{q^{2u}}\left[\frac{q^{2u-2}-1}{q^2-1}-\frac{q^{u-1}-1}{q-1}\right]\\ &\leq \frac{4(q-1)}{q^{2u}}\left[\frac{q^{2u-2}}{q^2-1}-\frac{q^{u-1}}{q-1}+1\right] \leq 4(q-1)\left[\frac{q^{-2}}{q^2-1}-\frac{q^{-u-1}}{q-1}+q^{-u}\right]\\ &\leq \frac{4(q-1)}{q^2(q^2-1)} = \frac{4}{q^2(q+1)}.\end{aligned}$$

Thus,

$$\begin{aligned}e_{-\widehat{q}_1}(q^j, q^K) &= \prod_{u=j}^{K-1}\left[1+(q-1)q^u\widehat{q}_1(q^u)\right]^{-1} \geq \prod_{u=j}^{K-1}\left[1+\frac{4(q-1)}{q^2(q+1)}\right]^{-1}\\ &= \left[1+\frac{4(q-1)}{q^2(q+1)}\right]^{-K+j} \geq \left[1+\frac{4(q-1)}{q^2(q+1)}\right]^{-K}.\end{aligned}$$

Therefore,

$$R_2(\tau(\nu)) \geq (q-1)^2 \sum_{k=0}^{K-1} q^k \sum_{j=0}^{k-1} q^j \left[1 + \frac{4(q-1)}{q^2(q+1)}\right]^{-K}$$
$$= (q-1)^2 \left[1 + \frac{4(q-1)}{q^2(q+1)}\right]^{-K} \sum_{k=0}^{K-1} q^k \sum_{j=0}^{k-1} q^j$$
$$= (q-1)^2 \left[1 + \frac{4(q-1)}{q^2(q+1)}\right]^{-K} \left(\frac{q^{2K}-1}{q^2-1} - \frac{q^K-1}{q-1}\right).$$

A straightforward verification shows that (3.29) *holds when* $m = 2$ *and* $q = 2$*. Also, it is easy to check* (3.26) *holds. So, by Corollary 3.1, every solution of* (4.45) *either oscillates or* $w(\nu) \to 0$ *as* $\nu \to \infty$*.*

5. Concluding Remark

By employing an iterative approach together with a method of comparison using first-order dynamic equations, we have derived sufficient conditions to guarantee the oscillatory and/or asymptotic behavior of all solutions to (1.1). In the special case where $\mathbb{T} = \mathbb{R}$, (1.1) reduces to the classical third-order linear delay differential equation discussed in [10], which aligns our results with previously reported findings. However, we found no references addressing the discrete analogues of the results in [10]. This observation motivated us to extend these findings dynamically within the time scales framework, thereby covering both continuous and discrete cases. Consequently, to the best of our knowledge, the results presented here are novel in the discrete setting (see, e.g., (4.34)) and are applicable to a broader class of time scales that satisfy the assumptions of our theorems. A natural direction for future research is to investigate equation (1.1) in the case when (1.2) is convergent. Also, it would be of interest to extend the results obtained here to the more general equations, e.g., to the equations studied in [16, 17].

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[1] Department of Mathematics and Statistics, Vignan's Foundation for Science, Technology and Research (Deemed to be University), Deshmukhi-501512, Hyderabad, India
Email address: c.gokulananda@gmail.com

[2] Department of Engineering Mathematics, Faculty of Engineering, Cairo University, Orman, Giza 12221, Egypt
Email address: saidgrace@yahoo.com

[3] Mathematical Institute, Slovak Academy of Sciences Gresakova 6, SK-04001 Kosice, Slovakia
Email address: jadlovska@saske.sk