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Amenable gyrogroups and their fixed-point property*

TEERAPONG SUKSUMRAN[1] AND JATURON WATTANAPAN[2,3]

ABSTRACT. An amenable group is a group that possesses a left-invariant mean and, furthermore, satisfies the Markov–Kakutani Fixed-Point Theorem (also known as Day's Fixed-Point Theorem). In this article, we examine gyrogroups that satisfy the Markov–Kakutani–Day fixed-point property, and also give a characterization of amenable groups related to a fixed-point property.

1. INTRODUCTION

The Brouwer fixed-point theorem, given in [2], is undeniably among the most renowned results in fixed-point theory. It says that any continuous function from a closed ball in n-dimensional Euclidean space to itself must have a fixed-point. This theorem has had far-reaching implications, with applications spanning differential equations, game theory, and economics. Its success paved the way for later generalizations. For example, the theorem is extended to infinite-dimensional Banach spaces and to locally convex topological vector spaces for continuous functions on compact convex subsets by Schauder [9] and Tychonoff [15], respectively. For a brief summary of the development of fixed-point theory, we refer the reader to [7].

In the 1930s, rather than a fixed-point of a single function, a common fixed-point of a family of functions was starting to gain attention as well. Early contributions by Markov and Kakutani [5] focused on fixed-points for families of commuting continuous affine transformations. This line of research later gained prominence as Day [3,4] demonstrated its connection to the concept of amenable (semi)groups.

The Markov–Kakutani Fixed-Point Theorem, also called Day's Fixed-Point Theorem, states in a modern approach that if a group G acts on a compact convex set K in a Hausdorff locally convex topological vector space by continuous affine transformations, that is, if the induced function $g : k \mapsto g \cdot k$, $k \in K$, is continuous and $g \cdot (tk_1 + (1-t)k_2) = t(g \cdot k_1) + (1-t)(g \cdot k_2)$ for all $k_1, k_2 \in K$, $0 \le t \le 1$ and for all $g \in G$, then G has a common fixed point in K. It turns out that this is equivalent to saying that G is amenable as stated in [14, Theorem 12.11]. Therefore, a strong connection between amenability and a fixed-point property exists. In [18], the authors extend the notion of being amenable to the case of gyrogroups, and then give a characterization of amenable gyrogroups related to Tarski's Theorem. This motivates us to continue to study the property of being amenable in connection with a fixed-point property for gyrogroups.

Loosely speaking, a gyrogroup, coined by Ungar, is an algebraic group-like structure whose operation is in general non-associative. Gyrogroups have been shown to have strong connections to groups, with many group-related results naturally extending to gyrogroups. One of the most prominent examples of a gyrogroup is the complex Möbius gyrogroup [17], which consists of the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex

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Corresponding author: Jaturon Wattanapan; jaturon.w@cmu.ac.th

plane endowed with Möbius addition $\oplus_M$ defined by $a \oplus_M b = \dfrac{a+b}{1+\overline{a}b}$ for all $a, b \in \mathbb{D}$. Its gyroautomorphisms represent rotations of the disk. Moreover, Möbius addition induces Möbius transformations on the disk, which justifies the name of the gyrogroup.

2. PRELIMINARIES

In this section, we collect terminology and basic facts that will be used later on. See, for instance, [10,16,18] for further details. Recall that a *homomorphism* between gyrogoups is defined as a function that preserves the gyrogroup operations. A subgyrogroup K of a gyrogroup G is *normal* if $K = \varphi^{-1}(\{e\})$, where φ is a gyrogroup homomorphism from G to H and e is the identity of H. Let $(G, \oplus)$ be a gyrogroup, and let a, b, c be elements in G. As in [11], the *associator* of a, b, and c is defined by $[a, b, c] = \ominus(a\oplus(b\oplus c))\oplus((a\oplus b)\oplus c)$. Denoted by G^{a} the smallest normal subgyrogroup containing all the associators of elements from G. It is proved in Proposition 3.1 of [11] that the quotient gyrogroup G/G^{a} is, in fact, a group called the *associativization* of G. The importance of the associativization of G lies in the following universal property.

Theorem 2.1 (Theorem 3.1 of [11]). *Suppose that φ is a homomorphism from a gyrogroup G to a group Γ. Then there exists a unique homomorphism $\bar{\varphi}$ from G/G^{a} to Γ such that $\bar{\varphi} \circ \pi = \varphi$. Here, π is the canonical projection given by $\pi(a) = a \oplus G^{\mathrm{a}}$ for all $a \in G$. In fact, $\bar{\varphi}$ is given by $\bar{\varphi}(a \oplus G^{\mathrm{a}}) = \varphi(a)$ for all $a \in G$.*

An *action* of a gyrogroup G on a non-empty set X is a function from $G \times X$ to X, written $a \cdot x$ for its image of $(a, x) \in G \times X$, such that $e \cdot x = x$ for all $x \in X$ and $a \cdot (b \cdot x) = (a \oplus b) \cdot x$ for all $a, b \in G, x \in X$. A *global fixed-point* of the action of G on X is defined as a point x in X such that $a \cdot x = x$ for all $a \in G$ [12]. One of the fundamental consequences of Theorem 2.1 is that we can study a gyrogroup action of a gyrogroup G via studying the group action of G/G^{a} and vice versa. More precisely, if a gyrogroup G acts on a non-empty set X, then the group G/G^{a} also acts on X by the formula

$$(a \oplus G^{\mathrm{a}}) \cdot x = a \cdot x \tag{2.1}$$

for all $a \in G$ and for all $x \in X$. Similarly, Equation (2.1) can be used to induce a gyrogroup action of G in a natural way whenever the group action of G/G^{a} is given.

The notion of amenability is extended to the case of gyrogroups in [18]. Recall that a gyrogroup G is *amenable* if it admits a left-invariant finitely additive probability measure. The following theorem collects basic properties of amenable gyrogroups, which will prove useful in the sequel.

Theorem 2.2 (Theorem 3.6 of [18]). *Let G be a gyrogroup.*

(1) Finite gyrogroups are amenable.
(2) Any non-zero measure subgyrogroup of an amenable gyrogroup is amenable.
(3) If N is a normal subgyrogroup of G and G is amenable, then G/N is amenable.
(4) If N is a normal subgyrogroup of G such that N and G/N are amenable, then G is amenable.
(5) Let $\{G_i\}_{i\in I}$ be a collection of amenable subgyrogroups of a gyrogroup H. If for all $i, j \in I$, there is an index $k \in I$ such that $G_i, G_j \subseteq G_k$, then the gyrogroup $G = \bigcup_{i\in I} G_i$ is amenable.

3. MAIN RESULTS

3.1. **Basic properties of amenable gyrogroups.** In this section, we follow [18] to prove a few elementary properties of amenable gyrogroups, including the existence and uniqueness of a maximal normal amenable subgyrogroup sitting inside a given gyrogroup.

Proposition 3.1. *Let $G_1, G_2, \dots, G_n$ be amenable gyrogroups. Then the direct product $\prod_{i=1}^{n} G_i$ is amenable.*

Proof. Recall that if G and H are gyrogroups, then $(G \times H)/(\{e\} \times H) \cong G$. Hence, if G and H are amenable, then $\{e\} \times H$ and $(G \times H)/(\{e\} \times H)$ are amenable. By part 4 of Theorem 2.2, $G \times H$ is amenable. Thus, the proposition follows from the principle of mathematical induction. □

Proposition 3.2. *Let $\{G_i \mid i \in I\}$ be a non-empty collection of amenable gyrogroups. Then the weak direct product $\prod_{i\in I}^{w} G_i$ is amenable.*

Proof. The collection $\mathcal{F}$ of finite product of G_i's, by Proposition 3.1, satisfies the condition in part 5 of Theorem 2.2. Therefore, $\prod_{i\in I}^{w} G_i = \bigcup \mathcal{F}$ is amenable. □

We remark that an arbitrary direct product $\prod_{i\in I} G_i$ of amenable gyrogroups need not be amenable because there is a counter-example in the case of groups; see, for instance, [6, p. 261]. The next theorem shows that an arbitrary gyrogroup contains a unique largest normal amenable subgyrogroup. Its proof applies the Second Isomorphism Theorem for gyrogroups as well as Zorn's Lemma. We begin with proving the following lemma, which is an important property of gyrogroups.

Lemma 3.1. *Let A and B be normal subgyrogroups of a gyrogroup G. Then $\langle A \cup B\rangle = A \oplus B$ and $A \oplus B \trianglelefteq G$.*

Proof. Obviously, $A \oplus B \subseteq \langle A \cup B\rangle$. Since $A, B \trianglelefteq G$, we obtain that

$$\begin{aligned}(A \oplus B) \oplus (A \oplus B) &= ((A \oplus B) \oplus A) \oplus B\\ &= ((A \oplus A) \oplus B) \oplus B\\ &= A \oplus (B \oplus B)\\ &= A \oplus B.\end{aligned}$$

Furthermore, $\ominus(A \oplus B) = \ominus B \ominus A = B \oplus A = A \oplus B$. This shows that $A \oplus B$ is a subgyrogroup of G. It follows that $\langle A \cup B\rangle = A \oplus B$. Now, let $g, h \in G$. Then

$$\begin{aligned}(g \oplus (A \oplus B)) \oplus h &= ((g \oplus B) \oplus A) \oplus h\\ &= ((g \oplus B) \oplus h) \oplus A\\ &= ((g \oplus h) \oplus B) \oplus A\\ &= (g \oplus h) \oplus (B \oplus A)\\ &= (g \oplus h) \oplus (A \oplus B).\end{aligned}$$

Similarly, one can show that $g \oplus ((A \oplus B) \oplus h) = (g \oplus h) \oplus (A \oplus B)$. Hence, $A \oplus B$ forms a normal subgyrogroup of G. □

Theorem 3.3. *Every gyrogroup G has the largest normal amenable subgyrogroup. In other words, there exists a normal amenable subgyrogroup H of G such that if K is a normal amenable subgyrogroup of G, then $K \subseteq H$.*

Proof. Let $\mathcal{N}$ be the collection of normal amenable subgyrogroups of G. For $A, B \in \mathcal{N}$, define $A \leq B$ if and only if $A \subseteq B$. Then $(\mathcal{N}, \leq)$ is a partially ordered set. Part 5 of Theorem 2.2 implies that every chain in $\mathcal{N}$ has an upper bound in $\mathcal{N}$. By Zorn's Lemma, there is a normal amenable subgyrogroup H of G not contained in another element of $\mathcal{N}$. Suppose that $X \in \mathcal{N}$, and assume that $X \nsubseteq H$. By Lemma 3.1, $X \oplus H$ is a normal subgyrogroup of G containing H. By the Second Isomorphism Theorem (see Theorem 33 of [10]), $(X \oplus H)/H \cong X/(X \cap H)$. By part 3 of Theorem 2.2, $X/(X \cap H)$ is amenable, and so is $(X \oplus H)/H$. By part 4 of Theorem 2.2, $X \oplus H$ is amenable, a contradiction. □

As a consequence of Theorem 3.3, the largest normal amenable subgyrogroup of the complex Möbius gyrogroup is either the trivial subgyrogroup or itself. This fact can be proved by applying the following proposition, which is important in its own right.

Proposition 3.3. *There is no non-trivial proper normal subgyrogroup of the complex Möbious gyrogroup* $\mathbb{D}$*.*

Proof. Suppose that N is a non-trivial normal subgyrogroup of $\mathbb{D}$. Hence, as in the proof of Proposition 35 of [10], $\operatorname{gyr}[a, b](N) = N$ for all $a, b \in \mathbb{D}$. Since the gyroautomorphisms of $\mathbb{D}$ generate the rotation group of $\mathbb{D}$ (see Example 1 of [13]), if $z \in N$, then $S(z) \subseteq N$, where $S(z)$ denotes the circle in $\mathbb{C}$ centered at 0 containing z. Next, let $x \neq 0$ be an element of N. Then $x \oplus S(x) \subseteq N$. Since $0 = x \ominus x$ and $2x = x \oplus x$, we have $0, 2x \in x \oplus S(x) \subseteq N$. Since the modulus function and left gyrotranslation L_x are continuous and $S(x)$ is connected, for each r with $0 < r < |2x|$, there is an element $y \in S(x)$ such that $|x \oplus y| = r$ by the Intermediate Value Theorem. This shows that $B(0, |2x|) \subseteq N$. It follows from Proposition 7 of [1] that N is an open and a closed subgyrogroup of $\mathbb{D}$, which implies that $N = \mathbb{D}$. □

In light of Proposition 3.3, it follows that the complex Möbious gyrogroup is indeed a concrete example of a simple gyrogroup, which is defined as a gyrogroup without non-trivial proper normal subgyrogroups.

3.2. **Fixed-point property and amenability.** Next, we will examine a strong connection between amenability and a fixed-point property of gyrogroups. Inspired by the Markov–Kakutani Fixed-Point Theorem as well as Day's Fixed-Point Theorem, we formulate the following definition for gyrogroups.

Definition 3.1. *A gyrogroup G is said to have the Markov–Kakutani–Day fixed-point property if whenever G acts on a compact convex set K in a Hausdorff locally convex topological vector space by continuous affine transformations, that is, if the induced function $g : k \mapsto g \cdot k$, $k \in K$, is continuous and $g \cdot (tk_1 + (1-t)k_2) = t(g \cdot k_1) + (1-t)(g \cdot k_2)$ for all $k_1, k_2 \in K$, $0 \leq t \leq 1$ and for all $g \in G$, then there is a global fixed-point of the action of G on K.*

In view of Definition 3.1, a group is amenable if and only if it satisfies the Markov–Kakutani–Day fixed-point property. In this section, we aim to generalize this result to the case of gyrogroups.

Theorem 3.4. *If a gyrogroup G is amenable, then it has the Markov–Kakutani–Day fixed-point property.*

Proof. Suppose that G is an amenable gyrogroup. By part 3 of Theorem 2.2, the group G/G^{a} is amenable. Let K be a compact convex set in a Hausdorff locally convex topological vector space. Suppose that $\varphi : G \to \operatorname{Sym}(K)$ is a gyrogroup action by continuous affine transformations of G on K. In a similar fashion to the proof of Theorem 3.4 of [8], the group G/G^{a} acts on K by continuous affine transformations. Thus, a global fixed-point k in K of the action of G/G^{a} on K exists as above. The point k is also a global fixed-point of the action of G on K, which completes the proof. □

It is an open question whether the converse of Theorem 3.4 holds. However, a partial converse is presented below.

Theorem 3.5. *If a gyrogroup G has the Markov–Kakutani–Day fixed-point property, then G/G^{a} is amenable.*

Proof. Since every group action of G/G^{a} induces a gyrogroup action of G on the same set by the same formula, it follows that G/G^{a} has the Markov–Kakutani–Day fixed-point property. Since G/G^{a} is a group, G/G^{a} is amenable as above. □

The main result of this section, which is a characterization of amenable gyrogroups related to a fixed-point property in a suitable way, is proved in the following theorem. It is proved in [18] that a gyrogroup G is amenable if and only if it satisfies the Fϕlner's condition. That is, G is amenable if and only if for each finite subset E of G and for each $\epsilon > 0$, there exists a non-empty finite subset F of G such that $|(g\oplus F)\Delta F| \leq \epsilon|F|$ for all $g \in E$. For any gyrogroup G, let $\mathcal{S}$ be the set of all finite subsets of G. Define a partial order on $\mathcal{S}\times\mathbb{N}$ by $(E_1, n_1) \leq (E_2, n_2)$ if $E_1 \subseteq E_2$ and $n_1 \leq n_2$. If G is amenable, for each $(E, n) \in \mathcal{S}\times\mathbb{N}$, let $F_{(E,n)}$ be a non-empty finite subset of G such that

$$|(g\oplus F_{(E,n)})\Delta F_{(E,n)}| \leq \frac{|F_{(E,n)}|}{n}$$

for all $g \in E$. Then $\{F_{(E,n)}\}_{(E,n)\in\mathcal{S}\times\mathbb{N}}$ is a net in $\mathcal{S}$ with the property

$$\frac{|(g\oplus F_{(E,n)})\Delta F_{(E,n)}|}{|F_{(E,n)}|} \to 0 \tag{3.2}$$

for all $g \in G$. A net in $\mathcal{S}$ that satisfies 3.2 is called a *Fϕlner's net* and it is not hard to see that a gyrogroup is amenable if and only if it admits a Fϕlner net.

Theorem 3.6. *Let G be a gyrogroup. Then the following statements are equivalent.*

(1) G is amenable.

(2) For any compact convex subset K of a Hausdorff locally convex topological vector space E, if a function $\cdot : G\times K \to K$ satisfies the following properties:

(i) the function $k \mapsto x\cdot k$ is a continuous affine function from K to K for all $x \in G$, and

(ii) there is a point $k_0 \in K$ such that $x\cdot(y\cdot k_0) = (x\oplus y)\cdot k_0$ for all $x, y \in G$,

then G admits a global fixed-point in K.

Proof. Suppose that G is amenable. Let $\{F_i\}_{i\in I}$ be a Fϕlner's net. For each $i \in I$, define

$$k_i = \frac{1}{|F_i|}\sum_{f\in F_i} f\cdot k_0.$$

Then $\{k_i\}_{i\in I}$ is a net in K. Since K is compact, we may assume that $\{k_i\}_{i\in I}$ converges to a point k of K. We will show that k is a global fixed-point of G. Now, let $g \in G$, to show that $g\cdot k = k$, it suffices to show that $\varphi(g\cdot k) = \varphi(k)$ for all $\varphi \in E^*$. Observe that

$$\left|\varphi\left(\frac{1}{|F_i|}\sum_{f\in(g\oplus F_i)\setminus F_i} f\cdot k_0\right)\right| = \left|\frac{1}{|F_i|}\sum_{f\in(g\oplus F_i)\setminus F_i}\varphi(f\cdot k_0)\right| \leq \frac{|(g\oplus F_i)\setminus F_i|M}{|F_i|}$$

for some constant $M > 0$. It follows that

$$\varphi\left(\frac{1}{|F_i|}\sum_{f\in(g\oplus F_i)\setminus F_i} f\cdot k_0\right) \to 0.$$

In similar fahsion, we obtain that

$$\varphi\left(\frac{1}{|F_i|}\sum_{f\in F_i\setminus(g\oplus F_i)} f\cdot k_0\right)\to 0.$$

Thus, the following series of equality holds:

$$\begin{aligned}
\varphi(g\cdot k) &= \lim_{i\to\infty}\varphi(g\cdot k_i)\\
&= \lim_{i\to\infty}\frac{1}{|F_i|}\sum_{f\in F_i}\varphi(g\cdot(f\cdot k_0))\\
&= \lim_{i\to\infty}\frac{1}{|F_i|}\sum_{f\in F_i}\varphi((g\oplus f)\cdot k_0)\\
&= \lim_{i\to\infty}\frac{1}{|F_i|}\sum_{f\in g\oplus F_i}\varphi(f\cdot k_0)\\
&= \lim_{i\to\infty}\left(\frac{1}{|F_i|}\sum_{f\in(g\oplus F_i)\setminus F_i}\varphi(f\cdot k_0)\right)+\lim_{i\to\infty}\left(\frac{1}{|F_i|}\sum_{f\in(g\oplus F_i)\cap F_i}\varphi(f\cdot k_0)\right)\\
&= \lim_{i\to\infty}\left(\frac{1}{|F_i|}\sum_{f\in F_i\setminus(g\oplus F_i)}\varphi(f\cdot k_0)\right)+\lim_{i\to\infty}\left(\frac{1}{|F_i|}\sum_{f\in(g\oplus F_i)\cap F_i}\varphi(f\cdot k_0)\right)\\
&= \lim_{i\to\infty}\frac{1}{|F_i|}\sum_{f\in F_i}\varphi(f\cdot k_0)\\
&= \lim_{i\to\infty}\varphi\left(\frac{1}{|F_i|}\sum_{f\in F_i} f\cdot k_0\right)\\
&= \varphi(k).
\end{aligned}$$

Conversely, assume that the above conditions hold. Recall that the set of all means on $l^\infty(G)$, denoted by $M(G)$, is a compact convex subset of $l^\infty(G)^*$ in weak*-topology. For each $x\in G$ and for each mean m on $l^\infty(G)$, define a function $x\cdot m: l^\infty(G)\to\mathbb{R}$ by the formula $(x\cdot m)(f)=m(f\circ L_x)$ for all $f\in l^\infty(G)$. It is not difficult to see that $x\cdot m$ is also a mean on $l^\infty(G)$, and that $m\mapsto x\cdot m$ is continuous affine. Now, let δ_e be the mean defined by $\delta_e(f)=f(e)$ for all $f\in l^\infty(G)$. Then we obtain that $x\cdot(y\cdot\delta_e)=(x\oplus y)\cdot\delta_e$ for all $x,y\in G$. By assumption, there exists a mean $\mu\in M(G)$ that that $x\cdot\mu=\mu$ for all $x\in G$, and so μ is a left-invariant mean on $l^\infty(G)$. □

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References

[1] Atiponrat, W. Topological gyrogroups: Generalization of topological groups. *Topology Appl.* **224** (2017), 73–82.

[2] Brouwer, L. E. J. Uber Abbildung von Mannigfaltigkeiten. *Math. Ann.* **71** (1912), 97–115.

[3] Day, M. Fixed-point theorems for compact convex sets. *Illinois J. Math.* **5** (1961), 585–590.

[4] Day, M. Correction to my paper "Fixed-point theorems for compact convex sets". *Illinois J. Math.* **8** (1964), 713.
[5] Kakutani, S. Two fixed-point theorems concerning bicompact convex sets. *Proc. Imp. Acad. Jap.* **14** (1938), 242–245.
[6] Keller, G. Amenable groups and varieties of groups. *Illinois J. Math.* **16** (1972), 257–269.
[7] Kumar, S. A Short Survey of the Development of Fixed Point Theory. *Surv. Math. Appl.* **8** (2013), 91–101.
[8] Panyanak, B.; Atiponrat, W.; et al. Extended Cartan fixed point theorem. *J. Fixed Point Theory Appl.* **24** (2022), no. 4, Article 76 (12 pages).
[9] Schauder, J. Der Fixpunktsatz in Funktionalraumen. *Studia Math.* **2** (1930), 171–180.
[10] Suksumran, T. The algebra of gyrogroups: Cayley's theorem, Lagrange's theorem, and isomorphism theorems, *Essays in mathematics and its applications* (Th. M. Rassias and P. M. Pardalos, eds.). Springer, Cham, 2016, 369–437.
[11] Suksumran, T. Associativization of gyrogroups and the universal property. *Asian-Eur. J. Math.* **17** (2024), no. 11, Article 2450066 (16 pages).
[12] Suksumran, T. Global-fixed-point property of gyrogroup actions. *Carpathian J. Math.* **41** (2025), no. 2, 517–523.
[13] Suksumran, T.; Wattanapan, J. On left-gyrotranslation groups of gyrogroups. *Algebra Discrete Math.* **38** (2024), no. 1, 115–126.
[14] Tomkowicz, G.; Wagon, S. *The Banach–Tarski Paradox*, 2nd ed., Encyclopedia of Mathematics and its Applications, **series no. 24**. Cambridge University Press, Cambridge, 2016.
[15] Tychonoff, A. Ein Fixpunktsatz. *Math. Ann.* **111** (1935), 767–776.
[16] Ungar, A. *Analytic hyperbolic geometry and Albert Einstein's Special Theory of Relativity*. World Scientific, Hackensack, NJ, 2008.
[17] Ungar, A. From Möbius to gyrogroups. *Amer. Math. Monthly.* **115** (2008), no. 2, 138–144.
[18] Wattanapan, J.; Suksumran, T. Characterizations of amenable gyrogroups related to Tarski's theorem. *Carpathian J. Math.* **41** (2025), no. 1, 223–235.

[1] CHIANG MAI UNIVERSITY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS
CHIANG MAI 50200, THAILAND
Email address: teerapong.suksumran@cmu.ac.th

[2] CHIANG MAI UNIVERSITY
FACULTY OF SCIENCE
CHIANG MAI 50200, THAILAND

[3] CHIANG MAI UNIVERSITY
OFFICE OF RESEARCH ADMINISTRATION
CHIANG MAI 50200, THAILAND
Email address: jaturon.w@cmu.ac.th