

The Solodov–Svaiter type proximal point algorithm on a complete geodesic space

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ABSTRACT. Approximation of zeros of monotone operators can be applied to solve some nonlinear problems on Banach spaces such as function spaces. Similarly, a concept of monotone set-valued vector fields on geodesic spaces includes classes of convex minimisation problems and equilibrium problems. In this paper, we prove a zero point approximation theorem with a projection method for a monotone vector field on complete geodesic spaces. This method guarantees us to generate a sequence converging strongly to a zero point of a given set-valued vector field.

1. INTRODUCTION

In Hilbert spaces, zero point approximation theorems for a monotone operator can be applied to solve some nonlinear problems. Based on the proximal point algorithm investigated by Rockafellar [24], Solodov and Svaiter [26] proved the following strong convergence theorem:

Theorem 1.1 (Solodov–Svaiter [26]). *Let H be a Hilbert space and A a maximal monotone operator on H which has a zero point. Let $\{r_n\}$ be a positive real sequence such that $\inf_{k \in \mathbb{N}} r_k > 0$. Let P_K be the metric projection onto a nonempty closed convex subset K of H . For a given point $u = x_1 \in H$, generate a sequence $\{x_n\}$ of H as follows:*

$$\begin{aligned}y_n &= (I + r_n A)^{-1} x_n; \\H_n &= \{v \in H \mid \langle v - y_n, x_n - y_n \rangle \leq 0\}; \\W_n &= \{w \in H \mid \langle w - x_n, u - x_n \rangle \leq 0\}; \\x_{n+1} &= P_{H_n \cap W_n} u\end{aligned}$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to the closest zero point of A to u .

Recently, convex minimisation problems and equilibrium problems are discussed on a geodesic space with curvature bounded above; see [1, 7, 9, 10, 11, 22]. Such problems are also discussed in the setting of Riemannian manifolds; see [19, 20, 28] and references therein. Similarly, various iterative schemes are generalised to the framework of geodesic spaces by some researchers. For instance, see [13, 16, 25]. In particular, two projection methods were investigated by Kimura and Satô [15] in the setting of Hilbert spheres. One of them is a modified iterative scheme of the Solodov–Svaiter type projection method, which is called the CQ projection method or the Nakajo–Takahashi [23] type projection method.

Recently, the notion of set-valued operators and their monotonicity has been generalised to the setting of geodesic spaces. For instance, Chaipunya, Kohsaka and Kumam [3] have investigated monotone vector fields on a complete CAT(0) space, and recently, the

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author has proposed a class of monotone vector fields on a $\text{CAT}(\kappa)$ space; see [27]. For related results, see [6] for instance.

In this paper, we prove a convergence theorem with the Solodov–Svaiter type proximal point algorithm for a monotone vector field on a $\text{CAT}(\kappa)$ space with a general real number κ . In Theorem 1.1, we need to employ an inner product to generate the approximate sequence. Nevertheless, in general, $\text{CAT}(\kappa)$ spaces do not have any inner product since spherical planes and hyperbolic planes are examples of such spaces. In this work, we introduce tangent spaces on a $\text{CAT}(\kappa)$ space and its metric, and we show a convergence theorem using techniques of Riemannian manifolds.

2. PRELIMINARIES

Let (M, d) be a metric space and $D \in]0, \infty]$. For $x, y \in M$ and $l = d(x, y)$, we call an isometric mapping γ_{xy} from $[0, l]$ into M a geodesic from x to y if $\gamma_{xy}(0) = x$ and $\gamma_{xy}(l) = y$. Additionally, M is said to be uniquely D -geodesic if for all $x, y \in M$ with $d(x, y) < D$, there exists a unique geodesic. In a uniquely D -geodesic space M , for $x, y \in M$ with $d(x, y) < D$ and $t \in [0, 1]$, we define convex combination of x and y with a ratio t by

$$tx \oplus (1-t)y = \gamma_{xy}((1-t)d(x, y)).$$

To define $\text{CAT}(\kappa)$ spaces, we introduce a function c_κ on \mathbb{R} as follows: For $a \in \mathbb{R}$,

$$c_\kappa(a) = \frac{1}{2}a^2 + \sum_{n=2}^{\infty} \frac{(-\kappa)^{n-1}a^{2n}}{(2n)!} = \begin{cases} \frac{1 - \cos(\sqrt{\kappa}a)}{\kappa} & (\kappa > 0); \\ \frac{a^2}{2} & (\kappa = 0); \\ \frac{\cosh(\sqrt{-\kappa}a) - 1}{-\kappa} & (\kappa < 0). \end{cases}$$

From the definition of $c_{\kappa'}$ for $a \in \mathbb{R}$,

$$c'_\kappa(a) = a + \sum_{n=2}^{\infty} \frac{(-\kappa)^{n-1}a^{2n-1}}{(2n-1)!} = \begin{cases} \frac{\sin(\sqrt{\kappa}a)}{\sqrt{\kappa}} & (\kappa > 0); \\ a & (\kappa = 0); \\ \frac{\sinh(\sqrt{-\kappa}a)}{\sqrt{-\kappa}} & (\kappa < 0) \end{cases}$$

and

$$c''_\kappa(a) = 1 + \sum_{n=2}^{\infty} \frac{(-\kappa)^{n-1}a^{2n-2}}{(2n-2)!} = \begin{cases} \cos(\sqrt{\kappa}a) & (\kappa > 0); \\ 1 & (\kappa = 0); \\ \cosh(\sqrt{-\kappa}a) & (\kappa < 0). \end{cases}$$

For more details about c_κ , see [8] for example.

Let M be a metric space. We define a function ϕ_κ from M^2 to $[0, \infty[$ by

$$\phi_\kappa(x, y) = c_\kappa(d(x, y))$$

for $x, y \in M$. We denote the diameter of the model spaces by $D_{\kappa'}$ and define it by

$$D_\kappa = \begin{cases} \frac{\pi}{\sqrt{\kappa}} & \text{if } \kappa > 0; \\ \infty & \text{if } \kappa \leq 0. \end{cases}$$

Further, for $l \in [0, D_\kappa[$, we define an adjuster function $(\cdot)_l^\kappa$ on $[0, 1]$ by

$$(t)_l^\kappa = \begin{cases} \frac{c'_\kappa(tl)}{c'_\kappa(l)} & \text{if } l \neq 0; \\ t & \text{if } l = 0 \end{cases}$$

for $t \in [0, 1]$.

We finally define a $\text{CAT}(\kappa)$ space. Let M be a uniquely D_κ -geodesic space. We call M a $\text{CAT}(\kappa)$ space if

$$\begin{aligned} \phi_\kappa(tx \oplus (1-t)y, z) &\leq (t)_l^\kappa \phi_\kappa(x, z) + (1-t)_l^\kappa \phi_\kappa(y, z) \\ &\quad - (t)_l^\kappa \phi_\kappa(x, tx \oplus (1-t)y) - (1-t)_l^\kappa \phi_\kappa(y, tx \oplus (1-t)y) \end{aligned}$$

for $x, y, z \in M$ with $d(y, z) + d(z, x) + l < 2D_\kappa$ and $t \in [0, 1]$, where $l = d(x, y)$. We call this inequality Stewart's inequality. Note that this definition is not the canonical one. For the usual definition of $\text{CAT}(\kappa)$ spaces, see [2] for instance; for the other one employed in this paper, see [17]. Moreover, a $\text{CAT}(\kappa)$ space M is said to be admissible [12] if

$$d(u, v) < \frac{D_\kappa}{2}$$

for all $u, v \in M$.

Let M be an admissible $\text{CAT}(\kappa)$ space. We call a subset C of M a convex set if

$$tx \oplus (1-t)y \in C$$

for $x, y \in C$ and $t \in [0, 1]$. For a nonempty closed convex subset C of an admissible complete $\text{CAT}(\kappa)$ space M , and for $x \in M$, there exists a unique point $P_C x \in C$ such that

$$d(x, P_C x) = \inf_{y \in C} d(x, y);$$

see [1, 4]. We call such a mapping P_C the metric projection onto C .

For a metric space M and a bounded sequence $\{x_n\}$ of M , we call $x \in M$ an asymptotic centre of $\{x_n\}$ if

$$\limsup_{n \rightarrow \infty} d(x_n, x) = \inf_{y \in M} \limsup_{n \rightarrow \infty} d(x_n, y).$$

A sequence $\{x_n\}$ of M is said to be κ -bounded if

$$\inf_{y \in M} \limsup_{n \rightarrow \infty} d(x_n, y) < \frac{D_\kappa}{2}.$$

We say that a bounded sequence $\{x_n\}$ of M Δ -converges to a Δ -limit $x \in M$ [21] if x is a unique asymptotic centre for any subsequence of $\{x_n\}$. Assume that M is an admissible complete $\text{CAT}(\kappa)$ space. It is well known that a κ -bounded sequence of M has a unique asymptotic centre, and that such a sequence has a Δ -convergent subsequence; refer to [1, 4]. If a κ -bounded sequence $\{x_n\}$ of M Δ -converges to $x \in M$, then

$$d(x, y) \leq \liminf_{n \rightarrow \infty} d(x_n, y)$$

for $y \in M$. If a κ -bounded sequence $\{x_n\}$ of M Δ -converges to $x \in M$ and

$$\lim_{n \rightarrow \infty} d(x_n, y) = d(x, y)$$

for some $y \in M$, then $\{x_n\}$ converges to x . For more details about Δ -convergence, refer to [1, 4, 5, 14] for example.

3. TANGENT SPACES AND MONOTONE VECTOR FIELDS

In what follows, we define tangent spaces on a $\text{CAT}(\kappa)$ space. For more details, see [2, 18].

Let M be an admissible $\text{CAT}(\kappa)$ space. For $p, x, y \in M$, we define the Alexandrov angle A_p at p as follows:

$$A_p(x, y) = \lim_{t \rightarrow 0^+} \arccos \left(1 - \frac{d(\gamma_{px}(t), \gamma_{py}(t))^2}{2t^2} \right) \in [0, \pi]$$

if $p \neq x$ and $p \neq y$; $A_p(x, p) = A_p(p, x) = \pi/2$ if $p \neq x$; $A_p(p, p) = 0$. We obtain the following:

Lemma 3.1. *Let M be an admissible $\text{CAT}(\kappa)$ space. Then,*

$$A_p(x, sy \oplus (1-s)p) = A_p(x, y)$$

for $p, x, y \in M$ and $s \in]0, 1]$.

Proof. Fix $p, x, y \in M$ and $s \in]0, 1]$, and let

$$w = sy \oplus (1-s)p.$$

If $p = x$ or $p = y$, then we obtain the desired result. Thus, we may assume that $p \neq x$ and $p \neq y$. We know that

$$A_p(x, y) = \lim_{t \rightarrow 0^+} \arccos \left(1 - \frac{d(\gamma_{px}(t), \gamma_{py}(t))^2}{2t^2} \right)$$

and

$$A_p(x, w) = \lim_{t \rightarrow 0^+} \arccos \left(1 - \frac{d(\gamma_{px}(t), \gamma_{pw}(t))^2}{2t^2} \right).$$

On the other hand, since

$$\gamma_{py}(d(p, w)) = \gamma_{py}(sd(p, y)) = sy \oplus (1-s)p = w,$$

we get $\gamma_{py}|_{[0, d(p, w)]}$ is a geodesic from p to w since M is uniquely D_κ -geodesic. Fix a real number t such that

$$0 < t < \min\{d(p, x), sd(p, y)\}.$$

Then, $\gamma_{py}(t) = \gamma_{pw}(t)$, and hence

$$1 - \frac{d(\gamma_{px}(t), \gamma_{py}(t))^2}{2t^2} = 1 - \frac{d(\gamma_{px}(t), \gamma_{pw}(t))^2}{2t^2}.$$

Letting $t \rightarrow 0^+$, we have $A_p(x, w) = A_p(x, y)$. □

For more details, refer to [2, Proposition 1.14 in Chapter I.1 and Proposition 3.1 in Chapter II.3] for instance.

Let M be an admissible $\text{CAT}(\kappa)$ space, and let $p \in M$. We define an equivalence relation \sim_p on M by $x \sim_p y$ if

$$A_p(x, y) = 0.$$

For $x \in M$, let us write

$$[x]_p = \{z \in M \mid x \sim_p z\}.$$

Notice that $[p]_p = \{p\}$ since $A_p(p, x) = \pi/2$ if $p \neq x$. Further, we define

$$D_p M = M / \sim_p = \{[x]_p \mid x \in M\}.$$

Then, $(D_p M, A_p)$ is a metric space, where the distance A_p is defined by

$$A_p([x]_p, [y]_p) = A_p(x, y)$$

for $[x]_p, [y]_p \in D_p M$. Additionally, we define a function ζ from $D_p M$ into $\{0, 1\}$ by

$$\zeta([x]_p) = \begin{cases} 0 & ([x]_p = [p]_p); \\ 1 & ([x]_p \neq [p]_p) \end{cases}$$

for $[x]_p \in D_p M$. We define an equivalence relation \simeq_p on a Cartesian product

$$[0, \infty[\times D_p M$$

by $(r_1, [x]_p) \simeq_p (r_2, [y]_p)$ if one of the following conditions is satisfied:

- $r_1 \zeta([x]_p) = r_2 \zeta([y]_p) = 0$;
- $r_1 \zeta([x]_p) = r_2 \zeta([y]_p) > 0$ and $[x]_p = [y]_p$.

Put

$$T_p M = ([0, \infty[\times D_p M) / \simeq_p.$$

Let us write $r[x]_p$ for $[(r, [x]_p)]_{\simeq_p} \in T_p M$, where $[(r, [x]_p)]_{\simeq_p}$ is an equivalence class of $(r, [x]_p)$ by \simeq_p . In particular, we denote $0[p]_p$ by 0_p . Define a bifunction d_p on $T_p M$ by

$$d_p(r[x]_p, s[y]_p) = \sqrt{r^2 \zeta([x]_p) + s^2 \zeta([y]_p) - 2rs \zeta([x]_p) \zeta([y]_p) \cos A_p(x, y)}$$

for $r[x]_p, s[y]_p \in T_p M$. Then, $(T_p M, d_p)$ is a metric space for $p \in M$. We call this metric space the tangent space on M at p . Set

$$TM = \bigsqcup_{p \in M} T_p M = \bigcup_{p \in M} \{(v_p, p) \mid v_p \in T_p M\},$$

and call it the tangent bundle of M . For more details about tangent spaces on geodesic spaces, refer to [2, 18].

Let M be an admissible $\text{CAT}(\kappa)$ space, and let $p \in M$. For $v_p = r[v]_p \in T_p M$ and $t \in [0, \infty[$, we denote a point $(tr)[v]_p$ of $T_p M$ by tv_p . In particular, for $t > 0$, we denote a point $(r/t)[v]_p$ of $T_p M$ by v_p/t . We define a logarithmic mapping $\log_{\kappa, p}$ from M to $T_p M$ by

$$\log_{\kappa, p} x = c'_\kappa(d(p, x))[x]_p \in T_p M$$

for $x \in M$. Similarly, we define a canonical logarithmic mapping \log_p from M to $T_p M$ by

$$\log_p x = d(p, x)[x]_p \in T_p M$$

for $x \in M$. This mapping \log_p is an analogous notion to the inverse mapping of the exponential mapping on Riemannian manifolds. Note that

$$\frac{d(p, x)}{c'_\kappa(d(p, x))} \log_{\kappa, p} x = \log_p x$$

for $x \in M$ with $p \neq x$. We further define a function g_p by

$$g_p(u_p, v_p) = \frac{d_p(u_p, 0_p)^2 + d_p(v_p, 0_p)^2 - d_p(u_p, v_p)^2}{2}$$

for $u_p, v_p \in T_p M$. We call a mapping $g: p \mapsto g_p$ a metric on M . Note that the following hold:

- $g_p(v_p, v_p) \geq 0$ for all $p \in M$ and $v_p \in T_p M$;
- $g_p(u_p, v_p) = g_p(v_p, u_p)$ for $p \in M$ and $u_p, v_p \in T_p M$;
- $tg_p(u_p, v_p) = g_p(u_p, tv_p)$ for $p \in M$, $u_p, v_p \in T_p M$ and $t \geq 0$;
- $g_p(v_p, 0_p) = 0$ for $p \in M$ and $v_p \in T_p M$;
- $c'_\kappa(d(x, y))^2 = g_x(\log_{\kappa, x} y, \log_{\kappa, x} y) = g_y(\log_{\kappa, y} x, \log_{\kappa, y} x)$ for $x, y \in M$;
- $d(x, y)^2 = g_x(\log_x y, \log_x y) = g_y(\log_y x, \log_y x)$ for $x, y \in M$.

We notice that for $p, x \in M$ and $v_p \in T_p M$,

$$g_p(\log_p x, v_p) \leq 0$$

if and only if

$$g_p(\log_{\kappa, p} x, v_p) \leq 0.$$

Recall that since $[p]_p = \{p\}$, we have $\zeta([x]_p) = 0$ if $x = p$, and $\zeta([x]_p) = 1$ if $x \neq p$.

Therefore, for $p, x, y \in M$, we obtain

$$\begin{aligned} g_p(\log_p x, \log_p y) &= \frac{d_p(\log_p x, 0_p)^2 + d_p(\log_p y, 0_p)^2 - d_p(\log_p x, \log_p y)^2}{2} \\ &= \frac{2d(p, x)d(p, y)\zeta([x]_p)\zeta([y]_p)\cos A_p(x, y)}{2} \\ &= d(p, x)d(p, y)\cos A_p(x, y). \end{aligned}$$

We also have

$$g_p(\log_{\kappa, p} x, \log_{\kappa, p} y) = c'_\kappa(d(p, x))c'_\kappa(d(p, y))\cos A_p(x, y)$$

for $p, x, y \in M$. We further have known the following propositions:

Theorem 3.2 (Kimura–Sudo [18]). *Let M be an admissible $\text{CAT}(\kappa)$ space. Then,*

$$g_p(\log_{\kappa, p} x, \log_{\kappa, p} y) \geq \phi_\kappa(p, x) + c''_\kappa(d(p, x))\phi_\kappa(p, y) - \phi_\kappa(x, y),$$

and hence

$$g_p(\log_{\kappa, p} x, \log_{\kappa, p} y) \geq \phi_\kappa(p, x) - \phi_\kappa(x, y)$$

for $p, x, y \in M$.

Theorem 3.3 (Kimura–Sudo [18]). *Let M be an admissible $\text{CAT}(\kappa)$ space. Then,*

$$\lim_{t \rightarrow 0^+} \frac{\phi_\kappa(p, y) - \phi_\kappa(tx \oplus (1-t)p, y)}{t} = g_p(\log_p x, \log_{\kappa, p} y)$$

for $p, x, y \in M$.

In what follows, we introduce monotone vector fields on $\text{CAT}(\kappa)$ spaces. For more details, refer to [27].

Let M be an admissible $\text{CAT}(\kappa)$ space and A a set-valued mapping from M to a subset of the tangent bundle TM . We call A a set-valued vector field if $Ax \subset T_x M$ for $x \in M$. For a set-valued vector field A on M , we denote the domain and the graph of A by

$$\text{Dom } A = \{x \in M \mid Ax \neq \emptyset\};$$

$$\text{Gph } A = \{(x, v_x) \in M \times TM \mid v_x \in Ax\},$$

respectively. We call a point $x \in M$ a zero point of A if

$$0_x \in Ax.$$

We denote the set of all zero points of A by

$$\text{Zero } A = \{x \in M \mid 0_x \in Ax\}.$$

Let A be a set-valued vector field on an admissible $\text{CAT}(\kappa)$ space M . For $r > 0$, we define a set-valued vector field rA on M by

$$rAx = \{rv_x \in T_x M \mid v_x \in Ax\}$$

for $x \in M$. Note that for $r > 0$,

$$\text{Dom}(rA) = \text{Dom } A;$$

$$\text{Zero}(rA) = \text{Zero } A.$$

We say that A is monotone if

$$g_x(\log_x y, u_x) + g_y(\log_y x, v_y) \leq 0$$

for $(x, u_x), (y, v_y) \in \text{Gph } A$. If A is monotone, then so is rA for $r > 0$.

Let M be an admissible $\text{CAT}(\kappa)$ space and A a set-valued vector field on M . We say that A is resolvably monotone if it is monotone, and

$$\left\{ z \in M \mid \frac{\log_{\kappa, z} x}{r} \in Az \right\} \neq \emptyset$$

for any $r > 0$ and any $x \in M$. Suppose that A is resolvably monotone. Then, for $x \in M$ and $r > 0$, from the monotonicity of rA , a set

$$\{z \in M \mid \log_{\kappa, z} x \in rAz\}$$

consists of exactly one point. We denote such a point by $J_{rA}x$, namely,

$$\{J_{rA}x\} = \{z \in M \mid \log_{\kappa, z} x \in rAz\}$$

for $x \in M$. We call the mapping J_{rA} from M to $\text{Dom } A$ the resolvent operator of rA . Note that if A is resolvably monotone, then the zero point set $\text{Zero } A$ coincides with the fixed point set $\text{Fix } J_{rA}$, and it is closed and convex; see [27].

In what follows, we see a convex minimisation problem on geodesic spaces as an example of monotone vector fields.

Let f be a function from an admissible complete $\text{CAT}(\kappa)$ space M to $]-\infty, \infty]$. We say that f is lower semicontinuous if

$$f(x) \leq \liminf_{n \rightarrow \infty} f(x_n)$$

for a sequence $\{x_n\}$ of M converging to $x \in M$. We say that f is proper if there exists $x \in M$ such that $f(x) \in \mathbb{R}$. We further say that f is convex if

$$f(tx \oplus (1-t)y) \leq tf(x) + (1-t)f(y)$$

for $x, y \in M$ and $t \in]0, 1[$. We call $x \in M$ a minimiser of f if

$$f(x) = \inf_{y \in M} f(y).$$

We denote the set of all minimisers of f by $\text{Min } f$, or

$$\underset{x \in M}{\text{Argmin}} f(x).$$

Let f be a proper convex function on an admissible complete $\text{CAT}(\kappa)$ space M . we define a set-valued vector field ∂f on M by

$$\partial f(x) = \left\{ v_x \in T_x M \mid f(x) \leq \inf_{w \in M} (f(w) - g_x(\log_x w, v_x)) \right\}$$

for $x \in M$.

Theorem 3.4 (Sudo [27]). *Let M be an admissible complete $\text{CAT}(\kappa)$ space and f a lower semicontinuous proper convex function from M to $]-\infty, \infty]$ which has a minimiser. Then, the following hold:*

- (i) *The subdifferential ∂f is resolvably monotone;*
- (ii) *for $x \in M$ and $r > 0$,*

$$\{J_{r\partial f}x\} = \underset{y \in M}{\text{Argmin}} \left(f(y) + \frac{1}{r} \phi_\kappa(y, x) \right),$$

where $J_{r\partial f}$ is the resolvent operator for $r\partial f$;

- (iii) *$\text{Zero } \partial f = \text{Min } f$ and $\text{cl } \text{Dom } \partial f = \text{cl } \text{dom } f$.*

4. THE SOLODOV–SVAITER TYPE PROXIMAL POINT ALGORITHM

In this section, we prove an approximation theorem with a projection method proposed by Solodov and Svaiter [26]. Such as Theorem 1.1, we need to employ two half spaces defined with an inner product to generate the iterative sequence. However, in general, $\text{CAT}(\kappa)$ spaces do not have any inner product. Thus, at first, we consider the closedness and the convexity of a subset corresponding to the half spaces used in Theorem 1.1. We first prove the following lemma:

Lemma 4.2. *Let M be an admissible $\text{CAT}(\kappa)$ space. Assume that a subset*

$$\{w \in M \mid d(p, w) \leq d(q, w)\}$$

of M is convex for any $p, q \in M$. Then, a subset

$$\{v \in M \mid g_x(\log_x v, \log_x y) \leq 0\}$$

of M is closed and convex for $x, y \in M$.

Proof. Fix $x, y \in M$ arbitrarily. If $x = y$, then since

$$\{v \in M \mid g_x(\log_x v, \log_x x) \leq 0\} = \{v \in M \mid g_x(\log_x v, 0_x) \leq 0\} = M,$$

it is closed and convex. We assume that $x \neq y$. Let

$$H = \{v \in M \mid g_x(\log_x v, \log_x y) \leq 0\}.$$

Fix $s \in]0, 1[$ arbitrarily. Then, we define a closed convex subset H_s of M by

$$H_s = \{w \in M \mid d(w, x) \leq d(w, sy \oplus (1-s)x)\}.$$

We first show that $H \subset H_s$. Fix $v \in H$ arbitrarily. If $v = x$, then we immediately get $v \in H_s$. Assume $v \neq x$. From the definition the metric g and Lemma 3.1,

$$\begin{aligned} & g_x(\log_{\kappa, x} v, \log_{\kappa, x}(sy \oplus (1-s)x)) \\ &= c'_\kappa(d(x, v))c'_\kappa(d(x, sy \oplus (1-s)x)) \cos A_x(v, sy \oplus (1-s)x) \\ &= c'_\kappa(d(x, v))c'_\kappa(sd(x, y)) \cos A_x(v, y) \\ &= \frac{c'_\kappa(d(x, v))c'_\kappa(sd(x, y))}{d(x, v)d(x, y)} d(x, v)d(x, y) \cos A_x(v, y) \\ &= \frac{c'_\kappa(d(x, v))c'_\kappa(sd(x, y))}{d(x, v)d(x, y)} g_x(\log_x v, \log_x y) \leq 0. \end{aligned}$$

Therefore,

$$\begin{aligned} 0 &\geq g_x(\log_{\kappa, x} v, \log_{\kappa, x}(sy \oplus (1-s)x)) \\ &\geq \phi_\kappa(x, v) + c''_\kappa(d(x, v))\phi_\kappa(x, sy \oplus (1-s)x) - \phi_\kappa(v, sy \oplus (1-s)x) \\ &\geq \phi_\kappa(x, v) - \phi_\kappa(v, sy \oplus (1-s)x), \end{aligned}$$

and thus

$$d(v, x) \leq d(v, sy \oplus (1-s)x).$$

It means that $v \in H_s$. Therefore, $H \subset H_s$ for $s \in]0, 1[$. It implies that

$$H \subset \bigcap_{s \in]0, 1[} H_s.$$

Fix $v \in \bigcap_{s \in]0, 1[} H_s$. Then, for any $s \in]0, 1[$, we have

$$d(v, x) \leq d(v, sy \oplus (1-s)x),$$

or equivalently

$$\phi_\kappa(v, x) - \phi_\kappa(v, sy \oplus (1-s)x) \leq 0.$$

Dividing both sides by s and letting $s \rightarrow 0+$, we have

$$0 \geq \lim_{s \rightarrow 0+} \frac{\phi_\kappa(v, x) - \phi_\kappa(v, sy \oplus (1-s)x)}{s} = g_x(\log_x y, \log_{\kappa, x} v),$$

which implies that $v \in H$, and therefore

$$H = \bigcap_{s \in]0, 1[} H_s.$$

Since H_s is closed and convex for each $s \in]0, 1[$, we conclude that H is closed and convex. \square

Now, we obtain the following zero point approximation theorem with the Solodov–Svaiter type iterative scheme:

Theorem 4.5. *Let M be an admissible complete $\text{CAT}(\kappa)$ space for $\kappa \in \mathbb{R}$. Suppose that for any $x, y \in M$, a subset*

$$\{z \in M \mid d(x, z) \leq d(y, z)\}$$

of M is convex. Let A be a resolvably monotone vector field on M which has a zero point, and let $\{r_n\}$ be a positive real sequence such that $\inf_{k \in \mathbb{N}} r_k > 0$. Let P_K be the metric projection onto a nonempty closed convex subset K of M . For a given anchor point $u = x_1 \in M$, generate a sequence $\{x_n\}$ of M as follows:

$$\begin{aligned} y_n &= J_{r_n A} x_n; \\ H_n &= \{v \in M \mid g_{y_n}(\log_{y_n} v, \log_{y_n} x_n) \leq 0\}; \\ W_n &= \{w \in M \mid g_{x_n}(\log_{x_n} w, \log_{x_n} u) \leq 0\}; \\ x_{n+1} &= P_{H_n \cap W_n} u \end{aligned}$$

for $n \in \mathbb{N}$. Then, the generated sequence $\{x_n\}$ converges to $P_{\text{Zero } A} u$.

Proof. To show $\{x_n\}$ is well defined, we show that $H_n \cap W_n$ is closed and convex, and

$$\text{Zero } A \subset H_n \cap W_n$$

for $n \in \mathbb{N}$. Henceforth, for the simplicity, we denote E_n instead of $H_n \cap W_n$. If x_n is defined, then for $p \in \text{Zero } A$, since

$$(p, 0_p), (y_n, \log_{\kappa, y_n} x_n) \in \text{Gph}(r_n A),$$

from the monotonicity of $r_n A$, we have

$$0 \geq g_p(\log_p y_n, 0_p) + g_{y_n}(\log_{y_n} p, \log_{\kappa, y_n} x_n) = g_{y_n}(\log_{y_n} p, \log_{\kappa, y_n} x_n).$$

It implies that

$$0 \geq g_{y_n}(\log_{y_n} p, \log_{y_n} x_n),$$

and hence $p \in H_n$. Thus, $\text{Zero } A \subset H_n$ if x_n is defined. Now, we prove that

$$\text{Zero } A \subset E_n$$

for $n \in \mathbb{N}$ by induction. Since $u = x_1$, we have

$$W_1 = \{w \in M \mid g_{x_1}(\log_{x_1} w, 0_{x_1}) \leq 0\} = M,$$

and thus $\text{Zero } A \subset E_1$. From the assumption, since H_1 is closed and convex, so is E_1 . For fixed $k \in \mathbb{N}$, we assume that $\text{Zero } A \subset E_k$ and E_k is a closed convex set. Recall that x_{k+1} is defined in this case. Therefore, $\text{Zero } A \subset H_{k+1}$. Let $q \in \text{Zero } A$ and $t \in]0, 1[$. Then, $q \in E_k$ from the assumption of the induction. Since E_k is convex and $x_{k+1} = P_{E_k} u$, we have

$$d(u, x_{k+1}) \leq d(u, tq \oplus (1-t)x_{k+1}),$$

and hence

$$\frac{\phi_\kappa(u, x_{k+1}) - \phi_\kappa(u, tq \oplus (1-t)x_{k+1})}{t} \leq 0.$$

Letting $t \rightarrow 0+$, we get

$$\begin{aligned} 0 &\geq \lim_{t \rightarrow 0+} \frac{\phi_\kappa(u, x_{k+1}) - \phi_\kappa(u, tq \oplus (1-t)x_{k+1})}{t} \\ &= g_{x_{k+1}}(\log_{x_{k+1}} q, \log_{\kappa, x_{k+1}} u). \end{aligned}$$

It means that $q \in W_{k+1}$, and therefore

$$\text{Zero } A \subset H_{k+1} \cap W_{k+1} = E_{k+1}.$$

On the other hand, from the assumption, since both of H_{k+1} and W_{k+1} are closed and convex, so is E_{k+1} . Therefore, the sequence $\{x_n\}$ is well defined since E_n is nonempty, closed and convex for all $n \in \mathbb{N}$. We also obtain $\text{Zero } A \subset E_n$ for $n \in \mathbb{N}$.

We next show that $\{x_n\}$ converges to $P_{\text{Zero } A}u$. Note that $P_{\text{Zero } A}u \in E_n$ for $n \in \mathbb{N}$. Since

$$d(u, P_{E_n}u) \leq d(u, P_{\text{Zero } A}u)$$

for $n \in \mathbb{N}$, we have

$$\sup_{n \in \mathbb{N}} d(u, x_n) = \sup_{n \in \mathbb{N}} d(u, x_{n+1}) = \sup_{n \in \mathbb{N}} d(u, P_{E_n}u) \leq d(u, P_{\text{Zero } A}u) < \frac{D_\kappa}{2},$$

and thus $\{x_n\}$ is κ -bounded. Note that

$$c = \inf_{n \in \mathbb{N}} c''_\kappa(d(u, x_n)) > 0.$$

For $n \in \mathbb{N}$, since $x_{n+1} \in W_n$, we obtain

$$\begin{aligned} 0 &\geq g_{x_n}(\log_{\kappa, x_n} x_{n+1}, \log_{\kappa, x_n} u) = g_{x_n}(\log_{\kappa, x_n} u, \log_{\kappa, x_n} x_{n+1}) \\ &\geq \phi_\kappa(x_n, u) + c''_\kappa(d(x_n, u))\phi_\kappa(x_n, x_{n+1}) - \phi_\kappa(u, x_{n+1}) \\ &\geq \phi_\kappa(x_n, u) - \phi_\kappa(u, x_{n+1}), \end{aligned}$$

which implies that $d(u, x_n) \leq d(u, x_{n+1})$, and that

$$\phi_\kappa(x_n, u) + c''_\kappa(d(x_n, u))\phi_\kappa(x_n, x_{n+1}) - \phi_\kappa(x_{n+1}, u) \leq 0.$$

Therefore, $\{d(u, x_n)\}$ is a convergent real sequence. Since

$$c \cdot \phi_\kappa(x_n, x_{n+1}) \leq c''_\kappa(d(x_n, u))\phi_\kappa(x_n, x_{n+1}) \leq \phi_\kappa(x_{n+1}, u) - \phi_\kappa(x_n, u),$$

we obtain

$$\phi_\kappa(x_n, x_{n+1}) \leq \frac{\phi_\kappa(x_{n+1}, u) - \phi_\kappa(x_n, u)}{c} \rightarrow 0$$

as $n \rightarrow \infty$, and hence

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0.$$

Additionally, for $n \in \mathbb{N}$, since $x_{n+1} \in H_n$, we obtain

$$\begin{aligned} 0 &\geq g_{y_n}(\log_{\kappa, y_n} x_{n+1}, \log_{\kappa, y_n} x_n) = g_{y_n}(\log_{\kappa, y_n} x_n, \log_{\kappa, y_n} x_{n+1}) \\ &\geq \phi_\kappa(y_n, x_n) + c''_\kappa(d(y_n, x_n))\phi_\kappa(y_n, x_{n+1}) - \phi_\kappa(x_n, x_{n+1}) \\ &\geq \phi_\kappa(y_n, x_n) - \phi_\kappa(x_n, x_{n+1}), \end{aligned}$$

and thus $d(y_n, x_n) \leq d(x_n, x_{n+1})$. Consequently, we get

$$\lim_{n \rightarrow \infty} d(y_n, x_n) = 0.$$

Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily. Then, we can take a Δ -convergent subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$. Let $w \in M$ be its Δ -limit. For the sake of simplicity, henceforth, let us write $x_{n_{i_j}}$ as x_j and $y_{n_{i_j}}$ as y_j for $j \in \mathbb{N}$. Note that

$$\lim_{j \rightarrow \infty} d(y_j, x_j) = 0.$$

Since

$$\begin{aligned} \limsup_{j \rightarrow \infty} d(y_j, w) &\leq \limsup_{j \rightarrow \infty} (d(x_j, w) + d(x_j, y_j)) \\ &= \limsup_{j \rightarrow \infty} d(x_j, w) = \inf_{y \in M} \limsup_{j \rightarrow \infty} d(x_j, y) < \frac{D_\kappa}{2}, \end{aligned}$$

the sequence $\{y_j\}$ is κ -bounded. Furthermore, for $v \in M$, we have

$$\begin{aligned} \limsup_{j \rightarrow \infty} d(y_j, w) &\leq \limsup_{j \rightarrow \infty} d(x_j, w) \leq \limsup_{j \rightarrow \infty} d(x_j, v) \\ &\leq \limsup_{j \rightarrow \infty} (d(y_j, v) + d(x_j, y_j)) = \limsup_{j \rightarrow \infty} d(y_j, v), \end{aligned}$$

which implies that w is a unique asymptotic centre of $\{y_j\}$. Now, we prove that w is a zero point of A . From the definition of resolvent operators, we have

$$(J_A w, \log_{\kappa, J_A w} w), \left(y_j, \frac{\log_{\kappa, y_j} x_j}{r_{n_{i_j}}} \right) \in \text{Gph } A$$

for $j \in \mathbb{N}$. From the monotonicity of A , for $j \in \mathbb{N}$, we get

$$\begin{aligned} 0 &\geq g_{J_A w}(\log_{\kappa, J_A w} y_j, \log_{\kappa, J_A w} w) + \frac{g_{y_j}(\log_{\kappa, y_j} J_A w, \log_{\kappa, y_j} x_j)}{r_{n_{i_j}}} \\ &\geq \phi_\kappa(J_A w, y_j) - \phi_\kappa(y_j, w) + \frac{\phi_\kappa(y_j, J_A w) - \phi_\kappa(J_A w, x_j)}{r_{n_{i_j}}} \\ &\geq \phi_\kappa(J_A w, y_j) - \phi_\kappa(y_j, w) - \frac{|\phi_\kappa(y_j, J_A w) - \phi_\kappa(J_A w, x_j)|}{r_{n_{i_j}}} \\ &\geq \phi_\kappa(J_A w, y_j) - \phi_\kappa(y_j, w) - \frac{|c_\kappa(d(y_j, J_A w)) - c_\kappa(d(J_A w, x_j))|}{\inf_{k \in \mathbb{N}} r_k}. \end{aligned}$$

Thus, for $j \in \mathbb{N}$, we obtain

$$(*) \quad \phi_\kappa(y_j, J_A w) \leq \phi_\kappa(y_j, w) + \frac{|c_\kappa(d(y_j, J_A w)) - c_\kappa(d(J_A w, x_j))|}{\inf_{k \in \mathbb{N}} r_k}.$$

Note that c_κ is uniformly continuous on a compact interval, and

$$|d(y_j, J_A w) - d(J_A w, x_j)| \leq d(y_j, x_j) \rightarrow 0$$

as $j \rightarrow \infty$. Therefore, letting $j \rightarrow \infty$ for the equation (*), we have

$$\limsup_{j \rightarrow \infty} \phi_\kappa(y_j, J_A w) \leq \limsup_{j \rightarrow \infty} \phi_\kappa(y_j, w).$$

Hence, $J_A w = w$ since w is a unique asymptotic centre of $\{y_j\}$, and therefore w is a zero point of A since $\text{Fix } J_A = \text{Zero } A$. Since

$$d(u, P_{\text{Zero } A} u) \leq d(u, w) \leq \liminf_{j \rightarrow \infty} d(u, x_j) \leq \limsup_{j \rightarrow \infty} d(u, x_j) \leq d(u, P_{\text{Zero } A} u),$$

we obtain

$$\lim_{j \rightarrow \infty} d(u, x_j) = d(u, P_{\text{Zero } A} u)$$

and $w = P_{Z_{\text{ero}} A} u$. Since $\{x_j\}$ is Δ -convergent to $P_{Z_{\text{ero}} A} u$ and

$$\lim_{j \rightarrow \infty} d(u, x_j) = d(u, P_{Z_{\text{ero}} A} u),$$

we have $\{x_j\}$ converges to $P_{Z_{\text{ero}} A} u$. Consequently, any subsequence of $\{x_n\}$ has a subsequence converging to $P_{Z_{\text{ero}} A} u$. It means that $\{x_n\}$ converges to $P_{Z_{\text{ero}} A} u$. \square

Recall that we only employ a subset

$$(**) \quad \{v \in M \mid g_x(\log_x v, \log_x y) \leq 0\}$$

to generate an approximate sequence in Theorem 4.5. The assumption ‘for any $x, y \in M$, a subset

$$\{z \in M \mid d(x, z) \leq d(y, z)\}$$

of M is convex’ in Lemma 4.2 and Theorem 4.5 merely guarantees us the convexity of the subset (**).

Actually, the following holds:

Theorem 4.6 (Bridson–Haefliger [2]). *Let M be an admissible $\text{CAT}(\kappa)$ space. Then, for fixed $p, x \in M$, a function $A_p(x, \cdot)$ on M is continuous at $y \in M$ with $p \neq y$.*

Using this result, we immediately obtain the following:

Theorem 4.7. *Let M be an admissible $\text{CAT}(\kappa)$ space. Then, for fixed $p \in M$ and $v_p \in T_p M$, a function $g_p(v_p, \log_p(\cdot))$ from M to \mathbb{R} is continuous. Therefore, for any $x, y \in M$, a subset*

$$\{v \in M \mid g_x(\log_x v, \log_x y) \leq 0\}$$

is closed.

Therefore, in the same way as Theorem 4.5, we obtain the following:

Theorem 4.8. *Let M be an admissible complete $\text{CAT}(\kappa)$ space. Suppose that for any $x, y \in M$, a subset*

$$\{v \in M \mid g_x(\log_x v, \log_x y) \leq 0\}$$

of M is convex. Let A be a resolvably monotone vector field on M which has a zero point, and let $\{r_n\}$ be a positive real sequence such that $\inf_{k \in \mathbb{N}} r_k > 0$. Let P_K be the metric projection onto a nonempty closed convex subset K of M . For a given anchor point $u = x_1 \in M$, generate a sequence $\{x_n\}$ of M as follows:

$$\begin{aligned} y_n &= J_{r_n A} x_n; \\ H_n &= \{v \in M \mid g_{y_n}(\log_{y_n} v, \log_{y_n} x_n) \leq 0\}; \\ W_n &= \{w \in M \mid g_{x_n}(\log_{x_n} w, \log_{x_n} u) \leq 0\}; \\ x_{n+1} &= P_{H_n \cap W_n} u \end{aligned}$$

for $n \in \mathbb{N}$. Then, the generated sequence $\{x_n\}$ converges to $P_{Z_{\text{ero}} A} u$.

In this paper, by imposing an additional assumption on a $\text{CAT}(\kappa)$ space, we obtain a zero point approximation theorem. For now, we have two assumptions corresponding to the convexity of half spaces on an admissible $\text{CAT}(\kappa)$ space M as follows:

- (H1) For any $x, y \in M$, a subset $\{z \in M \mid d(x, z) \leq d(y, z)\}$ is convex;
- (H2) for any $x, y \in M$, a subset $\{v \in M \mid g_x(\log_x v, \log_x y) \leq 0\}$ is convex.

Lemma 4.2 means that the condition (H1) implies the condition (H2). Nevertheless, we have not known about its opposite yet. The condition (H1) is investigated by [4] for instance; we have many examples and some counter-examples corresponding to the condition (H1). For instance, the condition (H1) is satisfied by the space forms such as spheres, hyperboloids, and Hilbert spaces. Further, it is satisfied by \mathbb{R} -tree. According to the results

in [4], a CAT(0) glueing space containing two spaces of constant but different curvature does not enjoy the condition (H1).

In this work, we have only obtained theoretical results about convergence theorem. Further research into computational cost and speed is needed for practical applications in computer science.

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